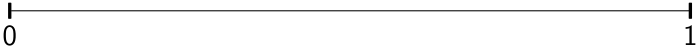
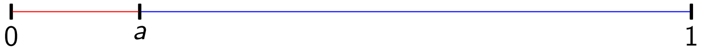
Discrete structures coding interval exchange transformations

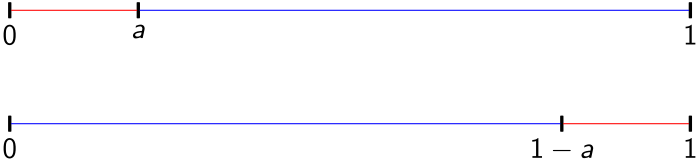
Luca Q. Zamboni
Université de Lyon 1
Reykjavik University

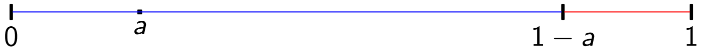
Rouen
19/09/2008

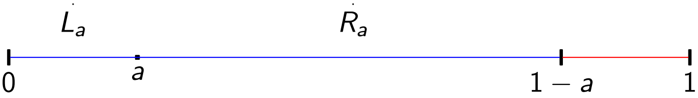
Euclidean Algorithm

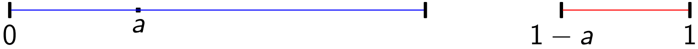



0
a
1

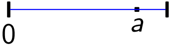

0
a
1
0
1 − a
1


0
a
1 − a
1


L_a
R_a
0
a
$1 - a$
1


0
a
1 − a
1


0
a
1 − a
1

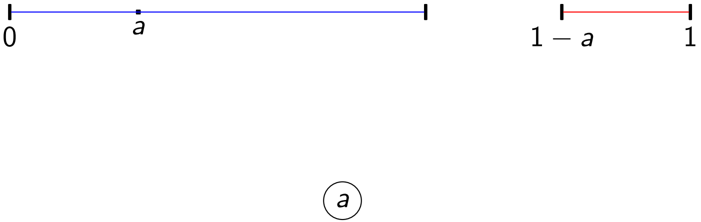

0
a

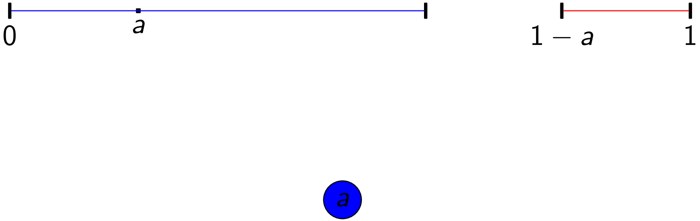

1 − a
1


0
a
1 − a
1

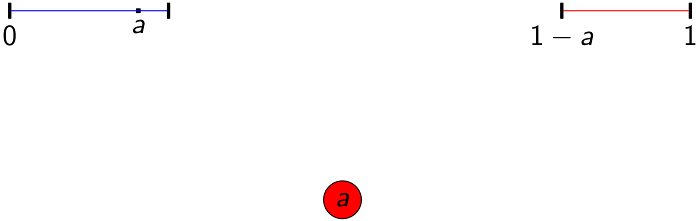

0
a
1 − a
1


0
a
1 − a
1


0
a
1 − a
1
a

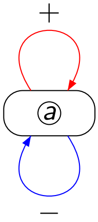


$$L_a - R_a < 0$$

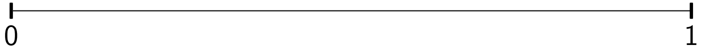


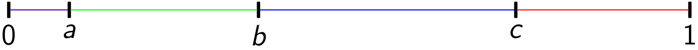
$$L_a - R_a > 0$$

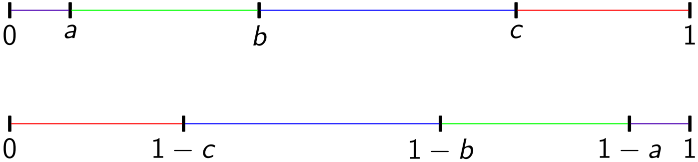
The Graph $G_0(1)$

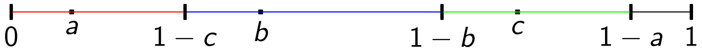


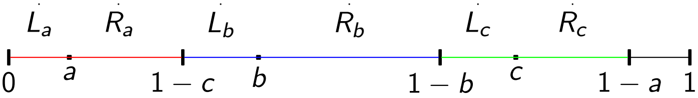
A Generalization



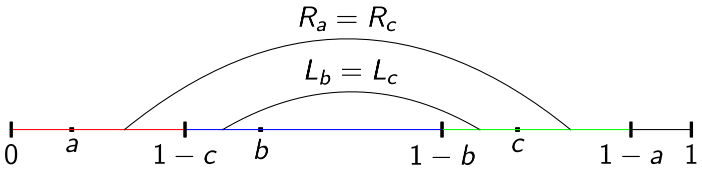

0
a
b
c
1

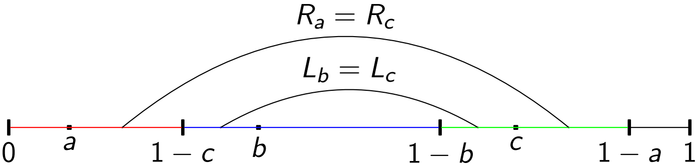

0
a
b
c
1
0
1 − c
1 − b
1 − a
1

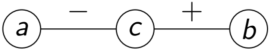

0
a
1 − c
b
1 − b
c
1 − a
1

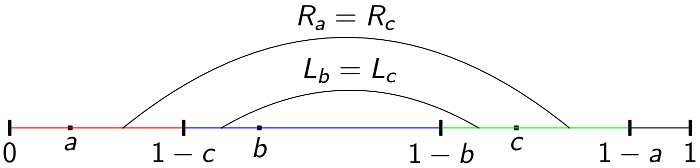


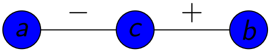
$\dot{L}_a$
$\dot{R}_a$
$\dot{L}_b$
$\dot{R}_b$
$\dot{L}_c$
$\dot{R}_c$
0
a
$1-c$
b
$1-b$
c
$1-a$
1


$R_a = R_c$
$L_b = L_c$
0
a
1 − c
b
1 − b
c
1 − a
1

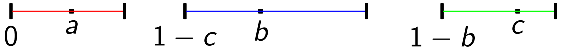

$R_a = R_c$
$L_b = L_c$
0
a
1 − c
b
1 − b
c
1 − a
1

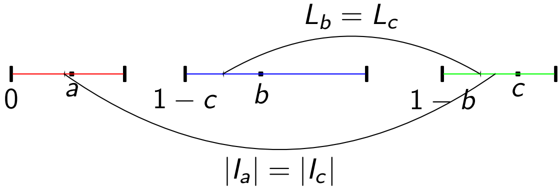

a
−
c
+
b

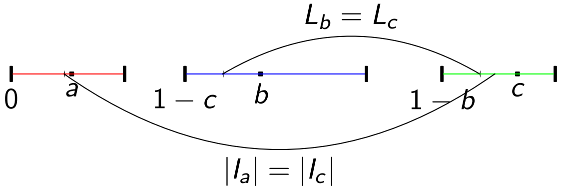


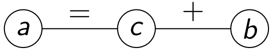


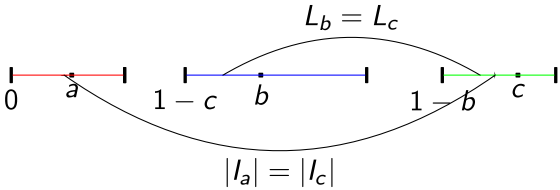
$$L_a - R_a < 0; \quad L_c - R_c < 0; \quad L_b - R_b < 0.$$

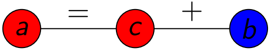

0
a
1 − c
b
1 − b
c


$L_b = L_c$
0
a
$1 - c$
b
$1 - b$
c
$|I_a| = |I_c|$

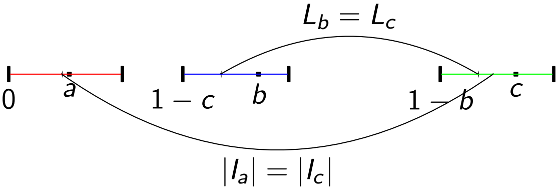

$L_b = L_c$
0
a
1 − c
b
1 − b
c
$|I_a| = |I_c|$

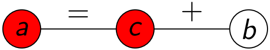

a
=
c
+
b

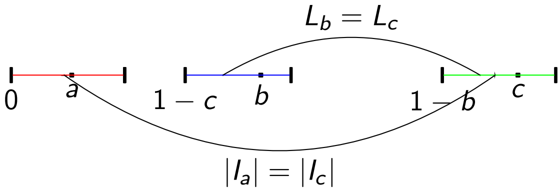


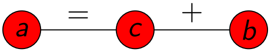


$$L_a - R_c > 0; \quad L_c - R_a > 0; \quad L_b - R_b < 0.$$

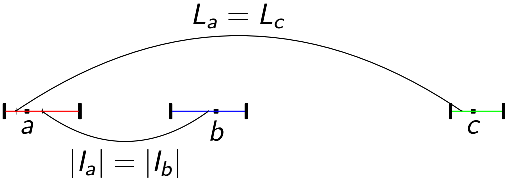

$L_b = L_c$
0
a
$1 - c$
b
$1 - b$
c
$|I_a| = |I_c|$

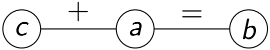

a
$=$
c
$+$
b

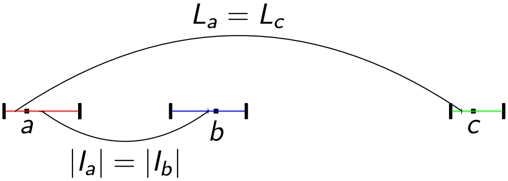


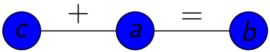


$$L_a - R_c > 0; \quad L_c - R_a > 0; \quad L_b - R_b > 0.$$

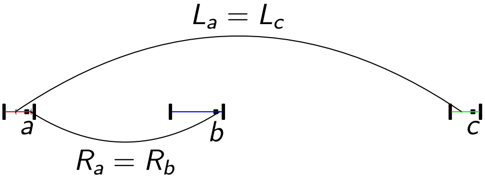

$L_a = L_c$
a
b
c
$|I_a| = |I_b|$


c
$+$
a
$=$
b

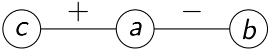




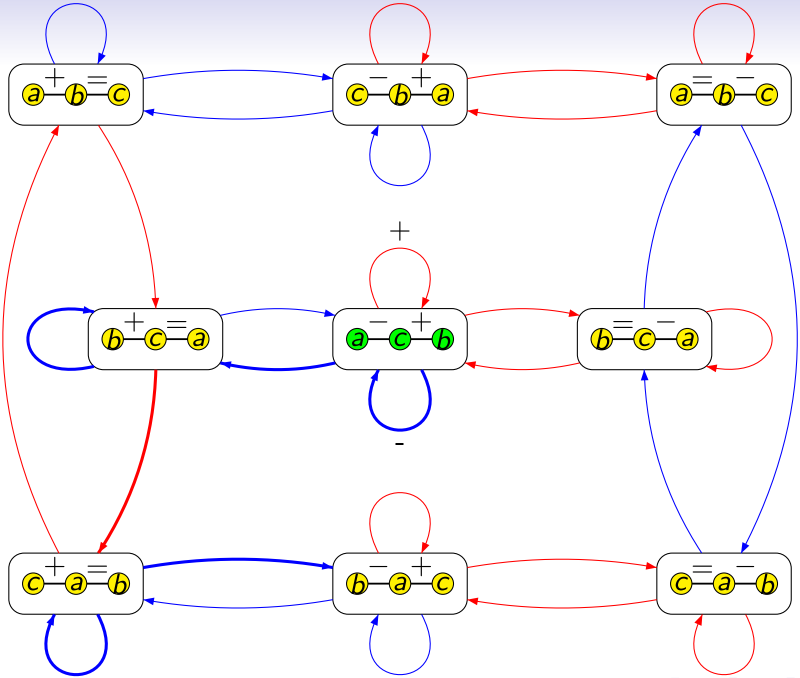
$$L_c - R_c < 0; \quad L_a - R_b < 0; \quad L_b - R_a < 0.$$



$L_a = L_c$
a
b
c
$R_a = R_b$



c
+
a
−
b


a + b = c
c − b + a
a = b − c
b + c = a
a − c + b
b = c − a
c + a = b
b − a + c
c = a − b
+
-

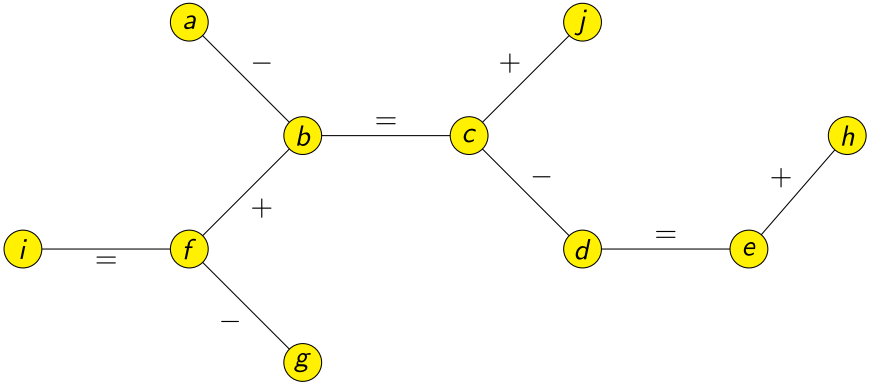
Rauzy trees

By a *tree* we mean a connected non-oriented graph without cycles.

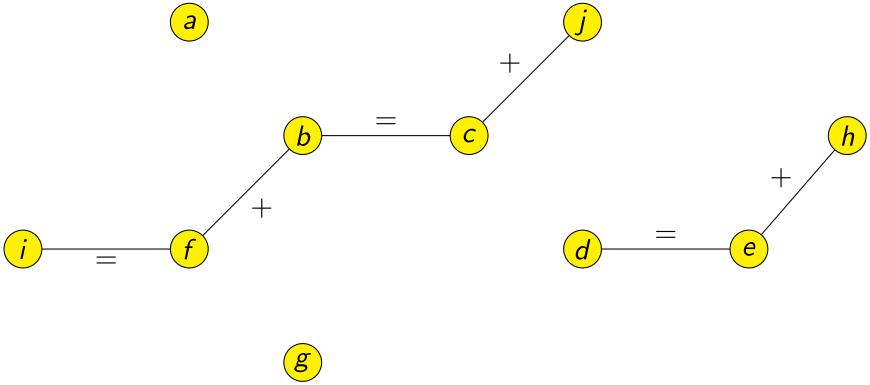
Definition : A *Rauzy tree* on a finite set K is a tree such that

- The vertices of T are the elements of K.
- Each edge of T is labeled with $\{+, =, -\}$.
- No two consecutive edges have the same label.

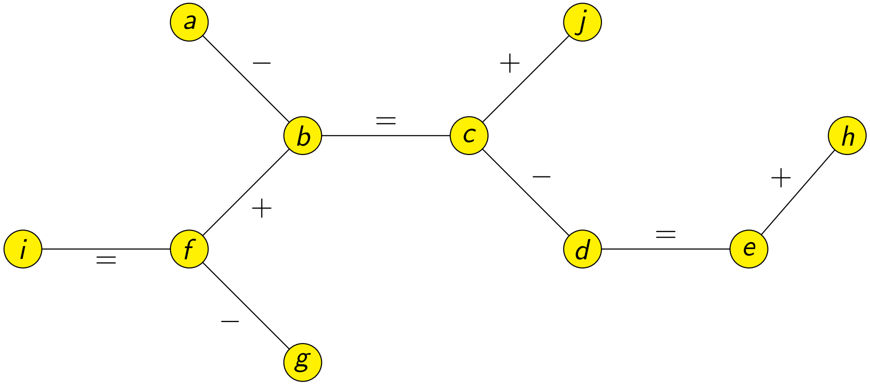
A Rauzy tree T with 10 vertices



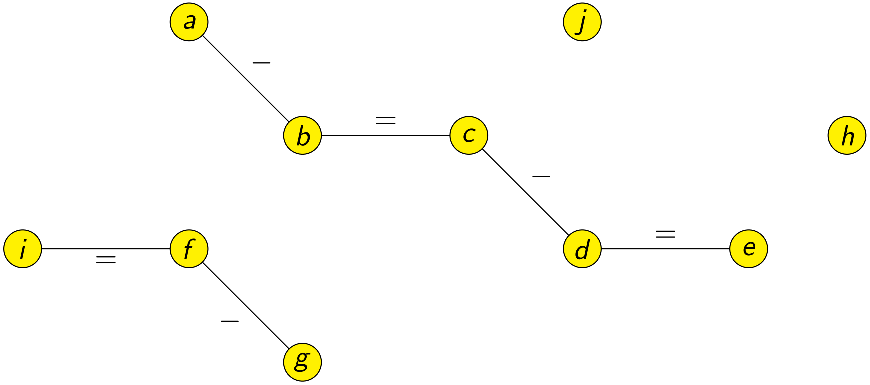
The positive chains of T



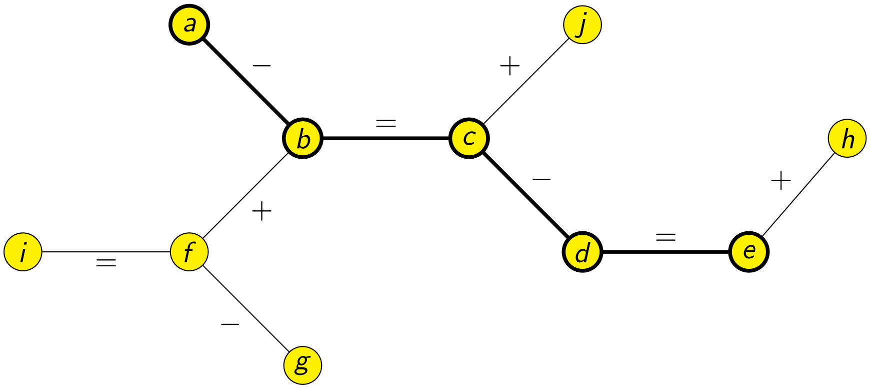
T



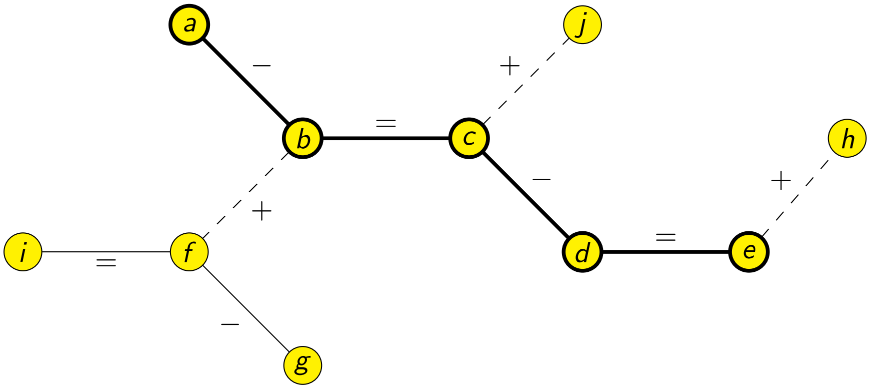
The negative chains of T



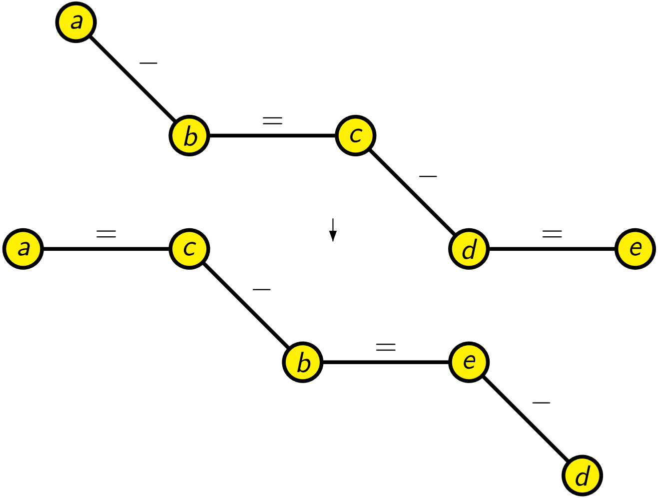
Induction : $(T, \gamma) \rightarrow T_\gamma$



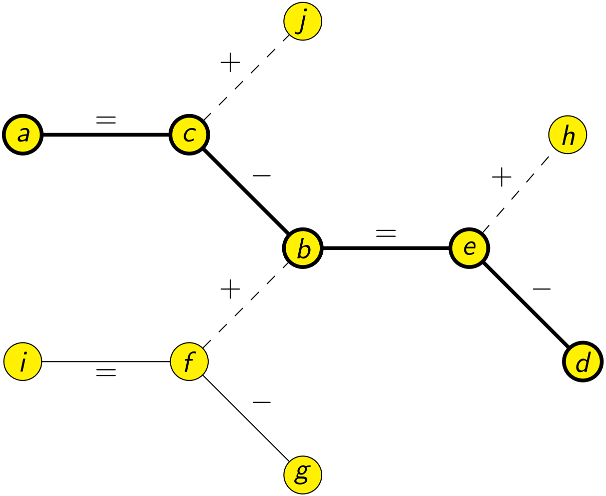
Step 1. Trimming



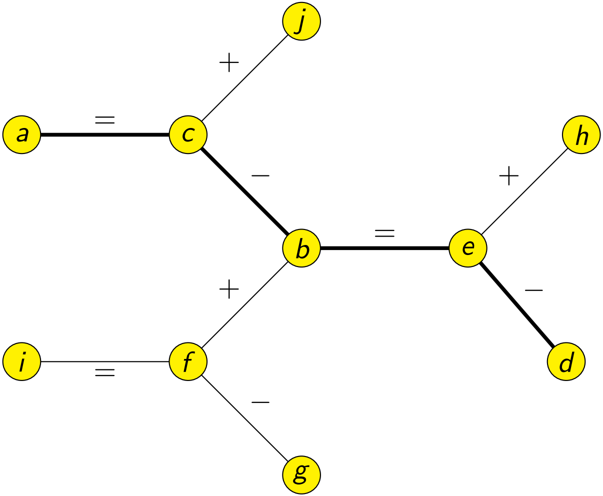
Step 2. Mutation $\gamma \to \gamma'$



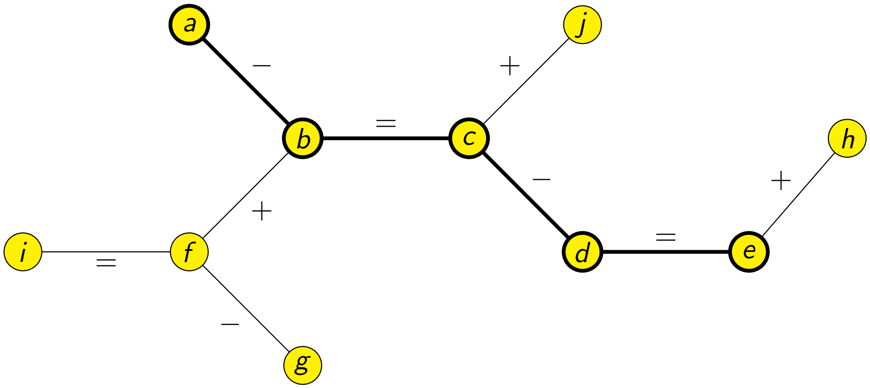
Step 3. Grafting



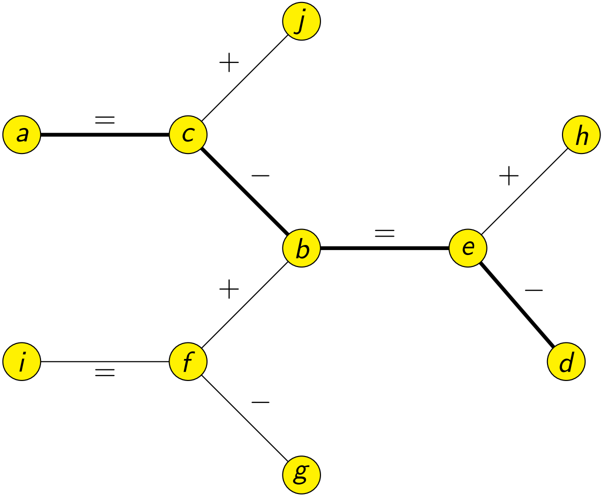
T_γ

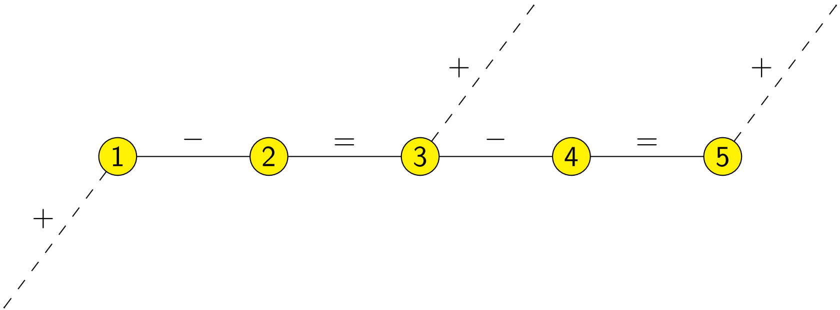


T

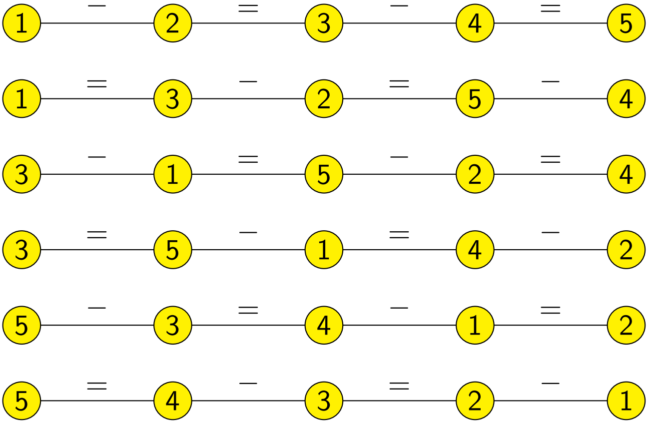


T_γ



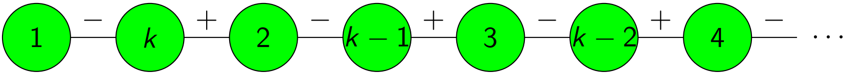


1
2
3
4
5
−
=
−
=
+
+
+

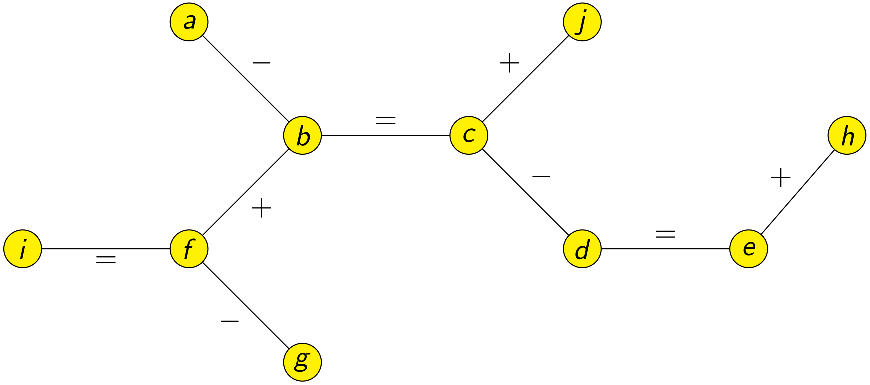

1 2 3 4 5
1 3 2 5 4
3 1 5 2 4
3 5 1 4 2
5 3 4 1 2
5 4 3 2 1

The graphs $G(k)$ and $G_0(k)$ for $k \geq 1$

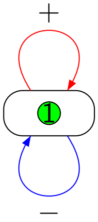
- The vertices of $G(k)$ are all Rauzy trees with vertices $\{1, 2, \ldots, k\}$.
- To each Rauzy tree T and to each signed chain γ of T, we put an oriented edge labeled γ from T to T_γ.
- $G_0(k)$ is the connected component of $G(k)$ containing the seed tree $T_0(k)$:



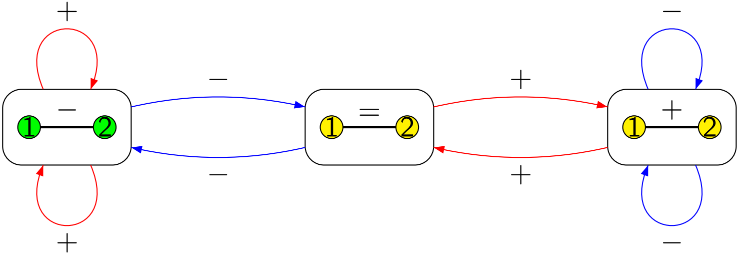
T revisited

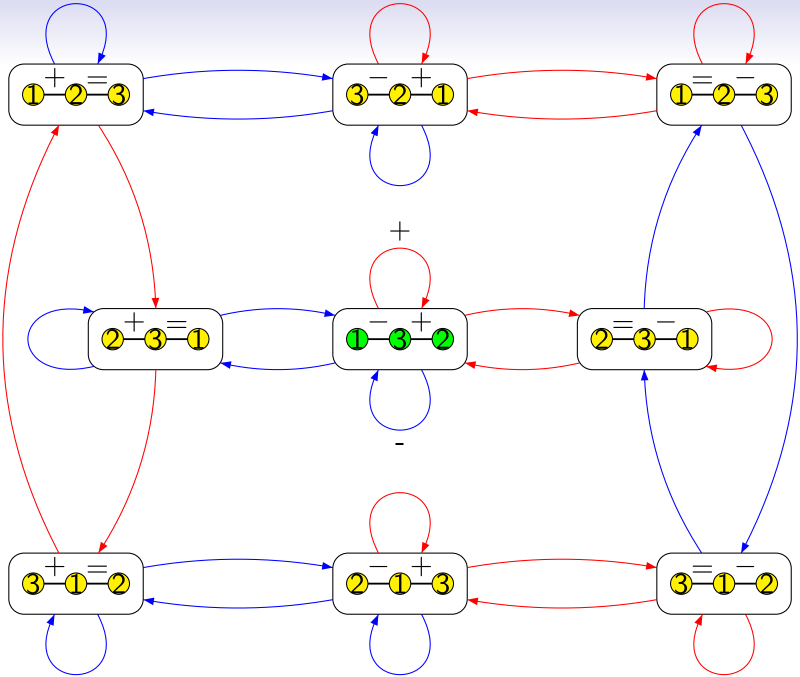


The Graph $G_0(1)$

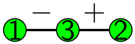


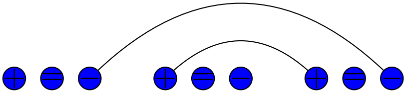
The Graph $G_0(2)$



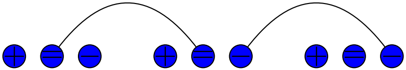

1 + 2 = 3
3 − 2 + 1
1 = 2 − 3
+
2 + 3 = 1
1 − 3 + 2
2 = 3 − 1
-
3 + 1 = 2
2 − 1 + 3
3 = 1 − 2

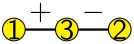
The intersection condition



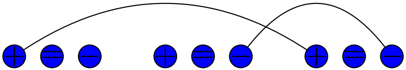



1
2
3





+
−
1
3
2

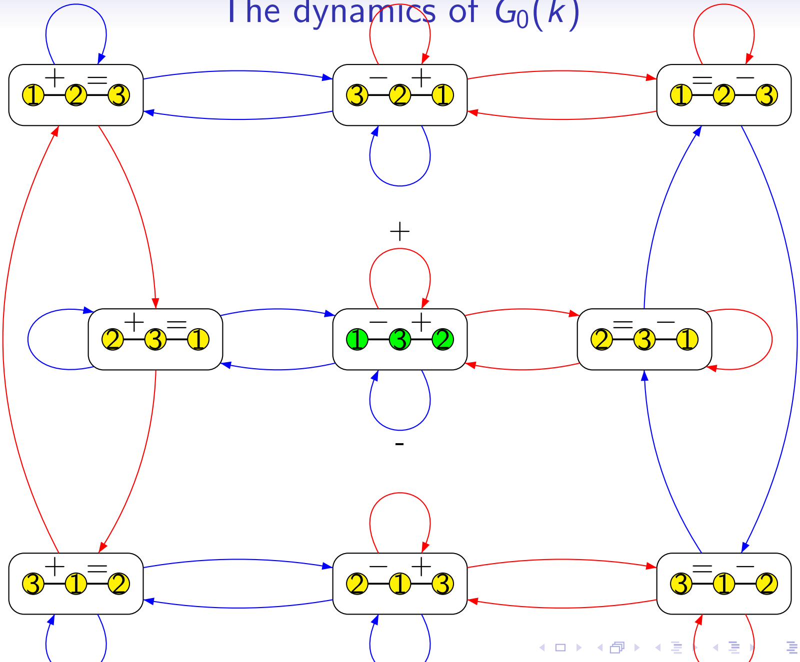


The cardinality of $G_0(k)$ for $k \geq 1$

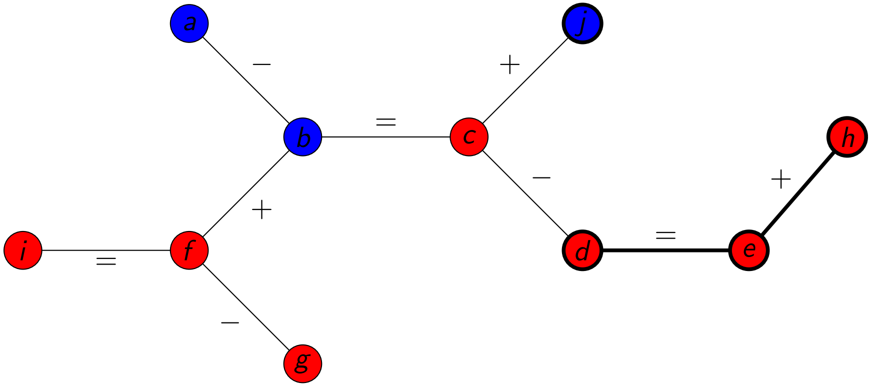
$$\#G_0(k) = \mathrm{Cat}_{k+1} - \mathrm{Cat}_k.$$

$$1, 3, 9, 28, 90, 297, 1001, 3432, 11934, 41990, \ldots$$

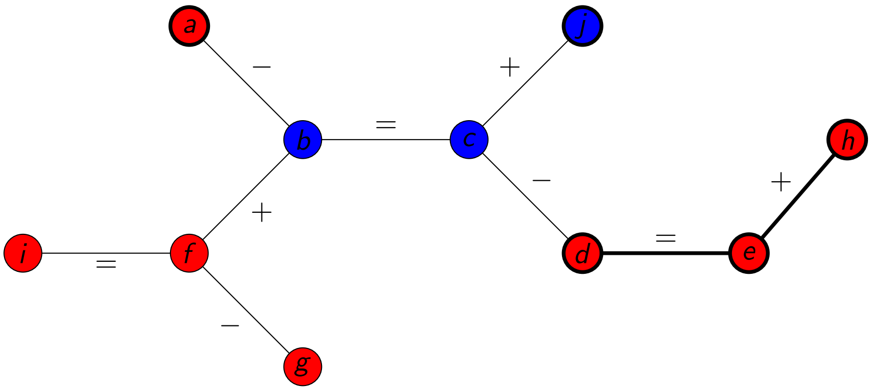
The dynamics of $G_0(k)$

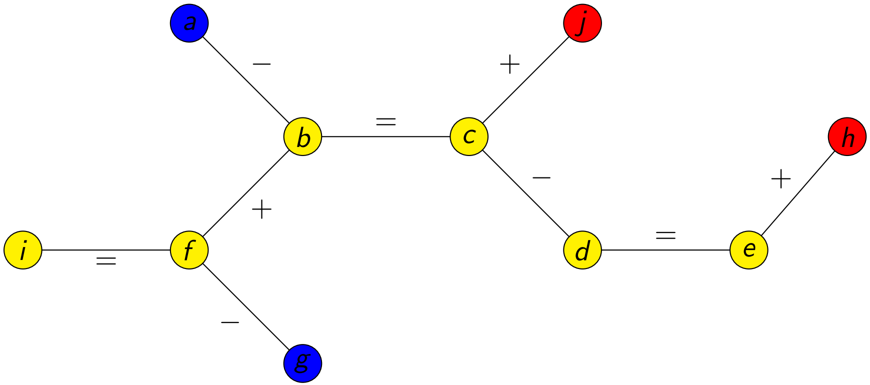


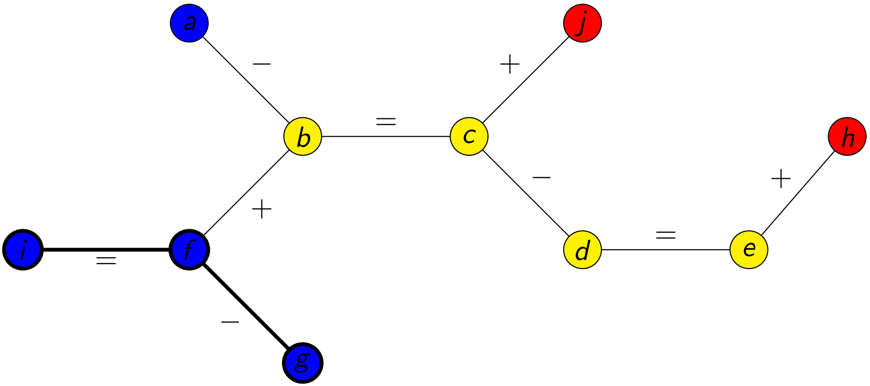
Coloring of T (validation of chains)

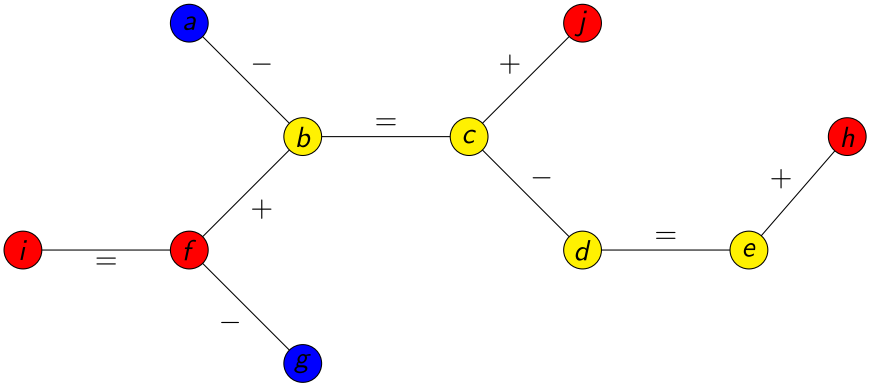


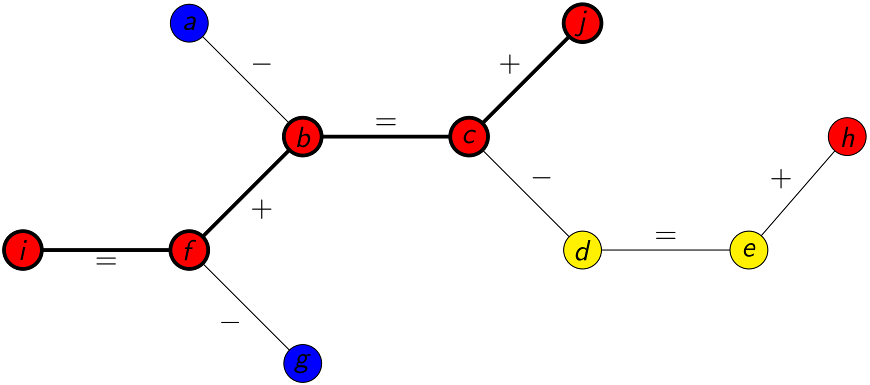
Proper coloring of T (validation of chains)

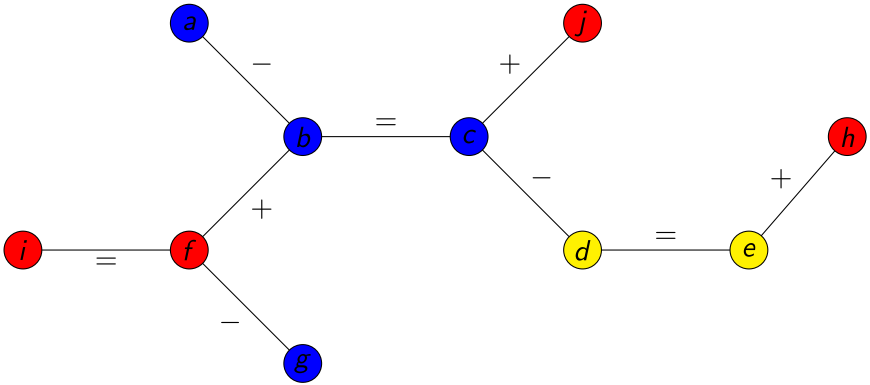


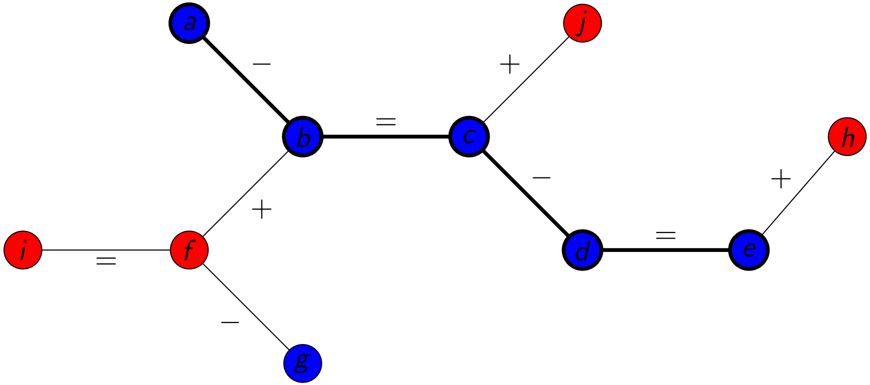

a
j
−
+
b
=
c
h
−
+
+
i
f
=
d
=
e
−
g

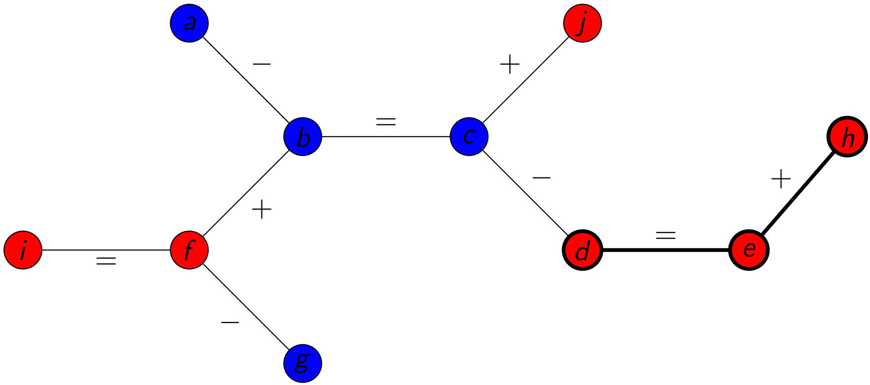

a
j
−
+
b
=
c
h
−
+
+
i
f
d
=
e
=
−
g

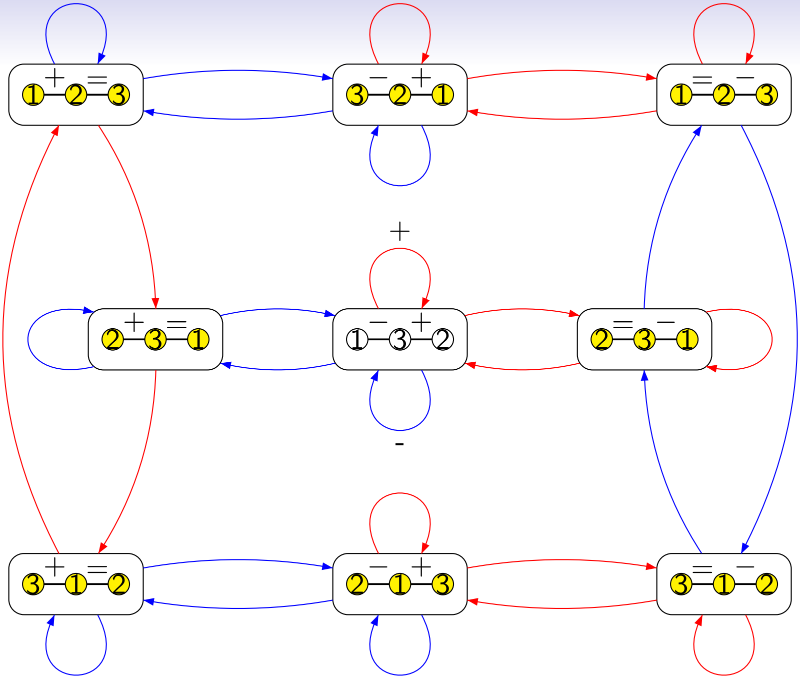

a
b
c
d
e
f
g
h
i
j
−
=
+
−
=
+
+
=
−

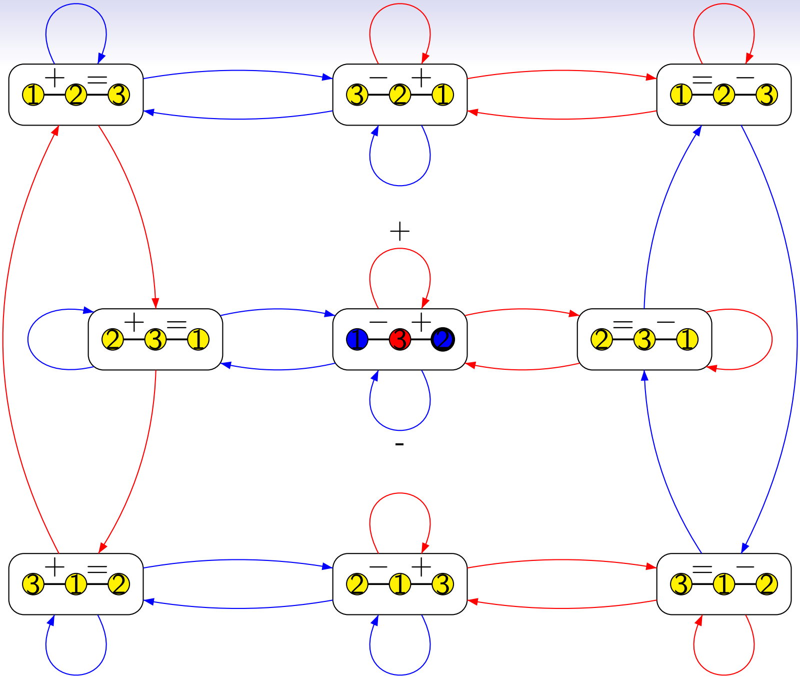

a
b
c
d
e
f
g
h
i
j
−
=
+
−
=
+
+
=
−

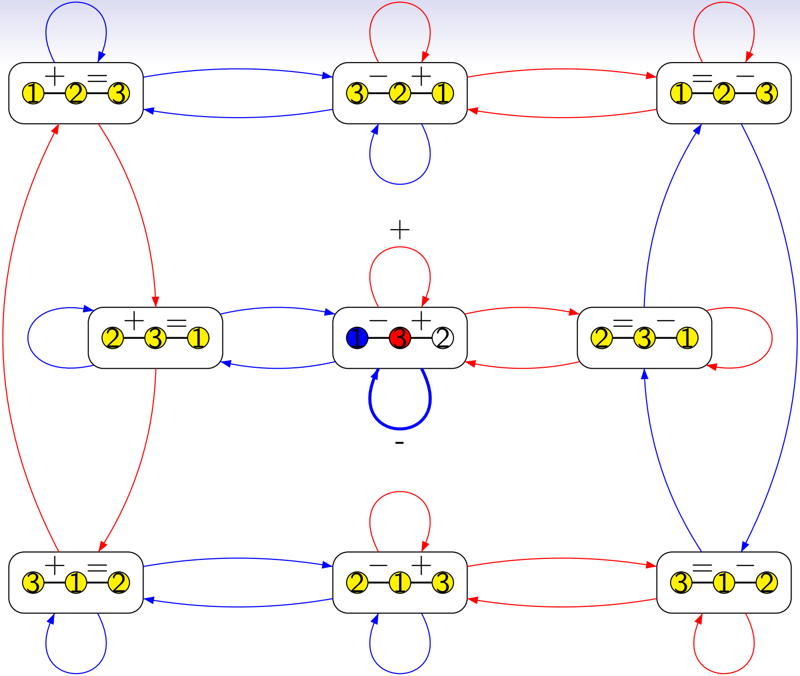

a
j
−
+
b
=
c
h
−
+
+
i
f
d
=
e
=
−
g


a
b
c
d
e
f
g
h
i
j
−
=
+
−
+
+
=
=
−

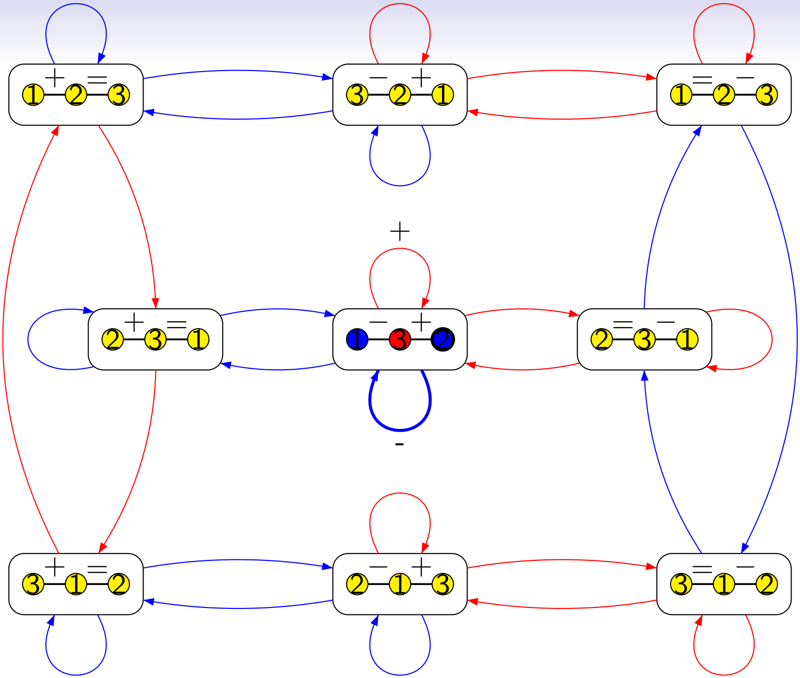

a
b
c
d
e
f
g
h
i
j
−
=
+
−
+
=
+
=
−



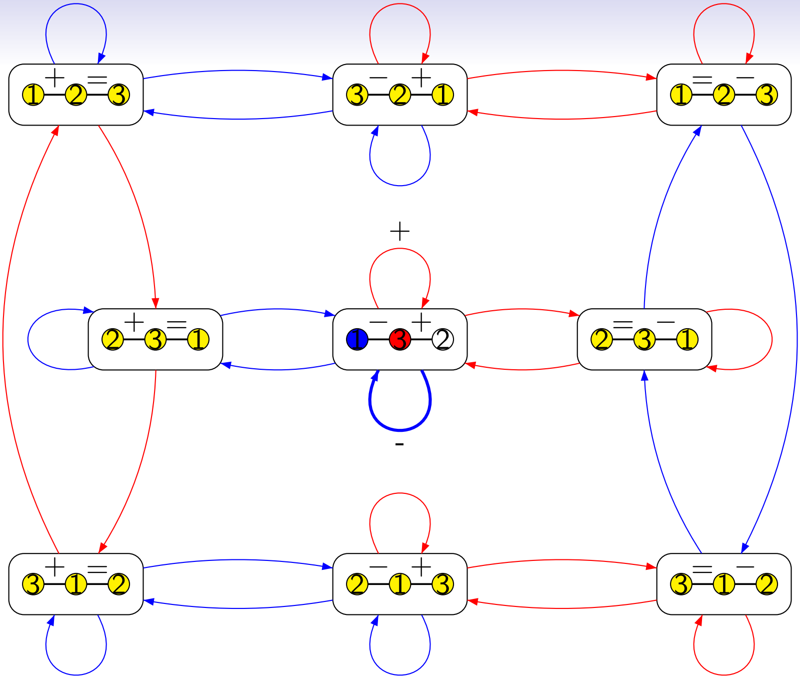




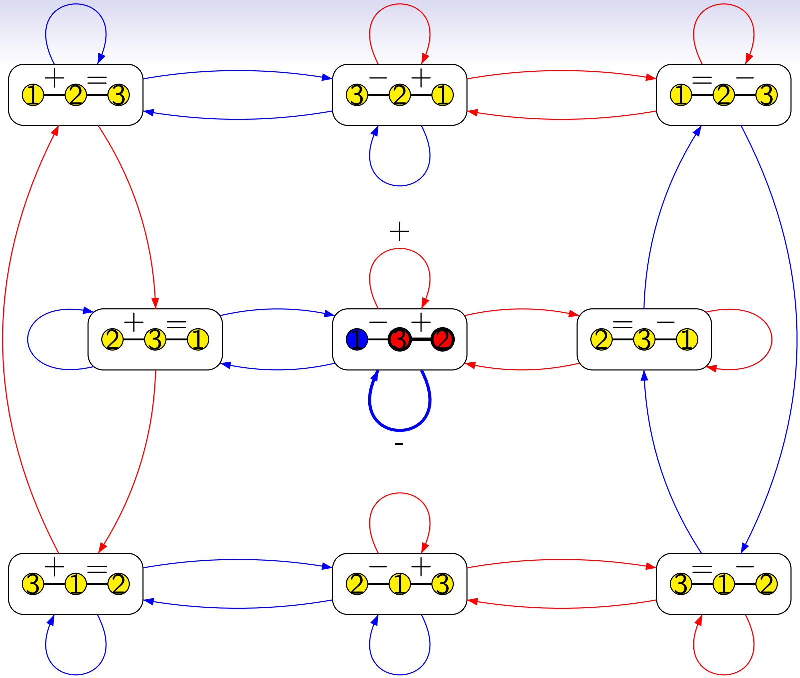
+ =
1 2 3
− +
3 2 1
= −
1 2 3
+ =
2 3 1
+
− +
1 3 2
-
= −
2 3 1
+ =
3 1 2
− +
2 1 3
= −
3 1 2



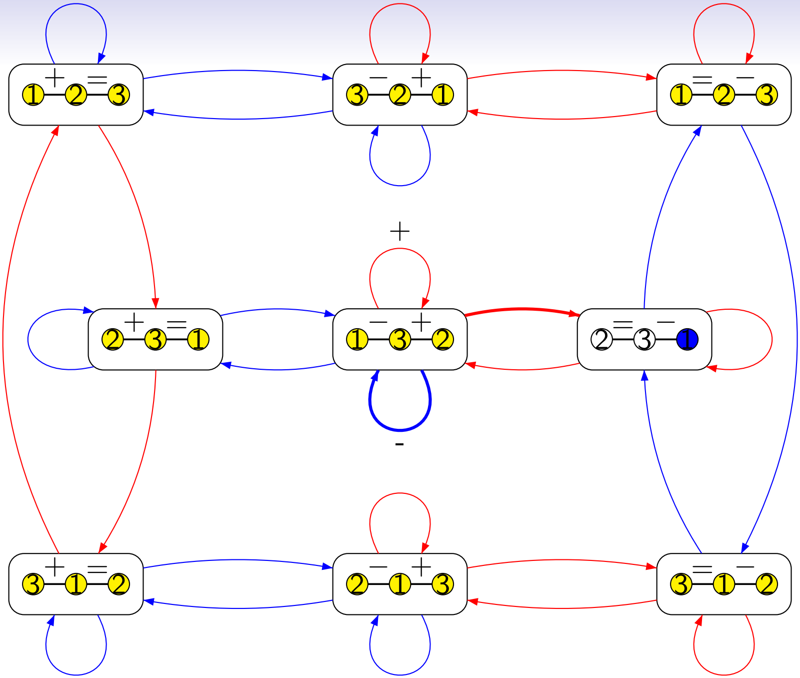
1 + 2 = 3
3 − 2 + 1
1 = 2 − 3
2 + 3 = 1
+
1 − 3 + 2
-
2 = 3 − 1
3 + 1 = 2
2 − 1 + 3
3 = 1 − 2

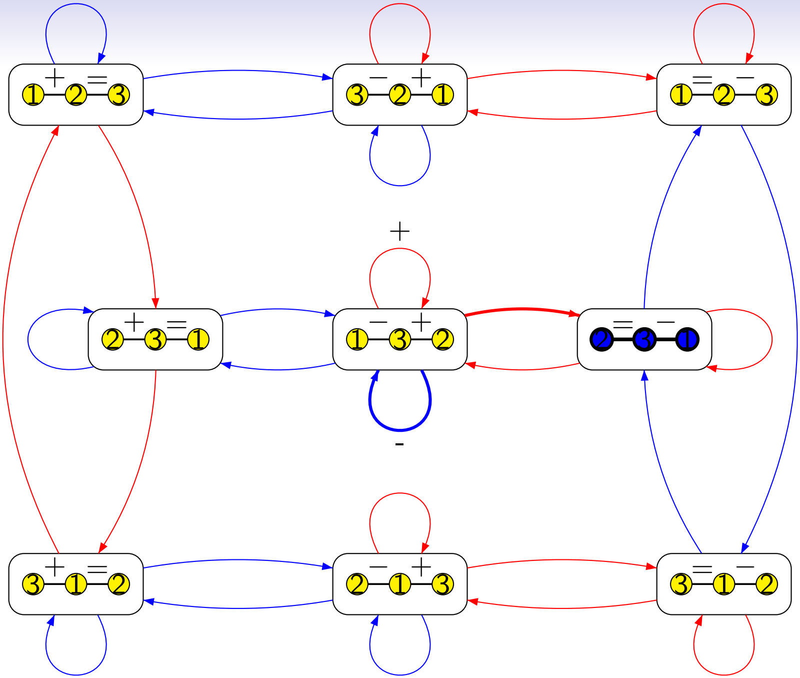


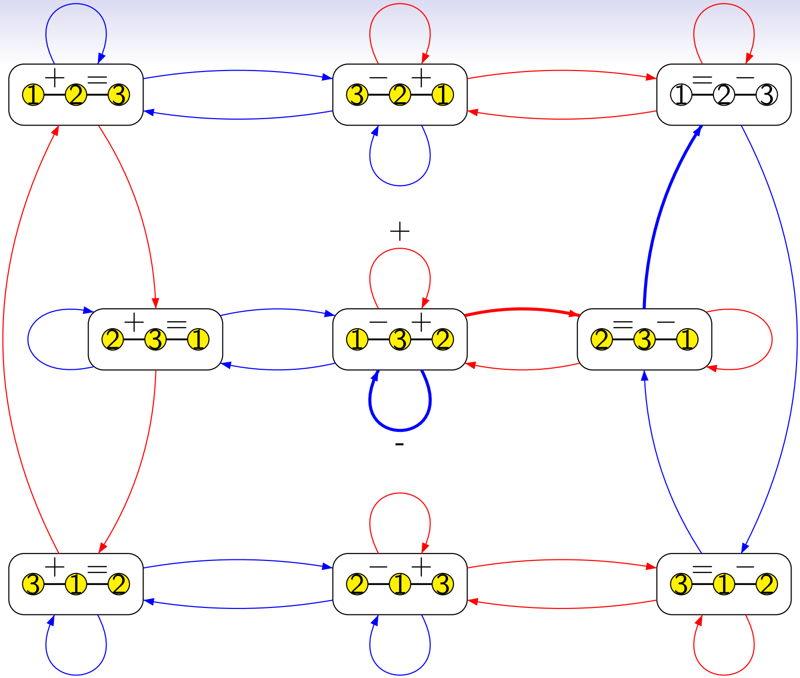
+ =
1 2 3
− +
3 2 1
= −
1 2 3
+
+ =
2 3 1
− +
1 3 2
= −
2 3 1
-
+ =
3 1 2
− +
2 1 3
= −
3 1 2

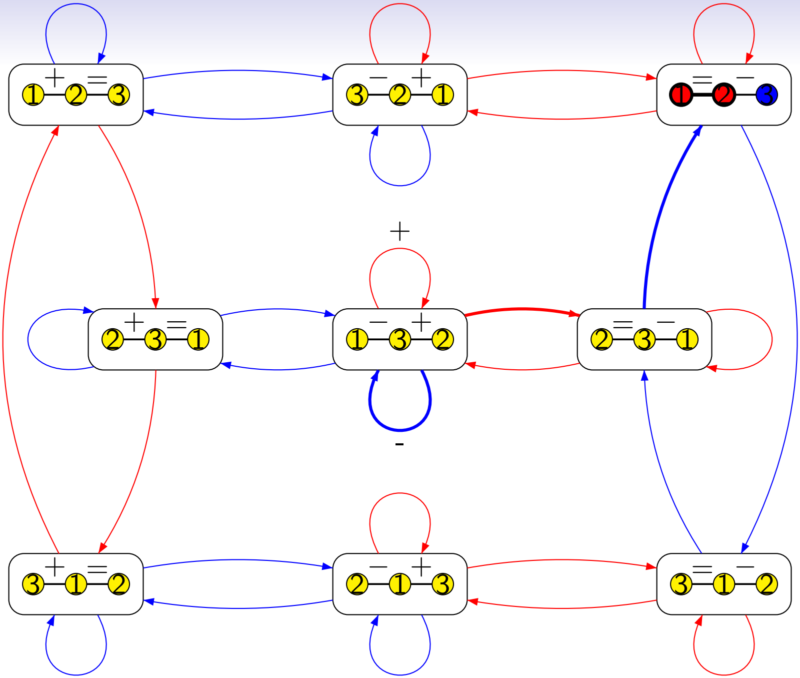


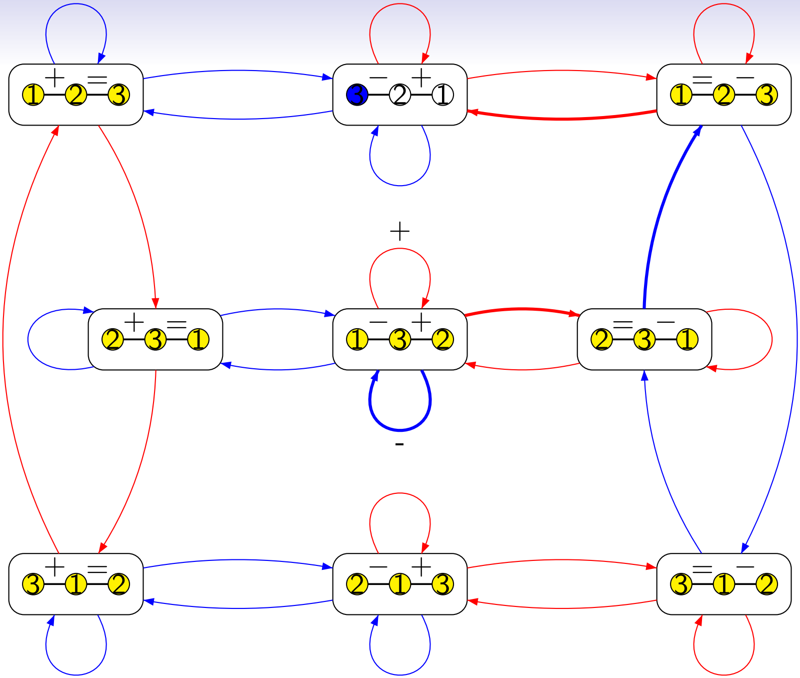
+ =
1 2 3
− +
3 2 1
= −
1 2 3
+
+ =
2 3 1
− +
1 3 2
= −
2 3 1
-
+ =
3 1 2
− +
2 1 3
= −
3 1 2



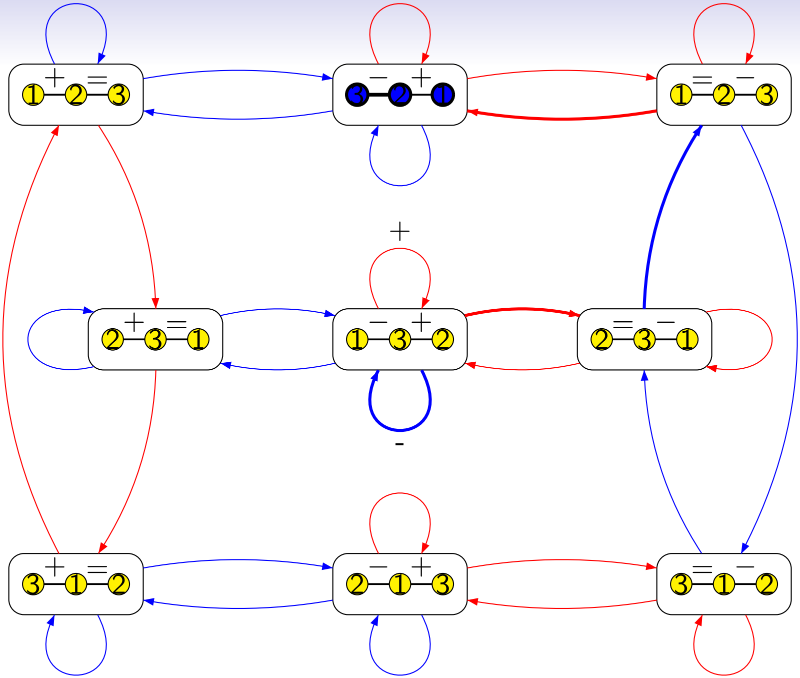


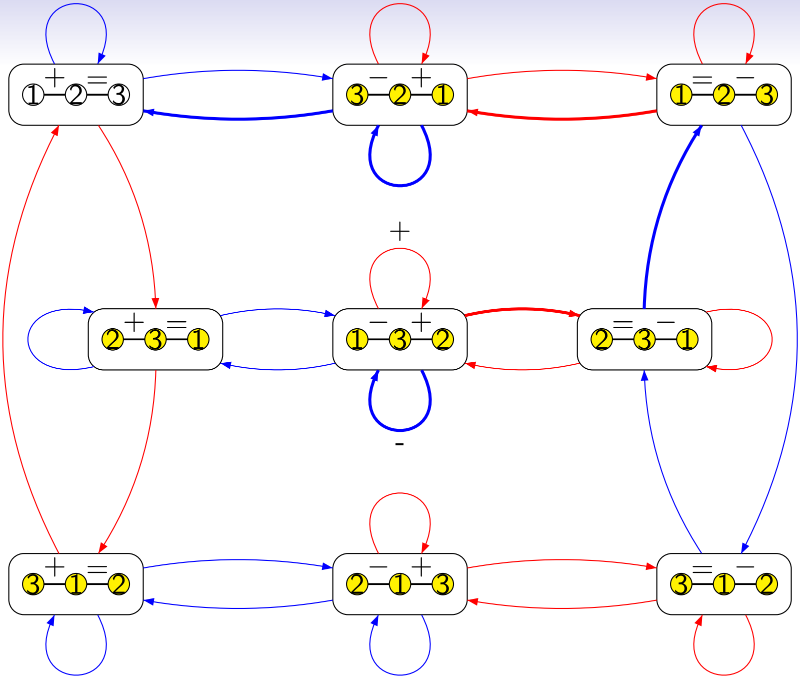



1 + 2 = 3
3 − 2 + 1
1 = 2 − 3
2 + 3 = 1
+
1 − 3 + 2
-
2 = 3 − 1
3 + 1 = 2
2 − 1 + 3
3 = 1 − 2



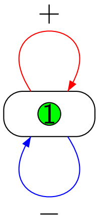
+ = 1 2 3
− + 3 2 1
= − 1 2 3
+ = 2 3 1
+
− + 1 3 2
-
= − 2 3 1
+ = 3 1 2
− + 2 1 3
= − 3 1 2


1 + 2 = 3
3 − 2 + 1
1 = 2 − 3
+
2 + 3 = 1
1 − 3 + 2
2 = 3 − 1
-
3 + 1 = 2
2 − 1 + 3
3 = 1 − 2



①+②=③
③−②+①
①=②−③
+
②+③=①
①−③+②
②=③−①
-
③+①=②
②−①+③
③=①−②

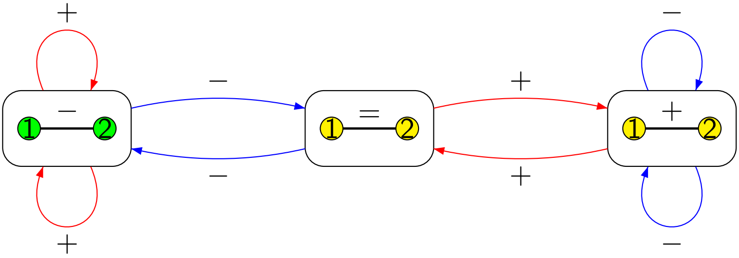
The Graph $G_0(1)$



$$(n_k)_{k\geq 1}$$

$$n_k \in \mathbb{Z}^{\geq 1}$$

The graph $G_0(2)$



$$(m_k, n_k, \epsilon_k)_{k\geq 1}$$

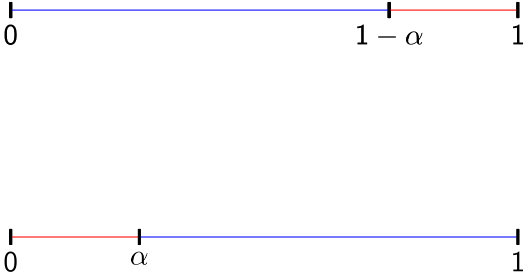
$$m_k, n_k \in \mathbb{Z}^{\geq 0}, \epsilon_k \in \{+1, -1\}$$

Analogue of Lagrange's theorem

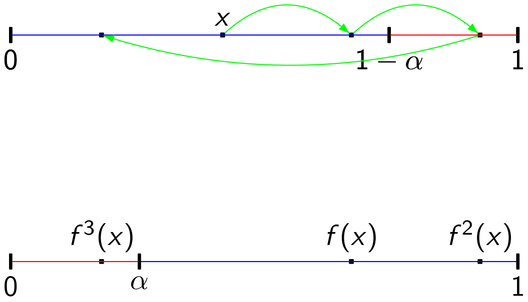
- $0 < a, b < 1$
- $a < 1 - a < b$
- $a, b, 1$ linearly independent over $\mathbb{Q}$.

Theorem (Ferenczi, Holton, Z) : The sequence $(m_k, n_k, \epsilon_k)_{k \geq 1}$ is ultimately periodic if and only if $a, b \in \mathbb{Q}(\sqrt{d})$.

A 2-interval exchange transformation



The orbit of a point



A 3-interval exchange transformation