# Sofic shifts and sychronizing words II

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# Content of the talk

- Shifts
- Sofic shifts, minimal cover
- Classification of sofic shifts
- Periodic finite type shifts (the periodic scenario).

# Subshifts

Let $\mathcal{F}$ be a set of words, $X_{\mathcal{F}}$ is the set of bi-finite sequences avoiding the words of $\mathcal{F}$ (depends on the alphabet).

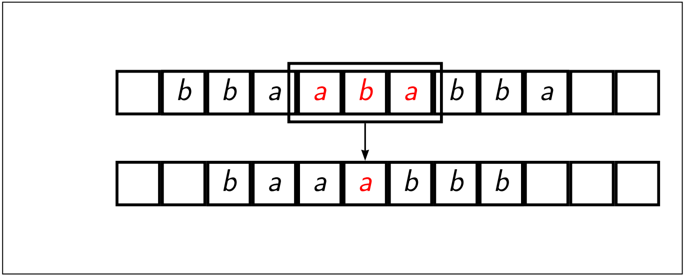
The shift map: $\sigma((x_i)_{i\in\mathbb{Z}} = (x_{i+1})_{i\in\mathbb{Z}}$.

$$\begin{aligned} x &= \ldots x_{-3}x_{-2}x_{-1}\cdot x_0x_1x_2\ldots \\ \sigma(x) &= \ldots x_{-3}x_{-2}x_0\cdot x_1x_2x_3\ldots \end{aligned}$$

$X_{\mathcal{F}}$ is invariant by $\sigma$ and closed for the product of the discrete topology.

# Maps

A sliding block map (or factor map) from a shift $X$ to a shift $Y$ is a continuous map which commutes with the shift $\sigma$.




A conjugacy is a one-to-one and onto factor map between two shifts. The inverse is also a block map (not with the same window size).

# Main classes

$\mathcal{F}$ finite $\leftrightarrow$ $X_{\mathcal{F}}$ of finite type
$\mathcal{F}$ regular $\leftrightarrow$ $X_{\mathcal{F}}$ sofic
$\mathcal{F}$ context-free $\leftrightarrow$ $X_{\mathcal{F}}$ algebraic shift

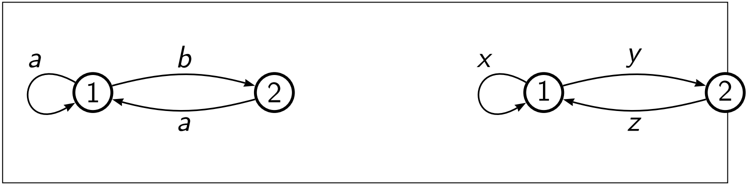
# Example: two conjugate shifts of finite type

Shift of finite type

$\mathcal{F} = \{bb\}$

Edge shift

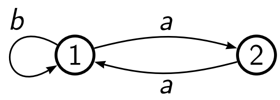
$\mathcal{F} = \{xz, yy, yx, zz\}$.



# Sofic shifts

Irreducible sofic shifts have a minimal deterministic cover (automaton) called its Shannon cover.

This minimal cover of an irreducible sofic shift has a synchronizing word.



# Local: all long enough words are synchronizing

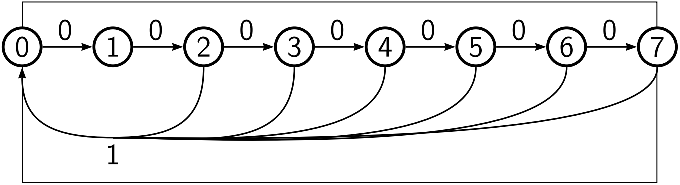
## Definition

Let $m, a$ be non-negative integers. An automaton is $(m, a)$-local (or $(m, a)$-definite) if whenever two paths of length $m + a$ have the same label, they go through the same state at time $m$.

Any deterministic $n$-state cover which is definite is $(m, 0)$-definite for some $m \leq n(n-1)/2$, for some $m \leq n-1$ if it is moreover complete.

# Shifts of finite type

The minimal deterministic cover of an irreducible shift of finite type is definite.




A local automaton for the $[2, 7]$-constraint ($(7, 0)$-definite).

$X_{\mathcal{F}}$ is the $[2, 7]$-constraint with $\mathcal{F} = \{11, 101, 00000000\}$.

# Periodic finite type channels

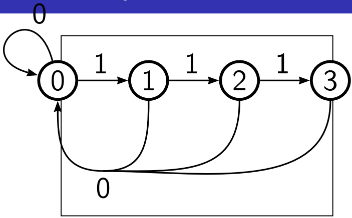
(introduced by Moision and Siegel 2001;
B., Crochemore, Fici, Gasieniec, Moision, Siegel)
Let $T$ be a positive integer (the *period*).
Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{T-1})$ be a list of $T$ possibly empty sets of words.
Let $X_0$ be the set of bi-infinite words $x$ such that, for each $i$, one has

$$u \prec_i x \Rightarrow u \notin \mathcal{F}_{i \bmod T}.$$

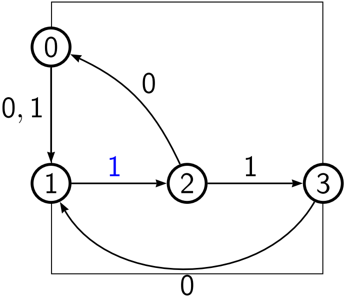
Hence, at position $i$, the word $x$ avoids the words in $\mathcal{F}_{i \bmod T}$, for all positions $i$.
The set of all bi-infinite sequences obtained by all integer shifts of words in $X_0$ defines a subshift $X = \mathrm{X}_{\{\mathcal{F},T\}}$.

A shift $X$ is PFT($T$) (periodic finite type) if $X = \mathrm{X}_{\{\mathcal{F},T\}}$ for some list $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{T-1})$ where all $\mathcal{F}_i$ are finite.

# PFT Examples




The MTR(3) channel $X_{\mathcal{F}}$ with $\mathcal{F} = \{1111\}$




The constraint $X_{\{\mathcal{F},T\}}$. with $T = 3$.

$$\begin{aligned} \mathcal{F}_0 &= \{1111\}, \\ \mathcal{F}_1 &= \{0, 1111\}, \\ \mathcal{F}_2 &= \{1111\}. \end{aligned}$$

The blue bit at position 1 mod 3 is "free" (flipping it does not violate the MTR(3) constraint). It can be used as parity bit.

# Almost of finite type (AFT) shifts

## AFT automaton

An automaton is almost of finite type (AFT) if

- It is weakly deterministic (or deterministic with finite delay) and weakly co-deterministic;
- It has a synchronizing word (or a reset sequence);
- It is irreducible (with a strongly connected graph).

## AFT shift

An almost of finite type shift is the set of bi-infinite sequences accepted by an AFT automaton.

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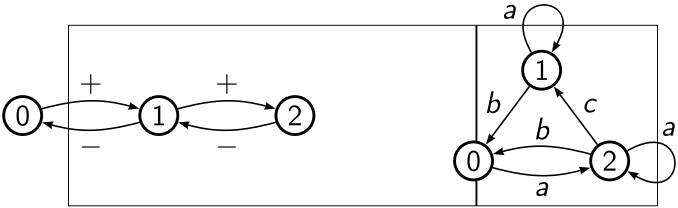
For irreducible shifts, SFT $\subsetneq$ PFT $\subsetneq$ AFT $\subsetneq$ Sofic

# Almost of finite type channels




Some charge constraint (AFT)

A constraint which is not AFT

# Checking the PFT Property

**Proposition**

*Let $X$ be an irreducible sofic shift and $p$ be the period of the Shannon cover $\mathcal{A}$ of $X$. Then the following assertions are equivalent.*

1. *$X$ is PFT.*
2. *$X$ is PFT($p$).*
3. *The irreducible components of $\mathcal{A}^p$ are definite.*

# PFT is a class of shifts

**Proposition**

*Let $X$ and $Y$ be two irreducible shifts. If $X$ and $Y$ are conjugate, then $X$ is PFT if and only if $Y$ is PFT.*

Proof: We prove this for elementary conjugate shifts.

# Elementary conjugacy for sofic shifts

## Definition

Let $A$ and $B$ be the symbolic adjacency matrices. We say that $A$ and $B$ are elementary strong shift equivalent if there are transition matrices $R, S$ such that, after recoding the alphabets of $A$ and $B$, we have $A = RS$ and $B = SR$.

Two symbolic adjacency matrices $A$ and $B$ are *strong shift equivalent within Shannon covers* if there is a sequence of symbolic adjacency matrices of Shannon covers

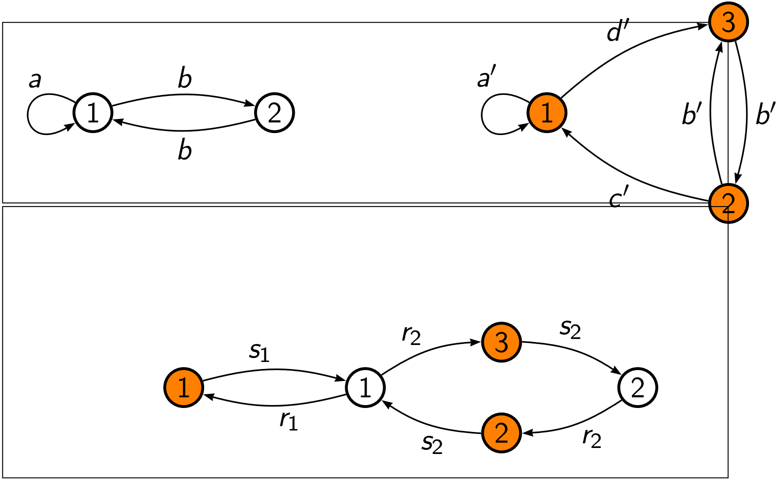
$$A = A_0, A_1, \ldots, A_{l-1}, A_l = B$$

such that $A_{i-1}$ and $A_i$ are elementary strong shift equivalent.

## Theorem (Nasu)

*Let $X$ and $Y$ be irreducible sofic shifts and let $A$ and $B$ be the symbolic adjacency matrices of their Shannon covers. Then $X$ and $Y$ are conjugate if and only if $A$ and $B$ are strong shift equivalent within Shannon covers.*

# PFT is a class of shifts



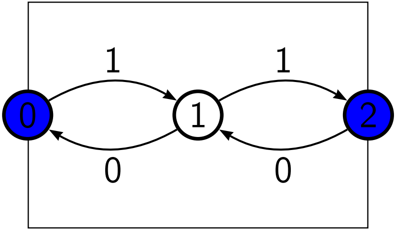

$a \leftrightarrow r_1 s_1 \quad a' \leftrightarrow s_1 r_1$

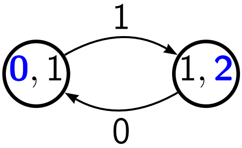
$b \leftrightarrow r_2 s_2 \quad b' \leftrightarrow s_2 r_2 \quad c' \leftrightarrow s_2 r_1 \quad d' \leftrightarrow s_1 r_2$

# PFT Properties

**Proposition**

*Let $X$ be an irreducible sofic shift presented by its n-state Shannon cover. It is decidable in time $O(n^2 \times \log|A|)$ whether $X$ is PFT.*




Checking if the biphase constraint is PFT. It is PFT.

There is no cycle going through a state $(p, q)$ with $p \neq q$ and $p, q$ having the same color.

# Minimal forbidden list for PFT shifts

Let $\mathcal{F}$ be a periodic forbidden list of a shift $X$ for some positive period $T$. We say that $\mathcal{F}$ is periodic anti-factorial if and only if for any $0 \leq i \leq T - 1$ and any $j \geq 0$,

$$w \in \mathcal{F}_i \text{ and } u \prec_j w \text{ with } u \neq w \Longrightarrow u \notin \mathcal{F}_{i+j \bmod T}.$$

## Example

The list

$$\mathcal{F}_0 = \{00, 11, 010\}$$
$$\mathcal{F}_1 = \{00, 10\},$$

with $T = 2$ is not periodic anti-factorial. Indeed, $010 \in \mathcal{F}_0$, $10 \in \mathcal{F}_1$, and $10 \prec_1 010$.

# Minimal forbidden list for PFT shifts

There is a notion of *Shannon periodic first offenders*: a list $\mathcal{F}(\mathcal{A}, c)$ where $\mathcal{A} = (V, E)$ is the Shannon cover of $X$, $p$ is the period of $\mathcal{A}$, and $c$ is $p$-coloring of the states of $\mathcal{A}$.
In this list, $\mathcal{F}_i$ are the sets of finite words $w$ such that

1. $w \notin \mathrm{Fut}(V_i)$,
2. for any $0 \leq j < |w| - 1$, $w_{[0,j]} \in \mathrm{Fut}(V_i)$,
3. for any $0 < j \leq |w| - 1$, $w_{[j,|w|-1]} \in \mathrm{Fut}(V_{i+j \bmod p})$.

# Applications: constrained channels with free positions

## Constrained channel stable by $U$-flips

Let $X \subset \{0,1\}^{\mathbb{Z}}$ be a channel of finite type, $T$ a period and $U$ a set of (free) positions modulus $T$.
Find a sub-channel $Y$ of $S$ stable by $U$-flips: one can flip any letter at position in $U$ modulus $T$ without violating the constaint..
One compute the $Y^1 \subset Y$ where the free positions are fixed to 1.

## Example

$T = 3$, $U = \{1\}$

$$0\ 0\ 1 \quad 0\ 0\ 0 \quad 0\ 1\ 1 \quad 0 \quad \ldots$$

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$T = 3$, $U = \{1\}$

0 1 1 0 1 0 0 0 1 0 ...

# Channel construction: algorithm

- Problem (B., Crochemore, Fici, 2005)
  - Input: a trie $\mathcal{T}$ of forbidden words describing the finite type constraint $X_{\mathcal{F}}$, a pair $T, U$ of period and free positions.
  - Output: a deterministic automaton accepting the largest channel that satisfies the constraint with free positions (channel $Y$ or $Y^1$)
  - Two steps:
    1. Build $T$ tries of a periodic list $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{T-1})$ such that $Y^1 = \mathsf{X}_{\{\mathcal{G}, T\}}$ in linear time. (B., Crochemore, Gasieniec, 2007)
    2. Build the channel $\mathsf{X}_{\{\mathcal{G}, T\}}$ from $\{\mathcal{G}, T\}$ (from the above tries) in linear time.
  - Linear time construction

# Step 1

## Algorithm

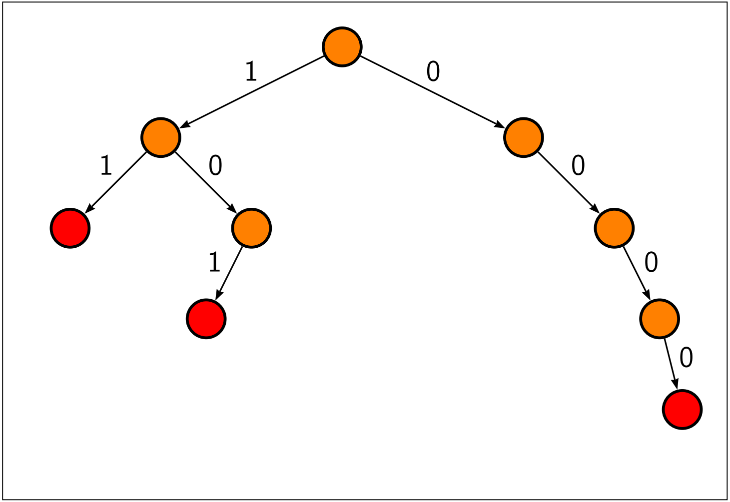
- Input: a trie of forbidden blocks $\mathcal{T}$, a pair $T, U$ of period and free positions
- Output: $T$ tries $\mathcal{T}_i$ of periodic forbidden words that define $T^1$.

## Method

- set $\mathcal{G}_i = \mathcal{F}$ for $0 \le i \le T - 1$, where $\mathcal{F}$ is accepted by $\mathcal{T}$,
- if $w \in \mathcal{G}_i$ add all words in $\text{flip}(w)$ to $\mathcal{G}_i$, where $\text{flip}(w)$ is the set of words obtained from $w$ by flipping all letters at positions $u - i \bmod T$, $u \in U$,
- add 0 to $\mathcal{T}_i$ for $i \in U$ to get $T^1$.

# The trie of forbidden words of the [2,4] constraint

$\mathcal{F} = \{11, 101, 00000\}$



# Periodic forbidden words of the [2,4] constraint, $T = 2, U = \{1\}$

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$\mathcal{F} = \{11, 101, 00000\}$,
$\mathcal{G}_0 = \{1x, 1y1, 0z0t0, x, y, z, t = 0, 1\}$

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$\mathcal{F} = \{11, 101, 00000\}$,
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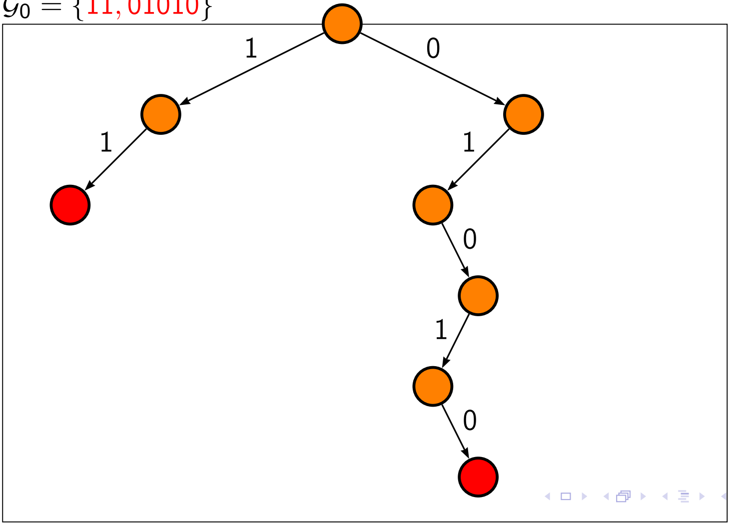

# Step 1

## Algorithm for building $\mathcal{G}_0$

- Change in $\mathcal{T}$ all the labels of the arcs at level $u$ with $u \bmod T \in U$ from 0 to 1.
- Add a node for 0 if $0 \in U$.
- Determinize the new tree and prune it.

# Step 2: the PFT channel from periodic tries

- Input: $T$ tries $\mathcal{T}_i$, $0 \leq i < T$, accepting $\mathcal{G}_i$, $(T, U)$.

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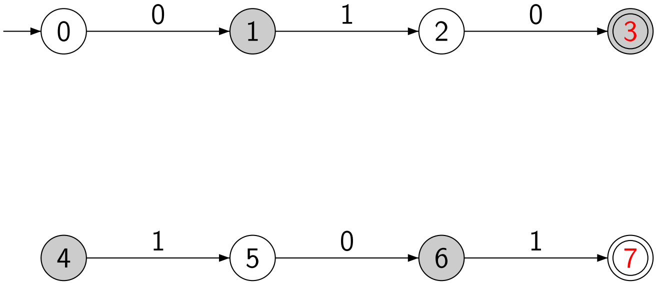

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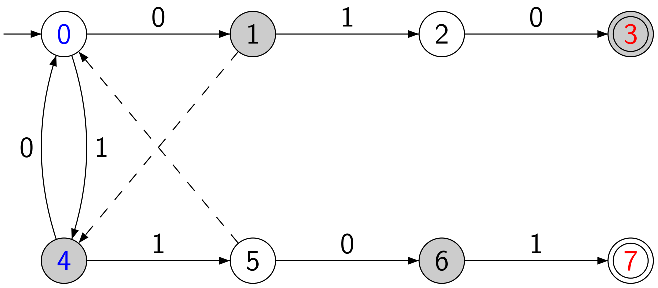

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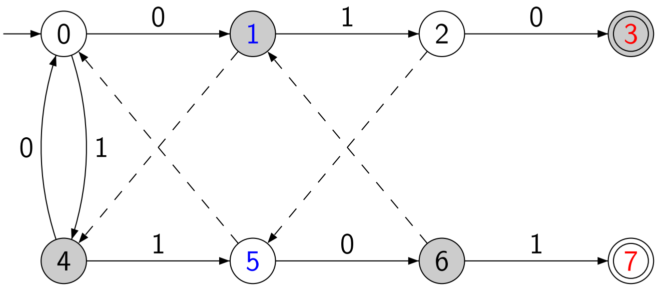

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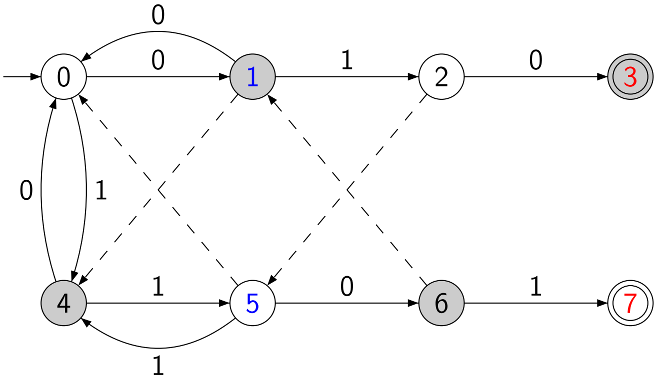

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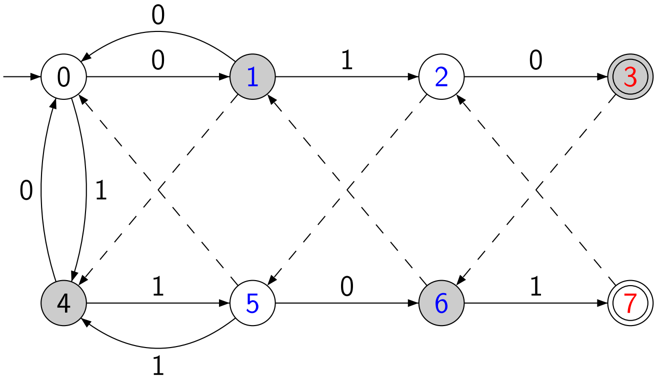

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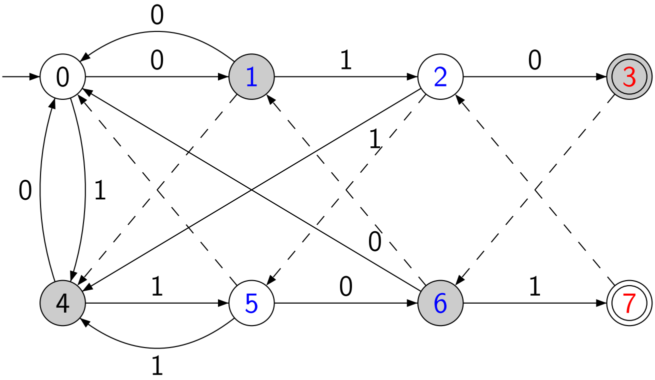

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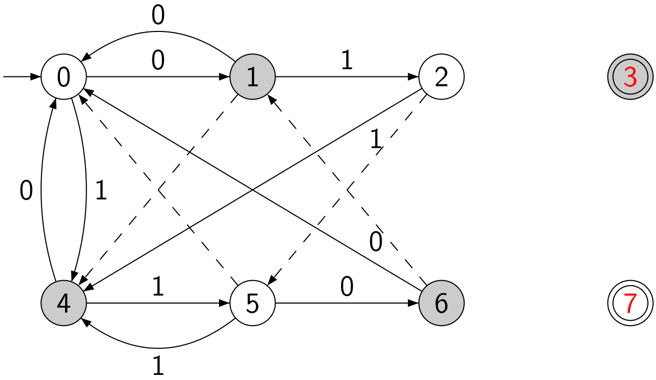
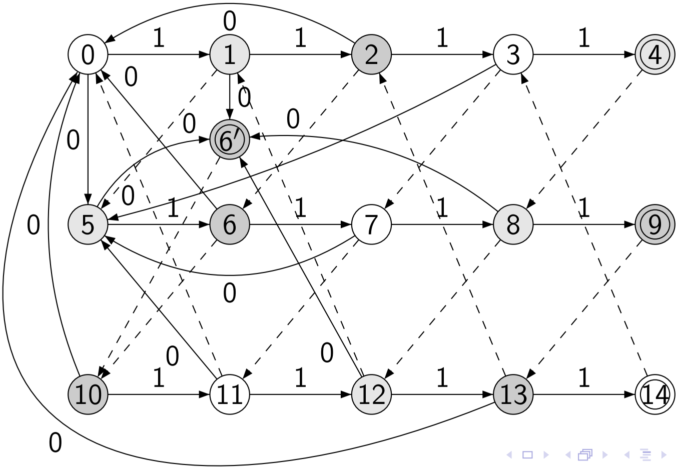


# Minimized MTR(3) constraint with free unconstained positions

With $T = 3$, $U = \{1\}$ $\mathcal{G}_0 = \{1111\}$, $\mathcal{G}_1 = \{0, 1111\}$, $\mathcal{G}_2 = \{1111\}$.



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