TRACE THEOREM ON THE HEISENBERG GROUP

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Abstract : We prove in this work the trace and trace lifting theorem for Sobolev spaces on the Heisenberg groups for hypersurfaces with characteristics submanifolds.

Résumé : Dans ce travail, nous démontrons des théorèmes de trace et de relèvement pour les espaces de Sobolev sur le groupe de Heisenberg pour des hypersurfaces dont l'ensemble caractéristique est une sous-variété.

Key words Trace and trace lifting, Heisenberg group, Hörmander condition, Hardy's inequality
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1. Introduction

In this work, we proceed with the study of the problem of restriction of functions that belongs to Sobolev spaces associated to left invariant vector fields for the Heisenberg group $\mathbb{H}^d$. We shall assume that $d \geq 2$. Let us recall that the Heisenberg group is the space $\mathbb{R}^{2d+1}$ of the (non commutative) law of product

$$w \cdot w' = (x, y, s) \cdot (s', x', y') = \big(x + x', y + y', s + s' + (y|x') - (y'|x)\big)\,.$$

The left invariant vector fields are

$$X_j = \partial_{x_j} + y_j \partial_s, \quad Y_j = \partial_{y_j} - x_j \partial_s, \ \ j = 1, \cdots, d \quad \text{and} \quad S = \partial_s = \frac{1}{2}[Y_j, X_j].$$

In all that follows, we shall denote by $\mathcal{Z}$ this family and state $Z_j = X_j$ and $Z_{j+d} = Y_j$ for j in $\{1, \cdots, d\}$. Moreover, for any C^1 function f, we shall state

$$\nabla_{\mathbb{H}} f \stackrel{\text{def}}{=} (Z_1 \cdot f, \cdots, Z_{2d} \cdot f).$$

The key point is that $\mathcal{Z}$ satisfies Hörmander's condition at order 2, which means that the family $(Z_1, \cdots, Z_{2d}, [Z_1, Z_{d+1}])$ spans the whole tangent space $T\mathbb{R}^{2d+1}$.

For $k \in \mathbb{N}$ and V an open subset of $\mathbb{H}^d$, we define the associated Sobolev space as following

$$H^k(\mathbb{H}^d, V) = \Big\{ f \in L^2(\mathbb{R}^{2d+1}) \,/\, \text{ Supp } f \subset V \quad \text{and} \quad \forall \alpha \,/\, |\alpha| \leq k\,,\ Z^\alpha f \in L^2(\mathbb{R}^{2d+1}) \Big\},$$

where if $\alpha \in \{1, \cdots, 2d\}^{k'}$, $|\alpha| \stackrel{\text{def}}{=} k'$ and $Z^\alpha \stackrel{\text{def}}{=} Z_{\alpha_1} \cdots Z_{\alpha_{k'}}$. As in the classical case, when s is any real number, we can define the function space $H^s(\mathbb{H}^d)$ through duality and complex interpolation, Littlewood-Paley theory on the Heisenberg group (see [1]), or Weyl-Hörmander calculus (see [8], [10] and [11]).

It turns out that these spaces have properties which look very much like the ones of usual Sobolev spaces, see [4] and their references.

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The purpose of this paper is the study of the problems of trace and trace lifting on a smooth hypersurface of $\mathbb{H}^d$ in the frame of Sobolev spaces. Let us point out that the problem of existence of trace appears only when s is less than or equal to 1. Indeed, under the sub-ellipicity of system $\mathcal{Z}$, the space $H^s(\mathbb{H}^d)$ is included locally in $H^{\frac{s}{2}}(\mathbb{R}^{2d+1})$. So if s is strictly larger than 1, this implies that the trace on any smooth hypersurface exists and belongs locally to the usual Sobolev space $H^{\frac{s}{2}-\frac{1}{2}}$ of the hypersurface. Thus the case when $s = 1$ appears as the critical one. It is the case we study here.

1.1. **Statement of the results.** Two very different cases then appear: the one when the hypersurface is non characteristic, which means that any point w_0 of the hypersurface Σ is such that $\mathcal{Z}_{|w_0} \not\subset T_{w_0}\Sigma$, and the one when some point w_0 of the hypersurface Σ is characteristic, which means that $\mathcal{Z}_{|w_0} \subset T_{w_0}\Sigma$.

The non characteristic case is now well understood. In [4], we give a full account of trace and trace lifting results on smooth non characteristic hypersurfaces for $s > 1/2$. This result generalize various previous results (see among others [7], [12] and [21]).

Let us recall this theorem in the case of H^1 (see [4] for the details). If w_0 is any non characteristic point of Σ, then there exists at last one of the vector fields $Z_1, \cdots Z_{2d}$ which is transverse to Σ at w_0. We denote by $\mathcal{X}_\Sigma$ the subspace of $T\Sigma$ define, for w in Σ, by $\mathcal{X}_{\Sigma|_w} = T_w\Sigma \cap \mathcal{X}|_w$ where $\mathcal{X}$ is the C^∞-module of vector fields spanned by $\{Z_1, \cdots, Z_{2d}\}$. It is easily checked that, if g is a local defining function of Σ, the family

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j$$

generates $\mathcal{X}_\Sigma$ and that it satisfies the Hörmander condition at order 2 (see for instance Lemma 4.1 of [4]). We define

$$H^k(\Sigma, Z_\Sigma) = \left\{ f \in L^2(\Sigma) \,/\, \text{ Supp } f \subset V \quad \text{and} \quad \forall (j,k)\,,\ R_{j,k}u \in L^2 \right\}.$$

We have proved the following trace and trace lifting theorem in [4]:

Theorem 1.1. *Let us suppose that Σ is non characteristic on an open subset V of $\mathbb{H}^d$, then the trace operator on Σ denoted by γ_Σ is an onto continuous map from $H^1(\mathbb{H}^d, V)$ onto $[H^1(\Sigma, Z_\Sigma), L^2(\Sigma)]_{\frac{1}{2}} \stackrel{\text{def}}{=} H^{\frac{1}{2}}(\Sigma, Z_\Sigma)$.*

Remark As the system $\mathcal{Z}_\Sigma$ satisfies the Hörmander's condition at order 2, Theorem 1.1 implies in particular that γ_Σ maps $H^1(\mathbb{H}^d, V)$ into $H^{1/4}(\Sigma, V)$.

We shall now consider the characteristic case. The set of characteristic points of Σ

$$\Sigma_c = \Big\{ w \in \Sigma \,/\, \mathcal{Z}_{|w} \subset T_w\Sigma \Big\},$$

may have a complicated structure. Let us introduce the following definition.

Definition 1.1. *A characteristic point w_0 of a hypersurface Σ is a regular point of order r if and only if*

i) for any 1-form $\theta \in T^\star\mathbb{R}^{2d+1}$ that vanishes on $T\Sigma$ and such that $\theta(w_0) \neq 0$, the system $(\mathcal{L}_{Z_j}\theta_{|T_{w_0}\Sigma})_{1\leq j\leq 2d}$ is of rank r;

ii) near w_0, the characteristic set Σ_c is a submanifold of Σ of codimension r in Σ.

Let us make some comments about this definition. A regular characteristic point of order $2d$ is exactly the familiar notion of non degenerate characteristic point. This notion of non degenerate characteristic point have been used in our preceeding work [4] to study this problem of trace.

As wee shall prove in forthcoming Proposition 2.1, if g is a local defining function of Σ, the condition i) means exactly that the matrix $(Z_i \cdot Z_j \cdot g)_{1\leq i,j\leq 2d}$ is of rank r at w_0. Let us notice that, because, if $i \in \{1, \cdots d\}$ and $j \neq i + d$,

$$(Z_i \cdot Z_{i+d} \cdot g)(w_0) - (Z_{i+d} \cdot Z_i \cdot g)(w_0) = -2\partial_s g(w_0) \neq 0 \quad \text{and} \quad (Z_i \cdot Z_j \cdot g) = (Z_j \cdot Z_i \cdot g),$$

the rank of the matrix $(Z_i \cdot Z_j \cdot g)_{1\leq i,j\leq 2d}$ is at least d at w_0.

Let us give some examples. First let us consider the case when the hypersurface Σ is given by an equation of the type $s - P(x, y)$ where P is a homogenenous polynomial of degree 2 on $\mathbb{R}^{2d}$. Let us observe that this equation is homogenenous of order 2 wih respect to the dilation of Heisenberg group $d_\lambda(x, y, s) \stackrel{\text{def}}{=} (\lambda x, \lambda y, \lambda^2 s)$. In this case $w_0 = (0, 0, 0)$ is always a regular characteristic point. Indeed the family $(Z_j \cdot g)_{1\leq j\leq 2d}$ is a family of linear form on $\mathbb{R}^{2d}$. As $X_{j|w_0} = \partial_{x_j}$ and $Y_{j|w_0} = \partial_{y_j}$, the rank of the family is exactly the rank of the matrix $(Z_i \cdot Z_j \cdot g)_{1\leq i,j\leq 2d}$ at point w_0. Thus Σ_c is obviously a submanifold of codimension r of Σ.

Now let us exhibit an example of non regular characteristic point. In the case when $d = 2$, let us define, for λ in $\mathbb{R}$,

$$\Sigma_\lambda = \Big\{(x_1, y_1, x_2, y_2, s) \in \mathbb{R}^5 \,/\, s = x_1 y_1 + \lambda(x_1^3 + y_1^3)\Big\}.$$

If $\lambda = 0$, as observe above, the origin is a regular characteristic point. A very easy computation shows that the rank of the matrix $(Z_i \cdot Z_j \cdot g)_{1\leq i,j\leq 4}$ is three. But the characteristic set $\Sigma_{\lambda,c}$ is the set of points of Σ_λ such that

$$3\lambda x_1^2 = -2x_1 + 3\lambda y_1^2 = y_2 = x_2 = 0.$$

If $\lambda \neq 0$, the characteristic set $\Sigma_{\lambda,c}$ reduces to the origin.

Let us introduce some rings of functions adapted to our situation.

Definition 1.2. *Let W be any open subset of Σ and F a closed subset of W. Let us denote by $C^\infty_F(W)$ the set of smooth functions a on $W \setminus F$ such that for any multi-index α, a constant C_α exists such that*

$$\forall \alpha \in \mathbb{N}^d \;\; |\partial^\alpha a(z)| \leq C_\alpha d(z, F)^{-|\alpha|},$$

where d denotes the distance on Σ induced by the euclian distance on $\mathbb{R}^{2d+1}$.

Now let us define the vector fields on Σ which will describe the regularity on Σ.

Definition 1.3. *Let w_0 a characteristic point of a hypersurface Σ. Let W be a neighhourhood of w_0. We denote by Z_Σ the $C^\infty_{\Sigma_c}(W)$ modulus spanned by the set vector fields of $\mathcal{Z} \cap T\Sigma_{|W}$ that vanish on Σ_c.*

As we shall see in Proposition 3.1, the modulus Z_Σ is of finite type (of course as a $C^\infty_{\Sigma_c}(W)$ modulus) if w_0 is a regular characteristic point and W is choosen small enough. If g is a local defining function of Σ, a generating system is given by

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j \quad \text{for} \quad 1 \leq j < k \leq 2d. \tag{1.1}$$

Now we are ready to introduce the space of traces.

Definition 1.4. *Let w_0 a regular characteristic point of a hypersurface Σ. Let W be a small enough neighbourhood of w_0. We denote by $H^1(\mathcal{Z}_\Sigma, W)$ the space of functions v of $L^2(\Sigma)$ supported in W such that*

$$\|v\|^2_{H^1(\mathcal{Z}_\Sigma)} \stackrel{\text{def}}{=} \|v\|^2_{L^2(\Sigma)} + \sum_{1\leq j,k\leq 2d} \|R_{j,k} v\|^2_{L^2(\Sigma)} < \infty.$$

where the family $(R_{j,k})_{1\leq j,k\leq 2d}$ *is given by (1.1). If* $s\in[0,1]$, *we define* $H^s(\mathcal{Z}_\Sigma,V)$ *by complex interpolation.*

Our theorem is the following.

Theorem 1.2. *Let* w_0 *a regular characteristic point of a hypersurface* Σ. *Let* V *be a small enough neighhourhood of* w_0. *Then the restriction map* γ_Σ *is an onto continuous map from* $H^1(\mathbb{H}^d,V)$ *onto* $H^{\frac12}(\mathcal{Z}_\Sigma,V\cap\Sigma)$.

Let us remark that, if w_0 is a non degenerate characteristic point (i.e. a regular characterisitic point of order $2d$) this theorem is Theorem 1.8 of [4].

1.2. **Structure of the proof.** In our paper [4], we use a blow up of the point w_0 (which is Σ_c in the case when the characteristic point w_0 is of order $2d$). Here we shall blow up the submanifold Σ_c. In order to do it, let us introduce a function $\varphi\in\mathcal{D}(\mathbb{R}_+\setminus\{0\})$ such that

$$\forall t\in[-1,1]\setminus\{0\}\,,\ \sum_{p=0}^{\infty}\varphi(2^pt)=1. \tag{1.2}$$

Let us define the function ρ_c by $\rho_c\stackrel{\rm def}{=}\bigl(g^2+|\nabla_{\mathbb{H}}g|^4\bigr)^{\frac14}$. Now writing that for any function u in $L^2(\rho_c\leq 1)$,

$$u=\sum_{p=0}^{\infty}\varphi_pu\quad\text{with}\quad \varphi_p(w)\stackrel{\rm def}{=}\varphi(2^p\rho_c(w)), \tag{1.3}$$

we apply Theorem 1.1 of trace and trace lifting to each piece φ_pu which is supported in a domain where Σ is non charactersitic because $\rho_c\sim 2^{-p}$ in this domain. This decomposition leads immediately to the problem of estimating the norm $H^1(\mathbb{H}^d)$ of each piece φ_pu. Leibnitz formula and the chain rule tell us that

$$\nabla_{\mathbb{H}}(\varphi_pu)=\varphi_p\nabla_{\mathbb{H}}u+2^p\varphi'(2^p\rho_c)u\nabla_{\mathbb{H}}\rho_c.$$

Let us observe that, as

$$Z_j\rho_c^4=2gZ_j\cdot g+4|\nabla_{\mathbb{H}}g|^2(Z_j\cdot g)\sum_{k=1}^{2d}Z_j\cdot(Z_k\cdot g)\,,$$

we have, for any real number s, $|\nabla_{\mathbb{H}}\rho_c^s|\leq C_s\rho_c^{s-1}$. As the support of $\varphi'(2^p\rho_c)$ included in $\rho_c\sim 2^{-p}$, the supports of $\varphi'(2^p\rho_c)$ and $\varphi'(2^{p'}\rho_c)$ are disjoint if $|p-p'|\leq N_0$ for some N_0. Thus, we get that

$$\sum_{p=0}^{\infty}2^{2p}\|\varphi'(2^p\rho_c)u\nabla_{\mathbb{H}}\rho_c\|_{L^2}^2\leq C\Big\|\frac{u}{\rho_c}\Big\|_{L^2}^2.$$

This leads to the proof of the following Hardy type inequality.

Theorem 1.3. *If* w_0 *is a regular characteristic point of* Σ, *a neighbourhood* V *of* w_0 *exists such that, for any* u *in the space* $H^1(\mathbb{H}^d,V)$ *of* $H^1(\mathbb{H}^d)$ *functions supported in* V,

$$\int_{\mathbb{H}^d}\frac{u^2}{\rho_c^2}dw\leq C\|\nabla_{\mathbb{H}}u\|_{L^2}^2.\quad\textit{with}\quad \rho_c=\bigl(g^2+|\nabla_{\mathbb{H}}g|^4\bigr)^{\frac14}\,.$$

This theorem implies that, for any u in $H^1(\mathbb{H}^d,V)$,

$$\sum_{p=0}^{\infty}\|\nabla_{\mathbb{H}}(\varphi_pu)\|_{L^2}^2\leq C\|\nabla_{\mathbb{H}}u\|_{L^2}^2. \tag{1.4}$$

The proof of this theorem, which is the core of this work, is the purpose of the second section.

In the third section, we first straighten the submanifolds Σ and Σ_c, and after dilation, we apply Theorem 1.1. This gives a rather unpleasant description on the trace space. Then, we prove an interpolation result which allows to conclude the proof of Theorem 1.2.

2. A Hardy type inequality

2.1. **The classical Hardy inequality.** As a warm up, let us recall briefly the usual proof of the classical Hardy inequality [1].

$$\int_{\mathbb{H}^d} \frac{u^2}{\rho^2} dw \leq C\|\nabla_{\mathbb{H}} u\|_{L^2}^2 \quad \text{with} \quad \rho(w) = \left(s^2 + (|x|^2 + |y|^2)^2\right)^{\frac{1}{4}}. \tag{2.5}$$

As $\mathcal{D}(\mathbb{H}^d \setminus \{0\})$ is dense $H^1(\mathbb{H}^d)$, we have restrict ourselves to functions u in $\mathcal{D}(\mathbb{H}^d \setminus \{0\})$. Then the proof mainely consists in an integration by parts with respect to the radial vector field $R_{\mathbb{H}}$ adapted to the structure of $\mathbb{H}^d$, namely

$$R_{\mathbb{H}} \stackrel{\text{def}}{=} 2s\partial_s + \sum_{j=1}^d \left(x_j \partial_{x_j} + y_j \partial_{y_j}\right) = s[Y_1, X_1] + \sum_{j=1}^d \left(x_j X_j + y_j Y_j\right)$$

once noticed that $R_{\mathbb{H}} \cdot \rho^{-2} = -2\rho^{-2}$ and $\operatorname{div} R_{\mathbb{H}} = 2d + 2$. More precisely, this gives

$$-d \int \frac{u^2}{\rho^2} dw = \int \sum_{j=1}^d \frac{u}{\rho}\Big(\frac{x_j}{\rho} X_j + \frac{y_j}{\rho} Y_j\Big) u dw - \int \Big(Y_1 \frac{s}{\rho^2}\Big) u(X_1 u) dw + \int \Big(X_1 \frac{s}{\rho^2}\Big) u(Y_1 u) dw.$$

As we have $\left|Z_j\Big(\frac{s}{\rho^2}\Big)\right| \leq C\rho^{-1}$, Cauchy-Schwarz inequality gives (2.5).

2.2. **Construction of substitute of ρ and $R_{\mathbb{H}}$.** Let us start with some remarks about the relations between Σ_c and the vector fileds Z_j in the case when w_0 is a regular characteristic point.

Proposition 2.1. *The condition i) of Definition 1.1 is equivalent to the fact that, for any defining function g of Σ, the rank of the matrix $(Z_i \cdot Z_j \cdot g)_{1 \leq i,j \leq 2d}$ is r.*

Proof of Proposition 2.1 Let g be a local defining function of Σ. Of course, Dg vanishes on $T\Sigma$. As $Z_j(w_0)$ belongs to $T_{w_0}\Sigma$, we have $\mathcal{L}_{Z_j}(Dg)(w_0) = D(Z_j \cdot g)(w_0)$. By definition of $\mathcal{Z}$, we infer that

$$D(Z_j \cdot g)(w_0) = \sum_{i=1}^{2d} (Z_i \cdot Z_j \cdot g)(w_0) dz_i.$$

Thus the rank of matrix $(Z_i \cdot Z_j \cdot g)_{1 \leq i,j \leq 2d}$ is the rank of $\mathcal{L}_{Z_j}(Dg)(w_0)$.

Conversevely, let θ be a 1-form that vanishes on $T\Sigma$ and such that $\theta(w_0) \neq 0$ and g a local defining function of Σ. A function a that does not vanish at w_0 exists such that $\theta = aDg$. Thanks to Leibnitz formula, $\mathcal{L}_{Z_j}(\theta)(w_0)_{|T_{w_0}\Sigma} = a(w_0) D(Z_j \cdot g)(w_0)_{|T_{w_0}\Sigma}$. The fact that the function a does not vanish at point w_0 implies the proposition. ∎

In all that follows, g will denote a defining function of Σ of the form $g(x, y, s) = s + f(x, y)$ (this is allowed by the implicit function theorem) near w_0, assumed to be the origin of $\mathbb{H}^d$ which is assumed to be a characterisitc regular point of order $r < 2d$.

[1] For a different approach based on Fourier analysis, see [3].

As the matrix $(Z_i\cdot Z_j\cdot g)_{1\leq i,j\leq 2d}$ is of rank r in w_0, and as $Z_{i|w_0}=\partial_{z_i}$, a family $(j_\ell)_{1\leq\ell\leq r}$ exists in $\{1,\dots,2d\}^r$ such that the linear forms $(D(Z_{j_\ell}\cdot g))_{1\leq\ell\leq r}$ are linearly independent near w_0. Moreover, the function Z_ig are independent of s and $Dg(w_0)=(ds,0,0)$. Thus the family of functions

$$(g\,,\ (Z_{j_1}\cdot g),\cdots,(Z_{j_r}\cdot g)) \tag{2.6}$$

is a family of $r+1$ independent functions. They vanish on the submanifold Σ_c which is by hypothesis a submanifold of $\mathbb{H}^d$ of codimension $r+1$. This implies that, near w_0,

$$\Sigma_c=\{w\,/\,g(w)=(Z_{j_1}\cdot g)(w)=\cdots=(Z_{j_r}\cdot g)(w)=0\}\,. \tag{2.7}$$

We shall keep these notations all along this text.

The definition of substitute to ρ and $R_{\mathbb{H}}$ relies on the following two lemmas.

Lemma 2.1. *A couple of vector fields $(Z_0,\overline{Z}_0)$ exists in $(\mathcal{Z}\setminus\{Z_{j_1},\cdots,Z_{j_r}\})\times(\pm\mathcal{Z})$ such that*

$$[Z_0,\overline{Z}_0]=2\partial_s\quad\textit{and}\quad D(\overline{Z}_0\cdot g)(w_0)\neq 0.$$

Proof of Lemma 2.1 Let us consider $Z_0\in\mathcal{Z}\setminus\{Z_{j_1},\cdots,Z_{j_r}\}$. and $\underline{Z}_0$ in $\pm\mathcal{Z}$ such that $[Z_0,\underline{Z}_0]=2\partial_s$. If $\pm\underline{Z}_0$ belongs to $\{Z_{j_1},\cdots,Z_{j_r}\}$, then (2.6) implies that $D(\underline{Z}_0\cdot g)(w_0)$ is different from 0 and then $\overline{Z}_0=\underline{Z}_0$ fits. If $\pm\underline{Z}_0$ is not in $\{Z_{j_1},\cdots,Z_{j_r}\}$, as

$$(Z_0\cdot(\underline{Z}_0\cdot g))(w_0)-(\underline{Z}_0\cdot(Z_0\cdot g))(w_0)=2,$$

either $D(Z_0\cdot g)(w_0)$ or $D(\underline{Z}_0\cdot g)(w_0)$ is different from 0. Thus if $D(\underline{Z}_0\cdot g)(w_0)=0$, we get the lemma interchanging the role of $\underline{Z}_0$ and Z_0. ∎

Using (2.6) and (2.7), the proof of the following lemma is very easy and thus omitted.

Lemma 2.2. *A neighhourhood V of w_0 and a family $(\alpha_\ell)_{1\leq\ell\leq r}$ of functions of $C^\infty(V)$ exist such that*

$$Z_0\cdot g=\sum_{\ell=1}^r\alpha_\ell(Z_{j_\ell}\cdot g).$$

Now let us state a Hardy inequality, which is obviously better than the one of Theorem 1.3 and which is surprisingly the one we are able to prove.

Theorem 2.1. *A neighbourhood V of w_0 exists such that, for any u in $H^1(\mathbb{H}^d,V)$,*

$$\int\frac{u^2}{\rho_0^2}dw\leq C\|\nabla_{\mathbb{H}}u\|_{L^2}^2\quad\textit{with}\quad\rho_0\stackrel{\rm def}{=}\big(g^2+(\overline{Z}_0\cdot g)^4\big)^{\frac14}\,.$$

Now the problem is to find an analogous of $R_{\mathbb{H}}$ is our situation. We do not manage to do it for ρ_c. For the function ρ_0, it is done by the following Lemma.

Lemma 2.3. *A neighbourhood V of w_0, two functions β and θ of $C^\infty(V)$ exist such that θ vanishes on Σ_c and which satisfy the following properties. Let us define*

$$R_1=2g\partial_s+\beta(\overline{Z}_0\cdot g)\widetilde{Z}_0\quad\textit{with}\quad\widetilde{Z}_0\stackrel{\rm def}{=}Z_0-\sum_{\ell=1}^r\alpha_\ell Z_{j_\ell}$$

where the functions $(\alpha_\ell)_{1\ell\leq r}$ are the functions which appear in Lemma 2.2. Then,

$$R_1\cdot\rho_0^4=4\rho_0^4\quad\textit{and}\quad\operatorname{div}R_1=3+\theta.$$

Proof of Lemma 2.3 The main point of the proof is the computation of the function β. By definition of the function ρ_0, we have

$$R_1 \cdot \rho_0^4 = 2g(R_1 \cdot g) + 4(\overline{Z}_0 \cdot g)^3 \left(R_1 \cdot (\overline{Z}_0 \cdot g)\right).$$

Lemma 2.2 implies that $\widetilde{Z}_0$ is tangent to Σ. Using that $\partial_s g \equiv 1$, this implies that $R_1 \cdot g = 2g$. Let us compute $R_1 \cdot (\overline{Z}_0 \cdot g)$. As $\partial_s(\overline{Z}_0 \cdot g) = 0$, we have

$$R_1 \cdot (\overline{Z}_0 \cdot g) = \beta(\overline{Z}_0 \cdot g) \left(\widetilde{Z}_0 \cdot (\overline{Z}_0 \cdot g)\right).$$

Let us notice that Z_0 does not belong to the family $(Z_{j_\ell})_{1\leq \ell\leq r}$. Thus $\overline{Z}_0$ commutes with the vector fields Z_{j_ℓ}. By definition of $\widetilde{Z}_0$, we infer

$$\begin{aligned}
[\widetilde{Z}_0, \overline{Z}_0] &= [Z_0, \overline{Z}_0] + \sum_{\ell=1}^{r} [\alpha_\ell Z_\ell, \overline{Z}_0] \\
&= 2\partial_s - \sum_{\ell=1}^{r} (\overline{Z}_0 \cdot \alpha_\ell) Z_\ell. \qquad (2.8)
\end{aligned}$$

Using that $\widetilde{Z}_0 \cdot g = 0$, we deduce

$$\begin{aligned}
\widetilde{Z}_0 \cdot (\overline{Z}_0 \cdot g) &= \overline{Z}_0 \cdot (\widetilde{Z}_0 \cdot g) + 2\partial_s g - \sum_{\ell=1}^{r} (\overline{Z}_0 \cdot \alpha_\ell)(Z_\ell \cdot g) \\
&= 2 + \widetilde{\theta} \quad \text{with} \quad \widetilde{\theta} \stackrel{\text{def}}{=} -\sum_{\ell=1}^{r} (\overline{Z}_0 \cdot \alpha_\ell)(Z_\ell \cdot g). \qquad (2.9)
\end{aligned}$$

It turns out that $R_1 \cdot \rho_1^4 = 4g^2 + 4(\overline{Z}_0 \cdot g)^4 \beta(2 + \widetilde{\theta})$. Choosing $\beta \stackrel{\text{def}}{=} (2 + \widetilde{\theta})^{-1}$ gives the first relation of Lemma 2.3. Now, let us compute $\operatorname{div} R_1$. We have

$$\operatorname{div} R_1 = 2\partial_s g + \beta \widetilde{Z}_0 \cdot (\overline{Z}_0 \cdot g) + (\overline{Z}_0 \cdot g) \operatorname{div} \widetilde{Z}_0.$$

Using that $\partial_s g \equiv 1$ and (2.9), we get

$$\begin{aligned}
\operatorname{div} R_1 &= 2 + \beta(2 + \widetilde{\theta}) + (\overline{Z}_0 \cdot g) \operatorname{div} \widetilde{Z}_0 \\
&= 3 + (\overline{Z}_0 \cdot g) \operatorname{div} \widetilde{Z}_0.
\end{aligned}$$

This proves the lemma with $\theta \stackrel{\text{def}}{=} (\overline{Z}_0 \cdot g) \operatorname{div} \widetilde{Z}_0$. ■

2.3. **Proof of Theorem 2.1.** Lemma 2.1 implies that, near w_0, the set $\rho_0^{-1}(0)$ is a submanifold of $\mathbb{H}^d$ of codimension 2. The following lemma will allow us to assume that u belongs to $\mathcal{D}(V \setminus \rho_0^{-1}(0))$.

Lemma 2.4. *Let V be a bounded domain of $\mathbb{H}^d$ and Γ a submanifold of codimension ≥ 2. Then $\mathcal{D}(V \setminus \Gamma)$ is dense in the space $H_0^1(\mathbb{H}^d, V)$ of functions of $H_0^1(\mathbb{H}^d)$ supported in V equipped with the norm*

$$\left(\|u\|_{L^2}^2 + \|\nabla_{\mathbb{H}} u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Proof of Lemma 2.4 As $H_0^1(\mathbb{H}^d, V)$ is a Hilbert space, it is enough to prove that the orthogonal of $\mathcal{D}(V \setminus \Gamma)$ is $\{0\}$. Let u be in this space. For any v in $\mathcal{D}(V \setminus \Gamma)$, we have

$$(u|v)_{L^2} + (\nabla_{\mathbb{H}} u|\nabla_{\mathbb{H}} v)_{L^2} = 0.$$

By integration by part, this implies that

$$\forall v \in \mathcal{D}(V \setminus \Gamma), \ \langle u - \Delta_{\mathbb{H}} u, v\rangle = 0.$$

Thus the support of $u - \Delta_{\mathbb{H}} u$ is included in Γ. As $Z_j u$ belongs to L^2, then $Z_j^2 u$ belongs to $H^{-1}(\mathbb{R}^{2d+1})$ (the classical Sobolev space). And except 0, no distribution of $H^{-1}(\mathbb{R}^{2d+1})$ can be supported in a submanifold of codimension greater than 1. Thus $u - \Delta_{\mathbb{H}} u = 0$. Taking the L^2 scalar product with u implies that $u \equiv 0$. ■

Thanks to Lemma 2.3, we have

$$\rho_0^{-2} = -\frac{1}{2} R_1 \cdot \rho_0^{-2}. \tag{2.10}$$

Thus by integration by part, we have, using Lemma 2.3,

$$\int \frac{u^2}{\rho_0^2} dw = \frac{3}{2} \int \frac{u^2}{\rho_0^2} dw + \int \theta \frac{u^2}{\rho_0^2} dw + I \quad \text{with} \quad I \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} (R_1 \cdot u) dw.$$

Assuming V small enough such that $\|\theta\|_{L^\infty(V)} \leq 1/4$, we get

$$\int \frac{u^2}{\rho_0^2} dw \leq 4|I|. \tag{2.11}$$

In order to estimate I, which contains terms of the type $g\partial_s u$, we have to compute the vector field R_1 in term of elements of $\mathcal{Z}$. Using (2.8), we infer that

$$R_1 = 2g[\widetilde{Z}_0, \overline{Z}_0] + g \sum_{\ell=1}^{r} (\overline{Z}_0 \cdot \alpha_\ell) Z_{j_\ell} + \beta(\overline{Z}_0 \cdot g) Z_0 - \beta(\overline{Z}_0 \cdot g) \sum_{\ell=1}^{r} \alpha_\ell Z_{j_\ell}.$$

In other terms, two families $(\beta_k)_{1\leq k\leq 2d}$ and $(\gamma_k)_{1\leq k\leq 2d}$ exist such that

$$R_1 = 2g[\widetilde{Z}_0, \overline{Z}_0] + \sum_{k=1}^{2d} \big(\beta_k g + \gamma_k(\overline{Z}_0 \cdot g)\big) Z_k. \tag{2.12}$$

We deduce that

$$\begin{aligned} I &= J_1 + J_2 \quad \text{with} \\ J_1 &\stackrel{\text{def}}{=} \sum_{k=1}^{2d} \int \frac{u}{\rho_0} \frac{\beta_k g + \gamma_k(\overline{Z}_0 \cdot g)}{\rho_0} (Z_k \cdot u) dw \quad \text{and} \\ J_2 &\stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g[\widetilde{Z}_0, \overline{Z}_0] \cdot u dw. \end{aligned}$$

As V is supposed bounded, we have that the functions

$$\frac{\beta_k g + \gamma_k(\overline{Z}_0 \cdot g)}{\rho_0}$$

are bounded. Cauchy Schwarz inequality yields

$$|J_1| \leq C \Big\| \frac{u}{\rho_0} \Big\|_{L^2} \|\nabla_{\mathbb{H}} u\|_{L^2}. \tag{2.13}$$

The estimate about J_2 is a little bit more difficult to obtain. Let us write that $J_2 = K_1 - K_2$ with

$$K_1 \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g \widetilde{Z}_0 \cdot (\overline{Z}_0 \cdot u) dw \quad \text{and} \quad K_2 \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g Z_0 \cdot (\widetilde{Z}_0 \cdot u) dw.$$

By integration by parts, we have $K_1 = -K_{11} - K_{12}$ with

$$K_{11} \stackrel{\text{def}}{=} \int \frac{g}{\rho_0^2}(\widetilde{Z}_0 \cdot u)(\overline{Z}_0 \cdot u)dw \quad \text{and}$$

$$K_{12} \stackrel{\text{def}}{=} \int f\frac{u}{\rho_0}(\overline{Z}_0 \cdot u)dw \quad \text{with} \quad f \stackrel{\text{def}}{=} (\operatorname{div} \widetilde{Z}_0)\frac{g}{\rho_0} + \rho_0\Big(\widetilde{Z}_0 \cdot \frac{g}{\rho_0^2}\Big)\cdot$$

By definition of ρ_0, it is obvious that

$$|K_{11}| \leq C\|\nabla_{\mathbb{H}} u\|_{L^2}^2. \tag{2.14}$$

As we can assume that V is included in $\rho_0^{-1}([0,1])$, we have that $\rho_0^{-1}g|\operatorname{div} \widetilde{Z}_0| \leq C$ on V. Moreover using that $\widetilde{Z}_0 \cdot g = 0$, we get

$$\Big|\widetilde{Z}_0 \cdot \frac{g}{\rho_0^2}\Big| = \frac{2g}{\rho_0^6}\Big|\widetilde{Z}_0 \cdot (\overline{Z}_0 \cdot g)\Big|\,|\overline{Z}_0 \cdot g|^3 \leq C\frac{g}{\rho_0^3} \leq \frac{C}{\rho_0}\cdot$$

This ensures that f is bounded on V and thus by Cauchy-Schwarz inequality,

$$\|K_{12}\| \leq C\Big\|\frac{u}{\rho_0}\Big\|_{L^2}\|\nabla_{\mathbb{H}} u\|_{L^2}.$$

Together with (2.14), this proves that

$$|K_1| \leq C\bigg(\Big\|\frac{u}{\rho_0}\Big\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2}\bigg)\|\nabla_{\mathbb{H}} u\|_{L^2}. \tag{2.15}$$

In order to estimate K_2, let us write that, by integration by parts,

$$K_2 = \int \frac{g}{\rho_0^2}(\overline{Z}_0 \cdot u)(\widetilde{Z}_0 \cdot u)dw + \int \rho_0\Big(\overline{Z}_0 \cdot \frac{g}{\rho_0^2}\Big)\frac{u}{\rho_0}(\widetilde{Z}_0 \cdot u)dw.$$

Using that

$$\overline{Z}_0 \cdot \rho_0^4 = 2g(\overline{Z}_0 \cdot g) + 4\left(\overline{Z}_0 \cdot (\overline{Z}_0 \cdot g)\right)(\overline{Z}_0 \cdot g)^3,$$

we immediatly get that the function $\rho_0\Big(\overline{Z}_0 \cdot \dfrac{g}{\rho_0^2}\Big)$ is bounded on V and we deduce that

$$|K_2| \leq C\bigg(\Big\|\frac{u}{\rho_0}\Big\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2}\bigg)\|\nabla_{\mathbb{H}} u\|_{L^2}.$$

Together with (2.11), (2.13) and (2.15), we infer that

$$\Big\|\frac{u}{\rho_0}\Big\|_{L^2}^2 \leq C\bigg(\Big\|\frac{u}{\rho_0}\Big\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2}\bigg)\|\nabla_{\mathbb{H}} u\|_{L^2}$$

which concludes the proof of Theorem 2.1.

3. The proof of the trace and trace lifting theorem

3.1. Some preliminary properties.

Proposition 3.1. *A neighbourhood W of w_0 exists such that the $C_{\Sigma_c}(W)$ modulus $\mathcal{Z}_\Sigma$ spanned by the vector fields of $\mathcal{Z} \cap T\Sigma_{|W}$ which vanish on the characterisitic submanifold Σ_c is of finite type and generated by*

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j.$$

Proof of Proposition 3.1 It is enough to prove that any element L of $\mathcal{Z}\cap T\Sigma$ which vanish on Σ_c is a combinaison (with coeffcients in $C^\infty_{\Sigma_c}(W)$) of the $R_{j,k}$. By definition

$$L=\sum_{j=1}^{2d}\alpha_j Z_j \quad\text{with}\quad \alpha_{j|\Sigma_c}=0 \quad\text{and}\quad \sum_{j=1}^{2d}\alpha_j(Z_j\cdot g)=0.$$

Let us introduce a partition of unity $(\widetilde{\psi}_j)_{1\leq j\leq 2d}$ of the sphere $\mathbb{S}^{2d-1}$ such that the support of $\widetilde{\psi}_j$ is included in the set of ζ of $\mathbb{S}^{2d-1}$ such that $|\zeta_j|\geq (4d)^{-1}$. Let us state

$$\psi_j \stackrel{\text{def}}{=} \widetilde{\psi}_j\Big(\frac{\nabla_{\mathbb{H}}g}{|\nabla_{\mathbb{H}}g|}\Big)\cdot$$

It is an exercice left to the reader to check that ψ_j belongs to $C^\infty_{\Sigma_c}(W)$. On $\Sigma\setminus\Sigma_c$, we have, for any j in $\{1,\cdots,2d\}$,

$$\psi_j(L\cdot g)=\sum_{k=1}^{2d}\psi_j\alpha_k(Z_k\cdot g)=0.$$

By definition of ψ_j, $(Z_j\cdot g)$ does not vanish on the support of ψ_j. Thus we have

$$\alpha_j\psi_j=-\frac{1}{(Z_j\cdot g)}\sum_{k\neq j}\psi_j\alpha_k(Z_k\cdot g).$$

From this, we deduce that

$$\begin{aligned}\psi_j L &= \sum_{k\neq j}\psi_j\alpha_k\Big(Z_k-\frac{(Z_k\cdot g)}{(Z_j\cdot g)}Z_j\Big)\\ &= \sum_{k\neq j}\frac{\psi_j\alpha_k}{(Z_j\cdot g)}\big((Z_j\cdot g)Z_k-(Z_k\cdot g)Z_j\big).\end{aligned}$$

Now the facts that $\alpha_k\in C^\infty_{\Sigma_c}$ and that $(Z_j\cdot g)$ does not vanish on the support of ψ_j ensure that

$$\alpha_{j,k}\stackrel{\text{def}}{=}\frac{\varphi_j\alpha_k}{(Z_j\cdot g)}\in C^\infty_{\Sigma_c}.$$

So we have

$$L=\sum_{1\leq j<k\leq 2d}\alpha_{j,k}\big((Z_j\cdot g)Z_k-(Z_k\cdot g)Z_j\big)$$

and the proposition is proved. ■

The blow up prodecure requires to straighten the submanifolds Σ and Σ_c.

Lemma 3.1. *A neighbourhood V of w_0 and a diffeomorphism χ from V onto $\chi(V)$ exist which satisfy the following properties.*

- *It straighten the submanifolds Σ and Σ_c, namely*

  $$\chi(\Sigma\cap V)=(s=0)\cap\chi(V)\quad\text{and}\quad \chi(\Sigma_c\cap V)=(s=z_1=\cdots z_r=0)\cap\chi(V).$$

- *The transported vector fields are of the form*

  $$\chi^\star(\partial_s)=\partial_s\quad\text{and}\quad Z_j^D\stackrel{\text{def}}{=}\chi^\star(Z_j)=\frac{\partial}{\partial e_j}+\Big(\sum_{\ell=1}^{r}\alpha_k^\ell(z)z_k\Big)\partial_s+h_j(z,\partial_z)$$

  where $(e_j)_{1\leq j\leq 2d}$ is a basis of $\mathbb{R}^{2d}$, the (α_k^ℓ) are smooth bounded functions on V such that, for $j\in\{1,\cdots,r\}$, $\alpha_j^\ell\equiv\delta_j^\ell$ and $(h_j)_{1\leq j\leq 2d}$ is a family of smooth vector fields which vanish at $z=0$.

Proof of Lemma 3.1 It is easily checked that the (local) diffeomorphism defined by

$$\chi(x,y,s) = \begin{pmatrix} g(x,y,s) = s + f(x,y) \\ z_k = (Z_{j_\ell} \cdot g)(x,y) \quad \text{if} \quad k \leq r \\ z_k = \langle L_k, (x,y) \rangle \quad \text{if} \quad k > r \end{pmatrix}$$

where the family of linear form $(L_k)_{r+1\leq k\leq 2d}$ is choosen such that

$$\big(D(Z_{j_\ell} \cdot g)(w_0)\big)_{1\leq \ell \leq r}, (L_k)_{r+1\leq k\leq 2d}$$

is a basis of the dual space of $\mathbb{R}^{2d}$. ■

From now on, we shall work only in the straighten situation and to avoid excessive heavyness of notations, we shall still denote Z_j^D by Z_j.

3.2. **The blow up procedure.** Let us write that, for any function u, we can write (at least in L^2) that

$$u = \sum_{p=0}^{\infty} \varphi_p u \quad \text{with} \quad \varphi_p(z,s) \stackrel{\text{def}}{=} \varphi\Big(2^p(s^2 + |z'|^4)^{\frac14}\Big) \quad \text{and} \quad z' \stackrel{\text{def}}{=} (z_1, \cdots, z_r, 0, \cdots 0)$$

where φ is the function introduced in (1.2). We shall proof the following theorem.

Theorem 3.1. *The restriction map on the hypersurface* $(s = 0)$ *can be extended in a continuous onto map from* $H^1(\mathcal{Z}; \{\rho_c \leq 1\})$ *onto the space* $T^{\frac12}$ *of function* $v \in L^2(|z'| \leq 1)$ *such that*

$$\|v\|^2_{T^{\frac12}} \stackrel{\text{def}}{=} \sum_{p=0}^{\infty} \|\varphi_p^\Sigma v\|^2_{H^{\frac12}(\mathcal{R},p)} < \infty \quad \textit{with} \quad H^s(\mathcal{R},p) \stackrel{\text{def}}{=} [L^2(2^{-p}\mathcal{C}_\Sigma), H^1(\mathcal{R}, (2^{-p}\mathcal{C}_\Sigma)]_s.$$

where $\mathcal{C}_\Sigma \stackrel{\text{def}}{=} \{c \leq |z'| \leq C,\ \varphi_p^\Sigma(z) \stackrel{\text{def}}{=} \varphi_p(z,0) = \varphi(2^p|z'|)$, $[A,B]_\theta$ *denotes the complex interpolation between* A *and* B *and* $H^1(\mathcal{R}, W)$ *the space of functions of* $H^1(\mathcal{R})$ *supported in* W.

Proof of Theorem 3.1 Once noticed that the Hardy inequality given by Theorem 1.3 becomes

$$\int \frac{u^2(z,s)}{(s^2 + |z'|^4)^{\frac12}} dzds \leq C \sum_{j=1}^{2d} \|Z_j u\|^2_{L^2}, \tag{3.16}$$

we get, by computations very similar to the ones done at the beginning of Subsection 1.2, an analogous of (1.4), namely

$$\sum_{p=0}^{\infty} \sum_{j=1}^{2d} \|Z_j(\varphi_p u)\|^2_{L^2} \leq C \sum_{j=1}^{2d} \|Z_j u\|^2_{L^2}. \tag{3.17}$$

Let us notice that outside $\Sigma_c = \{(z,s)\,/ s = 0\,,\ z' = 0\}$, thus in particular on the support of φ_p, the hypersurface Σ is non characteristic for $\mathcal{Z}$. Thus locally we can apply Theorem 1.1 to each piece $\varphi_p u$. The key point is the control of the constant when p tends to ∞. In order to do so, it is convenient to use the quasi-homogenenous dilations $\delta_p(z,s) \stackrel{\text{def}}{=} (2^p z, 2^{2p} s)$. Let us define

$$u_p(z,s) \stackrel{\text{def}}{=} \varphi_0(z,s) u(2^p z, 2^{2p} s) \quad \text{and} \quad Z_{j,p} \stackrel{\text{def}}{=} \frac{\partial}{\partial e_j} + \sum_{\ell=1}^{r} \alpha_j^\ell(2^{-p}z) z_\ell \partial_s + h_j(2^{-p}z, \partial_z).$$

It is obvious that a one to one map σ of $\{1, \cdots, 2d\}$ exists such that

$$[Z_{j,p}, Z_{k,p}] = 2\delta_{k,\sigma(j)}\partial_s. \tag{3.18}$$

Moreover, as $\|u_p\|_{L^2}^2 = 2^{2p(d+1)}\|\varphi_p u\|_{L^2}$, we have, thanks to Hardy inequality (3.16),

$$\sum_{p=0}^{\infty} 2^{-2pd}\|u_p\|_{L^2}^2 \le C\sum_{j=1}^{2d}\|Z_j u\|_{L^2}^2.$$

Applying (3.17), we infer

$$\sum_{p=0}^{\infty} 2^{-2pd}\Big(\|u_p\|_{L^2}^2 + \sum_{j=1}^{2d}\|Z_{j,p}u_p\|_{L^2}^2\Big) \le C\sum_{j=1}^{2d}\|Z_j u\|_{L^2}^2. \tag{3.19}$$

On the support of φ_0, the hypersurface $(s = 0)$ is non characteristic with respect to the family $(Z_{j,p})_{1\le j\le 2d}$ because, for j between 1 and r,

$$Z_{j,p} = \frac{\partial}{\partial e_j} + h_j(2^{-p}z, \partial_z) + z_j\partial_s.$$

Let us notice that the transverse component of $Z_{j,p}$ does not depend on p. Thus we can apply Theorem 4.6 of [4] together with a result of interpolation between Sobolev space (see Remarque 4.2 page 89 in [8]) to each u_p. Using that $\|u_p\|_{H^1(\mathcal{Z}_p)} = 2^{2pd}\|\varphi_p u\|_{H^1(\mathcal{Z})}$, this gives in particular that a constant C exists (independent of p) such that

$$\|\gamma(u_p)\|_{[L^2(\mathbb{R}^{2d}), H^1(\widetilde{\mathcal{R}}_p, \mathbb{R}^{2d})]_{\frac12}} \le C2^{2pd}\|\varphi_p u\|_{H^1(\mathcal{Z})} \tag{3.20}$$

with $\widetilde{\mathcal{R}}_p$ is the union of

$$\mathcal{R}_p = \big(\widetilde{\varphi}_0^{\Sigma}(Z_{j,p}\cdot s)Z_{k,p} - \varphi_0^{\Sigma}(Z_{k,p}\cdot s)Z_{j,p}\big)_{1\le j,k\le 2d} \quad \text{and} \quad \big((1-\widetilde{\varphi}_0^{\Sigma})\partial_j\big)_{1\le j\le 2d}$$

where $\widetilde{\varphi}_0^{\Sigma}$ is a smooth function supported in $\mathcal{C}_\Sigma$ such that $\widetilde{\varphi}_0^{\Sigma} \equiv 1$ near the support of φ_0^{Σ}.

At this point, let us recall the definition of complex interpolation. For details of this theory, we refer in particular to [6] and [18].

Definition 3.1. *Let $(\mathcal{H}_j, \|\cdot\|_j)_{j\in\{0,1\}}$ be two Hilbert spaces such that $\mathcal{H}_1$ is densely included in $\mathcal{H}_0$. Let $\mathcal{F}(\mathcal{H}_0, \mathcal{H}_1)$ be the space of holomorphic functions f from the strip $0 < \Re e\,\zeta < 1$ into $\mathcal{H}_0$ such that $f(j+it)$ is continuous and vanishes at infinity in $\mathcal{H}_j$. Then, for $\theta \in]0,1[$, the space $[\mathcal{H}_0, \mathcal{H}_1]_\theta$ is*

$$[\mathcal{H}_0, \mathcal{H}_1]_\theta \stackrel{def}{=} \{v \in L^2 \,/\, \exists f \in \mathcal{F}(\mathcal{H}_0, \mathcal{H}_1) \,/\, f(\theta) = v\}$$

equipped with the norm

$$\|v\|_{[\mathcal{H}_0,\mathcal{H}_1]_\theta} \stackrel{def}{=} \inf_{f\in\mathcal{F}(\mathcal{H}_0,\mathcal{H}_1)} \max_{j\in\{0,1\}} \sup_{t\in\mathbb{R}} \|f(j+it)\|_{\mathcal{H}_j}.$$

As the support of $\gamma(u_p)$ is included in the support of φ_0^{Σ}, let us consider a smooth function φ_1^{Σ} supported in the set where φ_0^{Σ} has value 1 and such that φ_1^{Σ} has value 1 near the support of φ_0^{Σ}. If f is a function in $\mathcal{F}(L^2(\mathbb{R}^{2d}), H^1(\widetilde{\mathcal{R}}_p, \mathbb{R}^{2d}))$ such that $f(1/2) = v$, then the function $\zeta \mapsto \varphi_1 f(\zeta)$ belongs to $\mathcal{F}(L^2(\mathcal{C}_\Sigma), H^1(\widetilde{\mathcal{R}}_p, \mathcal{C}_\Sigma))$ and $\varphi_1 f(1/2) = v$. As we obviously have that $H^1(\widetilde{\mathcal{R}}_p, \mathcal{C}_\Sigma) = H^1(\mathcal{R}_p, \mathcal{C}_\Sigma)$, Inequality (3.20) becomes

$$\|\gamma(u_p)\|_{[L^2(\mathcal{C}_\Sigma), H^1(\mathcal{R}_p, \mathcal{C}_\Sigma)]_{\frac12}} \le C2^{2pd}\|\varphi_p u\|_{H^1(\mathcal{Z})}. \tag{3.21}$$

Moreover, dilations on $\mathbb{R}^{2d}$ of ratio 2^{-p} maps $L^2(\mathcal{C}_\Sigma)$ (resp. $H^1(\mathcal{R}_p),\mathcal{C}_\Sigma)$ into $L^2(2^{-p}\mathcal{C}_\Sigma)$ (resp. $H^1(\mathcal{R},2^{-p}\mathcal{C}_\Sigma)$) with norm equal to 2^{-pd}. Thus by the functorial property of complex interpolation, Inequality (3.21) becomes

$$\|\gamma(\varphi_p u)\|_{H^{\frac12}(\mathcal{R},p)} \leq C\|\varphi_p u\|_{H^1(\mathcal{Z})}. \tag{3.22}$$

Inequality (3.17) implies that γ can be extended to a continuous linear map from $H^1(\mathcal{Z})$ into $T^{\frac12}$.

In order to prove that γ is onto, let us consider $v \in T^{\frac12}$. By definition of $T^{\frac12}$, and after dilation, we infer that

$$\|\varphi_0^\Sigma v(2^{-p}\cdot)\|_{[L^2(\mathcal{C}_\Sigma),H^1(\mathcal{R}_p,\mathcal{C}_\Sigma)]_{\frac12}} \leq C2^{pd}c_p\|v\|_{T^{\frac12}} \quad \text{with} \quad \sum_p c_p^2 = 1. \tag{3.23}$$

As $L^2(\mathcal{C}^\Sigma)$ (resp. $H^1(\mathcal{R}_p,\mathcal{C}_\Sigma)$) is a subspace of $L^2(\mathbb{R}^{2d})$ (resp. $H^1(\widetilde{\mathcal{R}}_p;\mathbb{R}^{2d})$), using Theorem 4.6 of [4] together with Remarque 4.2 of [8] and (3.23), we claim the existence of a function $\widetilde{u}_p$ in the space $H^1(\widetilde{Z}_p)$ such that a constant C (independent of p) exists which satisfies, for any p,

$$\|\widetilde{u}_p\|_{H^1(\widetilde{\mathcal{Z}}_p)} \leq C \leq C2^{pd}c_p\|v\|_{T^{\frac12}} \quad \text{with} \quad \sum_p c_p^2 = 1 \tag{3.24}$$

where $\widetilde{\mathcal{Z}}_p$ is the union of the families

$$(\widetilde{\varphi}_0 Z_{j,p})_{1\leq j,k\leq 2d}\,,\ ((1-\widetilde{\varphi}_0)\partial_j)_{1\leq j\leq 2d} \quad \text{and} \quad (1-\widetilde{\varphi}_0)\partial_s.$$

Let us consider a smooth function φ_1 supported in the domain where $\widetilde{\varphi}$ has value 1 and such that $\varphi_1 \equiv 1$ near the support of φ_0. Defining $u_p \stackrel{\text{def}}{=} \varphi_1\widetilde{u}_p$, we have, by definition of $\mathcal{R}_p$ and by (3.24)

$$u_p \in H^1(\widetilde{\mathcal{Z}}_p,\mathcal{C}_\Sigma) = H^1(\mathcal{Z}_p,\mathcal{C}_\Sigma) \quad \text{and} \quad \|u_p\|_{H^1(\mathcal{R}_p)} \leq C\|\widetilde{u}_p\|_{H^1(\widetilde{\mathcal{R}}_p,\mathcal{C}_\Sigma)}.$$

After dilation, this gives

$$\sum_p \|u_p(2^p\cdot)\|^2_{H^1(\mathcal{Z})} \leq C\|v\|^2_{T^{\frac12}}.$$

As an integer N_0 exists such that

$$|p-p'| \geq N_0 \Longrightarrow u_p(2^p\cdot) \perp u_{p'}(2^{p'}\cdot) \quad \text{in} \quad H^1(\mathcal{Z}),$$

the series $\left(u_p(2^p\cdot)\right)_p$ converge in $H^1(\mathcal{Z})$ to a function u the trace of which is obviously v. This concludes the proof of Theorem 3.1. ■

3.3. **The space of trace as an interpolation space.** The description given by Theorem 3.1 is not totally satisfactory. We want to describe this space of trace as an interpolation space to get Theorem 1.2. In order to do so, let us define, for $s \in [0,1]$, the space

$$T^s \stackrel{\text{def}}{=} \Big\{v \in L^2 \;/\; \|v\|^2_{T^s} \stackrel{\text{def}}{=} \sum_p \|\varphi_p^\Sigma v\|_{H^s(\mathcal{R},p)} < \infty\Big\}.$$

Let us start with the proof of the following lemma.

Lemma 3.2. *The space T^1 is equal to $H^1(\mathcal{R})$ and the norm are equivalents.*

Proof of Lemma 3.2 By definition of the norm on $H^1(\mathcal{R}_p)$, we have

$$\|(\varphi_p^\Sigma v)(2^{-p}\cdot)\|^2_{H^1(\mathcal{R}_p)} = \|(\varphi_p^\Sigma v)(2^{-p}\cdot)\|^2_{L^2} + \sum_{j,k} \|\mathcal{R}_{j,k,p}\left((\varphi_p^\Sigma v)(2^{-p}\cdot)\right)\|^2_{L^2}.$$

By definition of $R_{j,k,p}$, we have

$$2^{-2pd}\|\mathcal{R}_{j,k,p}\left((\varphi_p^\Sigma v)(2^{-p}\cdot)\right)\|_{L^2}^2 = \|\mathcal{R}_{j,k}(\varphi_p^\Sigma v)\|_{L^2}^2.$$

By Leibnitz formula and by definition of φ_p^Σ, we have

$$\begin{aligned} R_{j,k}(\varphi_p^\Sigma v)(z) &= \varphi_p^\Sigma R_{j,k}\cdot v(z) + (R_{j,k}\cdot\varphi_p^\Sigma)v(z)\\ &= \varphi_p^\Sigma(z)(R_{j,k}\cdot v)(z) + 2^p(R_{j,k}\cdot|z'|)\varphi'(2^p|z'|)v(z). \end{aligned}$$

As the vector fields $R_{j,k}$ vanishes at 0, we have

$$\sup_{p,j,k}\|R_{j,k}\varphi_p^\Sigma\|_{L^\infty} < \infty.$$

This gives that

$$\left|R_{j,k}(\varphi_p^\Sigma v)(z) - \varphi_p^\Sigma R_{j,k}v(z)\right| \leq C\varphi'(2^p|z'|)|v(z)|.$$

As, for some positive integer N_0, the support of the two functions $\varphi(2^p|z'|)$ and $\varphi(2^{p'}|z'|)$ are disjoint when $|p-p'|\geq N_0$, this gives the lemma. ■

Now Theorem 1.2 will be an easy consequence of the following abstract interpolation lemma.

Lemma 3.3. *Let us consider $(\mathcal{H}_j,\|\cdot\|_j)_{j\in\{0,1\}}$ two Hilbert spaces such that $\mathcal{H}_1$ is densely included in $\mathcal{H}_0$ and a family $(\mathcal{H}_{j,p})_{(j,p)\in\{0,1\}\times\mathbb{N}}$ such that, for any p, $\mathcal{H}_{j,p}$ is a closed subset of $\mathcal{H}_j$.*

Let us assume that a family of $(\Lambda_p)_{p\in\mathbb{N}}$ of (unbounded) selfadjoints operators on $\mathcal{H}_{0,p}$ exists such that $\mathcal{H}_{1,p}$ equals to the domain of Λ_p and

$$\forall u\in\mathcal{H}_{1,p}\,,\ \|u\|_{\mathcal{H}_1}\sim\|\Lambda_p u\|_{\mathcal{H}_0}. \tag{3.25}$$

Let us assume in addition that a family of operators $(A_p)_{p\in\mathbb{N}}$ exists such that, for any (j,p) in $\{0,1\}\times\mathbb{N}$, the operator A_p is continuous from $\mathcal{H}_j$ into $\mathcal{H}_{j,p}$ and

$$\forall v\in\mathcal{H}_j\,,\ \lim_{p\to\infty}\Big\|v-\sum_{p=0}^N v_p\Big\| = 0 \quad\text{and}\quad \|v\|_{\mathcal{H}_j}^2\sim\sum_p\|A_p v\|_{\mathcal{H}_j}^2. \tag{3.26}$$

Then,

$$[\mathcal{H}_0,\mathcal{H}_1]_s = \Big\{v\in\mathcal{H}_0\ /\ \|v\|_{T^s}^2 \stackrel{\text{def}}{=} \sum_{p=0}^\infty\|A_p v\|_{\mathcal{H}_{s,p}}^2\Big\} \quad\text{with}\quad \mathcal{H}_{s,p}\stackrel{\text{def}}{=}[\mathcal{H}_{0,p},\mathcal{H}_{1,p}]_s.$$

Proof of Lemma 3.3 It is enough to prove that the two norms are equivalent on the dense space of v such that

$$v=\sum_{p=0}^N v_p \quad\text{with}\quad v_p\in\mathcal{H}_{1,p}.$$

Let us first estimate $\|v\|_{[\mathcal{H}_0,\mathcal{H}_1]_s}$. By definition of the norm on $\mathcal{H}_{s,p}$, a function f_p exists in $\mathcal{F}(\mathcal{H}_{0,p},\mathcal{H}_{1,p})$ such that

$$f_p(s)=A_p v \quad\text{and}\quad \max_{j\in\{0,1\}}\sup_{t\in\mathbb{R}}\|f_p(j+it)\|_{\mathcal{H}_j}\leq 2\|A_p v\|_{\mathcal{H}_{s,p}}.$$

Now let us define

$$F_N(\zeta)\stackrel{\text{def}}{=} e^{\zeta^2-s^2}\sum_{p=0}^N f_p(\zeta).$$

As the sum is finite, this is obvious that F_N belongs to $\mathcal{F}(\mathcal{H}_0, \mathcal{H}_1)$. Because of (3.26), we have, for $j \in \{0, 1\}$,

$$\begin{aligned}\|F_N(j+it)\|^2_{\mathcal{H}_j} &\leq Ce^{-t^2} \sum_{p=0}^{N} \|f_p(j+it)\|^2_{\mathcal{H}_j} \\ &\leq Ce^{-t^2} \sum_{p=0}^{N} \|A_p v\|^2_{\mathcal{H}_{s,p}} \\ &\leq C\|v\|^2_{T^s}.\end{aligned}$$

Thus by definition of the complex interpolation norm, we deduce that

$$\|v\|_{[\mathcal{H}_0,\mathcal{H}_1]_s} \leq C\|v\|_{T^s}.$$

Now let us estimate $\|v\|_{T^s}$. In order to do so, let us consider F in $\mathcal{F}(\mathcal{H}_0, \mathcal{H}_1)$ such that

$$F(s) = v \quad \text{and} \quad \max_{j\in\{0,1\}} \sup_t \|F_N(j+it)\|_{\mathcal{H}_j} \leq 2\|v\|_{[\mathcal{H}_0,\mathcal{H}_1]_s}.$$

For a greater than 1, let us introduce

$$\mathcal{N}_a(\zeta) \stackrel{\text{def}}{=} e^{\zeta^2-s^2} \sum_{p=0}^{N} \int_1^a \lambda^{2\zeta} d\mu_p(A_pF(\zeta), A_pF(z))$$

where μ_p is the spectral measure of Λ_p. Then, by using (3.25) and (3.26),

$$\begin{aligned}|\mathcal{N}_a(j+it)| &\leq Ce^{-t^2} \left| \int_1^a \lambda^{2it}\lambda^{2j} d\mu_p\Big(A_pF(j+it), A_pF(j+it)\Big)\right| \\ &\leq Ce^{-t^2} \sum_{p=0}^{N} \int_1^a \lambda^{2j} d\mu_p\Big(A_pF(j+it), A_pF(j+it)\Big) \\ &\leq Ce^{-t^2} \sum_{p=0}^{N} \|A_pF(j+it)\|^2_{\mathcal{H}_j} \\ &\leq Ce^{-t^2} \|F_j(j+it)\|^2_{\mathcal{H}_j} \\ &\leq C\|v\|^2_{[\mathcal{H}_0,\mathcal{H}_1]_s}.\end{aligned}$$

Then using the Phragmen-Lindelöf principle, we get that

$$\mathcal{N}_a(s) \leq \sup_t |\mathcal{N}_a(it)|^{1-s} |\mathcal{N}_a(1+it)|^s \leq C\|v\|^2_{[\mathcal{H}_0,\mathcal{H}_1]_s}.$$

Thus a constant C exists such that, for any a,

$$\sum_{p=0}^{N} \int_1^a \lambda^{2s} d\mu_p(A_pv, A_pv) \leq C\|v\|^2_{[\mathcal{H}_0,\mathcal{H}_1]_s}. \tag{3.27}$$

By definition of $\mathcal{H}_{s,p}$ and using that

$$\|w\|^2_{\mathcal{H}_{s,p}} = \int_1^\infty \lambda^{2s} d\mu_p(w, w),$$

we infer by passing to the limit when a tends to infinity in (3.27) that

$$\sum_{p=0}^{N} \|A_pv\|^2_{\mathcal{H}_{s,p}} \leq C\|v\|^2_{[\mathcal{H}_0,\mathcal{H}_1]_s}.$$

This conclude the proof of Lemma 3.3. ■

3.4. **Conclusion of the proof of Theorem 1.2.** Theorem 1.2 follows, observing that the hypothesis of Lemma 3.3 are satisfied with $\mathcal{H}_0 = L^2$, $\mathcal{H}_1 = H^1(\mathcal{R})$, $\mathcal{H}_{j,p}$ is the set of v in $\mathcal{H}_j$ the support of which is included in $2^{-p}\mathcal{C}$ and Λ_p is the square root of Dirichlet realization on $2^{-p}\mathcal{C}$ of the operator

$$\mathrm{Id} + \Delta_\Sigma \quad \text{with} \quad \Delta_\Sigma \stackrel{\text{def}}{=} \sum_{j,k} \mathcal{R}^\star_{j,k} \mathcal{R}_{j,k}.$$

To be able to apply Lemma 3.3, and then to conclude the proof of Theorem 1.2, it is enough to prove the following proposition.

Proposition 3.2. *A neighbourhood V of w_0 exists such that the operator Δ_Σ is selfadjoint on $L^2(V)$ with domain*

$$\Big\{ v \in L^2(V) \,/\, \forall (j,k,j',k') \in \{1, \cdots, 2d\}^4 \,,\ R_{j,k} v \in L^2(V) \quad \text{and} \quad R_{j,k} R_{j',k'} v \in L^2(V) \Big\}.$$

Proof of Proposition 3.2 Up to an omitted regularization process, it is enough to prove that, for any $v \in \mathcal{D}(V)$,

$$\sum_{j,k} \|R_{j,k} v\|^2_{L^2} + \sum_{j,k,j',k'} \|R_{j,k} R_{j',k'} v\|^2_{L^2} \leq C \Big(\|v\|^2_{L^2} + \|\Delta_\Sigma v\|^2_{L^2} \Big). \tag{3.28}$$

Let us start with the observation that

$$\begin{aligned} \sum_{j,k} \|R_{j,k} v\|^2_{L^2} &\leq C \sum_{j,k} (R_{j,k} v | R_{j,k} v)_{L^2} \\ &\leq C \sum_{j,k} (R^\star_{j,k} R_{j,k} v | v)_{L^2} \\ &\leq C (\Delta_\Sigma v | v)_{L^2} \\ &\leq C \|\Delta_\Sigma v\|_{L^2} \|v\|_{L^2}. \end{aligned} \tag{3.29}$$

In order to estimate $\|R_{j,k} R_{j',k'} v\|_{L^2}$, we are going to proceed as in the proof of Lemma 3.2. Let us write that

$$R_{j,k} R_{j',k'} (\varphi_p v) - \varphi_p R_{j,k} R_{j',k'} v = (R_{j,k} \varphi_p)(R_{j',k'} v) + (R_{j',k'} \varphi_p)(R_{j,k} v) + \varphi_p (R_{j,k} R_{j',k'} v). \tag{3.30}$$

As the coefficients of the vector fields $R_{j,k}$ vanishes on Σ_c, we have

$$\sup_{p,j,k,j',k'} \|R_{j,k} \varphi_p\|_{L^\infty} + \|R_{j,k} R_{j',k'} \varphi_p\|_{L^\infty} < \infty.$$

Thus, using (3.29), we have

$$\|\varphi_p R_{j,k} R_{j',k'} v - R_{j,k} R_{j',k'} (\varphi_p v)\|_{L^2} \leq C c_p \|\Delta_\Sigma v\|^{\frac{1}{2}}_{L^2} \|v\|^{\frac{1}{2}}_{L^2} \quad \text{with} \quad \sum_{p=0}^{\infty} c_p^2 = 1. \tag{3.31}$$

We have

$$R_{j,k} R_{j',k'} (\varphi_p v) = R_{j,k,p} R_{j',k',p} \left(\varphi_0 v(2^p \cdot) \right)$$

Lemma 4.1 of [4] tells us that the sytems $(\mathcal{R}_{j,k,p})_{j,k}$ satisfy the Hörmander condition at order 2 uniformely with respect to p on $\mathcal{C}$. Thus, the classical maximal estimate tells us that

$$\|R_{j,k,p} R_{j',k',p} w\|_{L^2} \leq C \bigg(\Big\| \sum_{j,k} R^\star_{j,k,p} R_{j,k,p} w \Big\|_{L^2} + \|w\|^2_{L^2} \bigg).$$

Applied with $w = \varphi_0 v(2^p\cdot)$, this gives

$$\begin{aligned}\|R_{j,k}R_{j',k'}(\varphi_p v)\|_{L^2} &\leq C2^{pd}\Big(\Big\|\sum_{j,k} R^\star_{j,k,p}R_{j,k,p}\varphi_0 v(2^p\cdot)\Big\|_{L^2} + \|\varphi_0 v(2^p\cdot)\|^2_{L^2}\Big)\\ &\leq C\left(\|\Delta_\Sigma(\varphi_p v)\|_{L^2} + \|\varphi_p v\|_{L^2}\right). \end{aligned}\tag{3.32}$$

Then (3.31) implies that

$$\|\Delta_\Sigma(\varphi_p v) - \varphi_p\Delta_\Sigma v\|_{L^2} \leq Cc_p\|\Delta_\Sigma v\|^{\frac{1}{2}}_{L^2}\|v\|^{\frac{1}{2}}_{L^2} \quad \text{with} \quad \sum_{p=0}^\infty c_p^2 = 1.$$

Thus, by using (3.30) and (3.32) we infer that

$$\|\varphi_p R_{j,k}R_{j',k'}v\|_{L^2} \leq Cc_p(\|\Delta_\Sigma v\|_{L^2} + \|v\|_{L^2}) \quad \text{with} \quad \sum_{p=0}^\infty c_p^2 = 1.$$

This proves (3.28) and thus Proposition 3.2. ∎

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