Ultra-analytic effect of Cauchy problem for a class of kinetic equations

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Abstract

The smoothing effect of the Cauchy problem for a class of kinetic equations is studied. We firstly consider the spatially homogeneous non linear Landau equation with Maxwellian molecules and inhomogeneous linear Fokker-Planck equation to show the ultra-analytic effects of the Cauchy problem. Those smoothing effect results are optimal and similar to heat equation. In the second part, we study a model of spatially inhomogeneous linear Landau equation with Maxwellian molecules, and show the analytic effect of the Cauchy problem.

Key words: Landau equation, Fokker-Planck equation, ultra-analytic effect of Cauchy problem.

AMS Classification: 35A05, 35B65, 35D10, 35H20, 76P05, 84C40

1. Introduction

It is well known that the Cauchy problem of heat equation possesses the ultra-analytic effect phenomenon, namely, if $u(t,x)$ is the solution of the following Cauchy problem :

$$\begin{cases} \partial_t u - \Delta_x u = 0, & x \in \mathbb{R}^d; \quad t > 0 \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d), \end{cases}$$

then under the uniqueness hypothesis, the solution $u(t,\,\cdot\,) = e^{t\Delta_x} u_0$ is an ultra-analytic function for any $t > 0$. We give now the definition of function spaces $\mathcal{A}^s(\Omega)$ where Ω is an open subset of $\mathbb{R}^d$.

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Definition 1.1. *For $0 < s < +\infty$, we say that $f \in \mathcal{A}^s(\Omega)$, if $f \in C^\infty(\Omega)$, and there exists $C > 0, N_0 > 0$ such that*

$$\|\partial^\alpha f\|_{L^2(\Omega)} \leq C^{|\alpha|+1}(\alpha!)^s, \qquad \forall\, \alpha \in \mathbb{N}^d,\ |\alpha| \geq N_0.$$

If the boundary of Ω is smooth, by using Sobolev embedding theorem, we have the same type estimate with L^2 norm replaced by any L^p norm for $2 < p \leq +\infty$. On the whole space $\Omega = \mathbb{R}^d$, it is also equivalent to

$$e^{c_0(-\Delta)^{\frac{1}{2s}}}(\partial^{\beta_0} f) \in L^2(\mathbb{R}^d)$$

for some $c_0 > 0$ and $\beta_0 \in \mathbb{N}^d$, where $e^{c_0(-\Delta)^{\frac{1}{2s}}}$ is the Fourier multiplier defined by

$$e^{c_0(-\Delta)^{\frac{1}{2s}}} u(x) = \mathcal{F}^{-1}\Big(e^{c_0|\xi|^{\frac{1}{s}}}\hat{u}(\xi)\Big).$$

If $s = 1$, it is usual analytic function. If $s > 1$, it is Gevrey class function. For $0 < s < 1$, it is called ultra-analytic function. Notice that all polynomial functions are ultra-analytic for any $s > 0$.

It is obvious that if $u_0 \in L^2(\mathbb{R}^d)$ then, for any $t > 0$ and any $k \in \mathbb{N}$, we have $u(t,\cdot) = e^{-t(-\Delta_x)^k}u_0 \in \mathcal{A}^{\frac{1}{2k}}(\mathbb{R}^d)$, namely, there exists $C > 0$ such that for any $m \in \mathbb{N}$,

$$\begin{aligned}\|(t^m\partial_x^{2km})u(t,\cdot)\|_{L^2(\mathbb{R}^d)} &\leq C^{k\,m}\|(t(-\Delta_x)^k)^m u(t,\cdot)\|_{L^2(\mathbb{R}^d)}\\ &\leq \|u_0\|_{L^2(\mathbb{R}^d)}C^{k\,m}m\,! \leq \tilde{C}^{2k\,m+1}\big((2km)\,!\big)^{\frac{1}{2k}}\,,\end{aligned}$$

where $\partial_x^{2km} = \sum_{|\alpha|=2km,\alpha\in\mathbb{N}^d}\partial_x^\alpha$. We say that the diffusion operators $(-\Delta_x)^k$ possess the ultra-analytic effect property if $k > 1/2$, the analytic effect property if $k = 1/2$ and the Gevrey effect property if $0 < k < 1/2$.

We study the Cauchy problem for spatially homogeneous Landau equation

$$\begin{cases} f_t = Q(f,\,f) \equiv \nabla_v\big(\bar{a}(f)\cdot\nabla_v f - \bar{b}(f)f\big), & v \in \mathbb{R}^d,\ t > 0,\\ f|_{t=0} = f_0, \end{cases} \tag{1.1}$$

where $\bar{a}(f) = (\bar{a}_{ij}(f))$ and $\bar{b}(f) = (\bar{b}_1(f), \cdots, \bar{b}_d(f)\,)$ are defined as follows (convolution is w. r. t. the variable $v \in \mathbb{R}^d$)

$$\bar{a}_{ij}(f) = a_{ij}\star f, \quad \bar{b}_j(f) = \sum_{i=1}^d\big(\partial_{v_i}a_{ij}\big)\star f\,, \quad i,j = 1,\cdots,d,$$

with

$$a_{ij}(v) = \left(\delta_{ij} - \frac{v_iv_j}{|v|^2}\right)|v|^{\gamma+2}, \qquad \gamma \in [-3,1].$$

We consider hereafter only the Maxwellian molecule case which corresponds to $\gamma = 0$. We introduce also the notation, for $l \in \mathbb{R}$, $L^p_l(\mathbb{R}^d) = \{f;\,(1+|v|^2)^{l/2}f \in L^p(\mathbb{R}^d)\}$ is the weighted function space.

We prove the following ultra-analytic effect results for the non linear Cauchy problem (1.1).

Theorem 1.1. *Let $f_0 \in L^2(\mathbb{R}^d)\bigcap L^1_2(\mathbb{R}^d)$ and $0 < T \le +\infty$. If $f(t,x) > 0$ and $f \in L^\infty(]0,T[;\, L^2(\mathbb{R}^d)\bigcap L^1_2(\mathbb{R}^d))$ is a weak solution of the Cauchy problem (1.1), then for any $0 < t < T$, we have*

$$f(t,\,\cdot\,) \in \mathcal{A}^{1/2}(\mathbb{R}^d),$$

and moreover, for any $0 < T_0 < T$, there exists $c_0 > 0$ such that for any $0 < t \le T_0$

$$\left\| e^{-c_0 t \triangle_v} f(t,\,\cdot\,)\right\|_{L^2(\mathbb{R}^d)} \le e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^d)}. \tag{1.2}$$

In [17], they proved the Gevrey regularity effect of the Cauchy problem for linear spatially homogeneous non cut-off Boltzmann equation. By a careful revision for the proof of Theorem 1.2 of [17], one can also prove that the solution of the Cauchy problem (1.10) in [17] belongs to $\mathcal{A}^{\frac{1}{2\alpha}}(\mathbb{R}^d)$ for any $t > 0$, where $0 < \alpha < 1$ is the order of singularity of collision kernel of Boltzmann operator. Hence, if $\alpha \ge 1/2$, there is also the ultra-analytic effect phenomenon. Now the above Theorem 1.1 shows that, for Landau equation, the ultra-analytic effect phenomenon holds in non linear case, which is an optimal regularity result.

The ultra-analytic effect property is also true for the Cauchy problem of the following generalized Kolmogorov operators

$$\begin{cases} \partial_t u + v\,\cdot\,\nabla_x u + (-\Delta_v)^\alpha\, u = 0, \quad (x,v) \in \mathbb{R}^{2d};\quad t > 0 \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^{2d}), \end{cases}$$

where $0 < \alpha < \infty$, and the classical Kolmogorov operators is corresponding to $\alpha = 1$. By Fourier transformation, the explicit solution of above Cauchy problem is given by

$$\hat{u}(t,\eta,\xi) = e^{-\int_0^t |\xi+s\eta|^{2\alpha}ds}\hat{u}_0(\eta,\xi+t\eta).$$

Since there exits $c_\alpha > 0$ (see Lemma 3.1 below) such that

$$c_\alpha\left(t|\xi|^{2\alpha} + t^{2\alpha+1}|\eta|^{2\alpha}\right) \le \int_0^t |\xi+s\eta|^{2\alpha}ds, \tag{1.3}$$

we have

$$e^{c_\alpha(t(-\Delta_v)^\alpha + t^{2\alpha+1}(-\Delta_x)^\alpha)}u(t,\,\cdot,\,\cdot\,) \in L^2(\mathbb{R}^{2d}),$$

i. e. $u(t,\,\cdot,\,\cdot\,) \in \mathcal{A}^{1/(2\alpha)}(\mathbb{R}^{2d})$ for any $t > 0$.

Notice that this ultra-analytic (if $\alpha > 1/2$) effect phenomenon is similar to heat equations of (x,v) variables. That is, this means $v\,\cdot\,\nabla_x + (-\Delta_v)^\alpha$ is equivalent to $(-\Delta_x)^\alpha + (-\Delta_v)^\alpha$ by time evolution in "some sense", though the equation is only transport for x variable.

We consider now a more complicate equation, the Cauchy problem for linear Fokker-Planck equation :

$$\begin{cases} f_t + v\cdot\nabla_x f = \nabla_v\,\cdot\,\big(\nabla_v f + vf\big), \;\; (x,v) \in \mathbb{R}^{2d},\;\; t > 0\,; \\ f|_{t=0} = f_0\,. \end{cases} \tag{1.4}$$

This equation is a natural generalization of classical Kolmogorov equation, and a simplified model of inhomogeneous Landau equation (see [20]). The local property of this equation is the same as classical Kolmogorov equation since the add terms $\nabla_v \cdot (vf)$ is a first order term, but for the studies of kinetic equation, v is velocity variable, and hence it is in whole space $\mathbb{R}^d_v$. Then there occurs additional difficulty for analysis of this equation.

The definition of weak solution in the function space $L^\infty(]0,T[;L^2(\mathbb{R}^{2d}_{x,v}))\cap L^1_1(\mathbb{R}^{2d}_{x,v}))$ for the Cauchy problem is standard in the distribution sense, where for $1\le p<+\infty, l\in\mathbb{R}$

$$L^p_l(\mathbb{R}^{2d}_{x,v})=\Big\{f\in\mathcal{S}'(\mathbb{R}^{2d});(1+|v|^2)^{l/2}f\in L^p(\mathbb{R}^{2d}_{x,v})\Big\}.$$

The existence of weak solution is similar to full Landau equation (see [1, 13]). We get also the following ultra-analytic effect result.

Theorem 1.2. *Let $f_0\in L^2(\mathbb{R}^{2d}_{x,v})\cap L^1_1(\mathbb{R}^{2d}_{x,v})$,$0<T\le+\infty$. Assume that $f\in L^\infty(]0,T[$-$;L^2(\mathbb{R}^{2d}_{x,v})\cap L^1_1(\mathbb{R}^{2d}_{x,v}))$ is a weak solution of the Cauchy problem (1.4). Then, for any $0<t<T$, we have*

$$f(t,\cdot,\cdot)\in\mathcal{A}^{1/2}(\mathbb{R}^{2d}).$$

Furthermore, for any $0<T_0<T$ there exists $c_0>0$ such that for any $0<t\le T_0$, we have

$$\Big\|e^{-c_0(t\triangle_v+t^3\triangle_x)}f(t,\cdot,\cdot)\Big\|_{L^2(\mathbb{R}^{2d})}\le e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^{2d})}. \tag{1.5}$$

Remark 1.1. *The ultra-analyticity results of above two theorems are optimal for the smoothness properties of solutions. From these results, we obtain a good understanding for the hypoellipticity of kinetic equations (see [11, 14]), and also the relationship, established by Villani [19] and Desvilletes-Villani [10], between the nonlinear Landau equation (with Maxwellian molecules) and the linear Fokker-Planck equation.*

We consider now the spatially inhomogeneous Landau equation

$$\begin{cases} f_t+v\cdot\nabla_x f=Q(f,f), & (x,v)\in\mathbb{R}^{2d},\quad t>0\,;\\ f|_{t=0}=f_0(x,v). \end{cases} \tag{1.6}$$

The problem is now much more complicate since the solution f is the function of (t,x,v) variables. We consider it here only in the linearized framework around the normalized Maxwellian distribution

$$\mu(v)=(2\pi)^{-\frac{d}{2}}e^{-\frac{|v|^2}{2}},$$

which is the equilibrium state because $Q(\mu,\mu)=0$. Setting $f=\mu+g$, we consider the diffusion part of linear Landau collision operators,

$$Q(\mu,\ g)=\nabla_v\Big(\bar a(\mu)\cdot\nabla_v g-\bar b(\mu)g\Big)$$

where

$$\bar a_{ij}(\mu)=a_{ij}\star\mu=\delta_{ij}(|v|^2+1)-v_iv_j,$$

$$\bar b_j(\mu)=\sum_{i=1}^d\big(\partial_{v_i}a_{ij}\big)\star\mu=-v_j,\quad i,\,j=1,\cdots,d.$$

In particular, it follows that

$$\sum_{ij=1}^{d} \bar{a}_{ij}(\mu)\xi_i\xi_j \geq |\xi|^2, \quad \text{for all } (v,\, \xi) \in \mathbb{R}^{2d}. \tag{1.7}$$

We then consider the following Cauchy problem

$$\begin{cases} g_t + v \cdot \nabla_x g = \nabla_v \Big(\bar{a}(\mu) \cdot \nabla_v g - \bar{b}(\mu) g \Big), \quad (x,\, v) \in \mathbb{R}^{2d}, \quad t > 0\,; \\ g|_{t=0} = g_0. \end{cases} \tag{1.8}$$

We can also look this equation as a linear model of spatially inhomogeneous Landau equation, which is much more complicate than linear Fokker-Planck equation (1.4), since the coefficients of diffusion part are now variables. The existence and C^∞ regularity of weak solution for the Cauchy problem have been considered in [1]. We prove now the following;

Theorem 1.3. *Let $g_0 \in L^2(\mathbb{R}^{2d}_{x,\,v}) \cap L^1_2(\mathbb{R}^{2d}_{x,\,v})$, $0 < T \leq +\infty$. Assume that $g \in L^\infty(]0,\, T[$-$;\ L^2(\mathbb{R}^{2d}_{x,\,v}) \cap L^1_2(\mathbb{R}^{2d}_{x,\,v}))$ is a weak solution of the Cauchy problem (1.8). Then, for any $0 < t < T$, we have*

$$g(t,\, \cdot,\, \cdot\,) \in \mathcal{A}^1(\mathbb{R}^{2d}).$$

Furthermore, for any $0 < T_0 < T$ there exist $C,\ c > 0$ such that for any $0 < t \leq T_0$, we have

$$\Big\| e^{c(t(-\triangle_v)^{1/2} + t^2(-\triangle_x)^{1/2})} g(t,\, \cdot,\, \cdot\,) \Big\|_{L^2(\mathbb{R}^{2d})} \leq e^{C\,t}\, \|g_0\|_{L^2(\mathbb{R}^{2d})}. \tag{1.9}$$

In this theorem, we only consider the analytic effect result for the Cauchy problem (1.8), neglecting the symmetric term $Q(g, \mu)$ in the linearized operators of Landau collision operator (cf.,(1.15) of [1]) because of the technical difficulty, see the remark in the end of section 4.

There have been many results about the regularity of solutions for Boltzmann equation without angular cut-off and Landau equation, see [1, 2, 3, 6, 7, 9, 12, 15, 16] for the C^∞ smoothness results, and [4, 5, 8, 18, 17] for Gevrey regularity results for Boltzmann equation and Landau equation in both cases : the spatially homogeneous and inhomogeneous cases. As for the analytic and Gevrey regularities, we remark that the propagation of Gevrey regularities of solutions is investigated in [5] for full non-linear spatially homogeneous Landau equations, including non-Maxwellian molecule case, and the local Gevrey regularity for all variables t, x, v is considered in [4] for some semi-linear Fokker-Planck equations. Comparing those results, the ultra-analyticity for x, v variables showed in Theorem 1.1 is strong although the Maxwellian molecule cased is only treated. As a related result for spatially homogeneous Boltzmann equation in the Maxwellian molecule case, we refer [8], where the propagation of Gevrey and ultra-analytic regularity is studied uniformly in time variable t. Throughout the present paper, we focus the smoothing effect of the Cauchy problem, and the uniform smoothness estimate near to $t = 0$. Concerning further details of the analytic and Gevrey regularities of solutions for Landau equations and Boltzmann equation without angular cutoff, we refer the introduction of [5] and references therein.

2. Spatially homogeneous Landau equations

We consider the Cauchy problem (1.1) and prove Theorem 1.1 in this section. We refer to the works of C. Villani [19, 20] for the essential properties of homogeneous Landau equations. We suppose the existence of weak solution $f(t,v) > 0$ in $L^\infty(]0,T[; L^1_2(\mathbb{R}^d)$-$\bigcap L^2(\mathbb{R}^d))$. The conservation of mass, momentum and energy reads,

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,v)\begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv \equiv 0.$$

Without loss of generality, we can suppose that

$$\begin{array}{cl}
\int_{\mathbb{R}^d} f(t,v)dv = 1, & \text{unit mass} \\
\int_{\mathbb{R}^d} f(t,v)v_j dv = 0, \quad j=1,\cdots,d; & \text{zero mean velocity} \\
\int_{\mathbb{R}^d} f(t,v)|v|^2 dv = T_0, & \text{unit temperature} \\
\int_{\mathbb{R}^d} f(t,v)v_j v_k dv = T_j\,\delta_{jk}, & \sum_j^d T_j = T_0 \\
T_j = \int_{\mathbb{R}^d} f(t,v)v_j^2 dv > 0, \quad j=1,\cdots,d; & \text{directional temperatures .}
\end{array}$$

Then we have,

$$\bar{a}_{jk}(f) = \delta_{jk}(|v|^2 + T_0 - T_j) - v_j v_k \ ; \tag{2.1}$$

$$\bar{b}_j(f) = -v_j \ ; \tag{2.2}$$

$$\sum_{j,k}^d \bar{a}_{jk}(f)\xi_j\xi_k \geq C_1|\xi|^2 \ , \ \ \forall\ (v,\xi) \in \mathbb{R}^{2d}\,. \tag{2.3}$$

where $C_1 = \min_{1\leq j\leq d}\{T_0 - T_j\} > 0$.

Now for $N > \frac{d}{4} + 1$ and $0 < \delta < 1/N$, $c_0 > 0$, $t > 0$, set

$$G_\delta(t,|\xi|) = \frac{e^{c_0 t|\xi|^2}}{\left(1+\delta e^{c_0 t|\xi|^2}\right)(1+\delta c_0 t|\xi|^2)^N}.$$

Since $G_\delta(t,\,\cdot\,) \in L^\infty(\mathbb{R}^d)$, we can use it as Fourier multiplier, denoted by

$$G_\delta(t,D_v)f(t,v) = \mathcal{F}^{-1}\Big(G_\delta(t,|\xi|)\hat{f}(t,\xi)\Big).$$

Then, for any $t > 0$,

$$G_\delta(t) = G_\delta(t,D_v) : L^2(\mathbb{R}^d) \ \to \ H^{2N}(\mathbb{R}^d) \subset C^2_b(\mathbb{R}^d).$$

The object of this section is to prove the uniform bound (with respect to $\delta > 0$) of

$$\|G_\delta(t,D_v)f(t,\,\cdot\,)\|_{L^2(\mathbb{R}^d)}.$$

Since $f(t,\,\cdot\,) \in L^2(\mathbb{R}^d)\bigcap L^1_2(\mathbb{R}^d)$ is a weak solution, we can take

$$G_\delta(t)^2 f(t,\,\cdot\,) = G_\delta(t,D_v)^2 f(t,\,\cdot\,) \in H^{2N}(\mathbb{R}^d),$$

as test function in the equation of (1.1), whence we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|G_\delta(t)f(t,\,\cdot\,)\|^2_{L^2(\mathbb{R}^d)} + \sum_{j,k=1}^d \int_{\mathbb{R}^d} \bar a_{j\,k}(f)\Big(\partial_{v_j} G_\delta(t) f(t,v)\Big) \\
&\qquad\qquad\qquad\qquad \times \overline{\Big(\partial_{v_k} G_\delta(t) f(t,v)\Big)} dv \\
= \;& \frac{1}{2}\Big(\Big(\partial_t G_\delta(t)\Big) f, G_\delta(t) f\Big)_{L^2(\mathbb{R}^d)} + \sum_{j=1}^d \int_{\mathbb{R}^d} \Big(\partial_{v_j}\big(v_j f(t,v)\big)\Big)\, \overline{G_\delta(t)^2 f(t,v)}\, dv \\
&+ \sum_{j,k=1}^d \int_{\mathbb{R}^d} \Big\{\bar a_{jk}(f)\Big(G_\delta(t)\partial_{v_j} f(t,v)\Big) - G_\delta(t)\Big(\bar a_{jk}(f)\partial_{v_j} f(t,v)\Big)\Big\} \\
&\qquad\qquad\qquad\qquad \times \overline{\Big(\partial_{v_k} G_\delta(t) f(t,v)\Big)}\; dv.
\end{aligned}
$$

To estimate the terms in the above equality, we prove the following two propositions.

Proposition 2.1. *We have*

$$
\begin{aligned}
C_1\|\nabla_v G_\delta(t) f(t)\|^2_{L^2(\mathbb{R}^d)} \le \sum_{j,k=1}^d \int_{\mathbb{R}^d} \bar a_{jk}(f)\Big(\partial_{v_j} G_\delta(t, D_v) f(t,v)\Big) \\
\times \overline{\Big(\partial_{v_k} G_\delta(t, D_v) f(t,v)\Big)}\; dv. \qquad (2.4)
\end{aligned}
$$

$$
\left|\Big(\Big(\partial_t G_\delta(t)\Big) f,\; G_\delta(t) f\Big)_{L^2}\right| \le c_0\|\nabla_v G_\delta(t) f(t)\|^2_{L^2}. \qquad (2.5)
$$

$$
\begin{aligned}
\mathrm{Re}\sum_{j=1}^d \int_{\mathbb{R}^d} \Big(\partial_{v_j}\big(v_j f(t,v)\big)\Big)\, \overline{G_\delta(t)^2 f(t,v)}\, dv \\
\le \frac{d}{2}\|G_\delta(t) f(t)\|^2_{L^2} + 2c_0 t\|\nabla_v G_\delta(t) f(t)\|^2_{L^2}. \qquad (2.6)
\end{aligned}
$$

Proof : The estimate (2.4) is exactly the elliptic condition (2.3). By using the Fourier transformation, (2.5) is deduced from the following calculus

$$
\partial_t G_\delta(t,|\xi|) = c_0|\xi|^2 G_\delta(t,|\xi|)\left(\frac{1}{1+\delta e^{c_0 t|\xi|^2}} - \frac{N\delta}{1+\delta c_0 t|\xi|^2}\right) = c_0|\xi|^2 G_\delta(t,|\xi|)\, J_{N,\delta}
$$

where

$$
|J_{N,\delta}| = \left|\frac{1}{1+\delta e^{c_0 t|\xi|^2}} - \frac{N\delta}{1+\delta c_0 t|\xi|^2}\right| \le 1.
$$

To treat (2.6), we use

$$
\partial_{\xi_j} G_\delta(t,|\xi|) = 2c_0 t\xi_j G_\delta(t,|\xi|)\, J_{N,\delta}. \qquad (2.7)
$$

Then, we have

$$\begin{aligned}
&\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\Big(\partial_{v_j}\big(v_j f(t,v)\big)\Big)\,\overline{G_\delta(t,D_v)^2 f(t,v)}\,dv\\
=\;& -\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} v_j G_\delta(t,D_v)f(t,v)\overline{\Big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\Big)}\,dv\\
&-\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\Big([G_\delta(t,D_v),v_j]f(t,v)\Big)\overline{\Big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\Big)}\,dv\\
=\;& \frac{d}{2}\|G_\delta(t)f(t,\,\cdot)\|^2_{L^2(\mathbb{R}^d)}-\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\Big([G_\delta(t,D_v),v_j]f(t,v)\Big)\\
&\qquad\qquad\times\overline{\Big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\Big)}\,dv.
\end{aligned}$$

Using Fourier transformation and (2.7), we have that for $t>0$,

$$\begin{aligned}
&-\sum_{j=1}^{d}\int_{\mathbb{R}^3}\Big([G_\delta(t,D_v),v_j]f(t,v)\Big)\overline{\Big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\Big)}dv\\
=\;& -\sum_{j=1}^{d}\int_{\mathbb{R}^d}\Big(G_\delta(t,D_v)v_jf(t,v)-v_jG_\delta(t,D_v)f(t,v)\Big)\overline{\Big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\Big)}dv\\
=\;& \sum_{j=1}^{d}\int_{\mathbb{R}^d}\Big\{i\partial_{\xi_j}\big(G_\delta(t,|\xi|)\hat f(t,\xi)\Big)-G_\delta(t,|\xi|)\big(i\partial_{\xi_j}\hat f(t,\xi)\big)\Big\}G_\delta(t,|\xi|)\;\overline{i\xi_j\hat f(t,\xi)}d\xi\\
=\;& \sum_{j=1}^{d}\int_{\mathbb{R}^3}\Big(\partial_{\xi_j}G_\delta(t,|\xi|)\Big)\hat f(t,\xi)\xi_jG_\delta(t,|\xi|)\overline{\hat f(t,\xi)}d\xi\\
=\;& 2c_0t\int_{\mathbb{R}^d}|\xi|^2|G_\delta(t,|\xi|)\hat f(t,\xi)|^2J_{N,\delta}\,d\xi\le 2c_0t\int_{\mathbb{R}^d}|\xi|^2|G_\delta(t,|\xi|)\hat f(t,\xi)|^2d\xi\,,
\end{aligned}$$

which give (2.6). The proof of Proposition 2.1 is now complete.

For the commutator term, the special structure of the operator implies

Proposition 2.2.

$$\begin{aligned}
&\sum_{j,k=1}^{d}\int_{\mathbb{R}^d}\Big\{\bar a_{jk}(f)\Big(G_\delta(t,D_v)\partial_{v_j}f(t,v)\Big)-G_\delta(t,D_v)\Big(\bar a_{jk}(f)\partial_{v_j}f(t,v)\Big)\Big\}\\
&\qquad\qquad\times\overline{\Big(\partial_{v_k}G_\delta(t,D_v)f(t,v)\Big)}\,dv=0.
\end{aligned}$$

Proof : We introduce now polar coordinates on $\mathbb{R}^d_\xi$ by setting $r=|\xi|$ and $\omega=\xi/|\xi|\in\mathbb{S}^{d-1}$. Note that $\partial/\partial\xi_j=\omega_j\partial/\partial r+r^{-1}\Omega_j$ where Ω_j is a vector field on $\mathbb{S}^{d-1}$, and (see

[14], Proposition 14.7.1)

$$\sum_{j=1}^{d} \omega_j \Omega_j = 0, \ \sum_{j=1}^{d} \Omega_j \omega_j = d-1 \ . \tag{2.8}$$

By using Fourier transformation, we have

$$\begin{aligned} &- \sum_{j,k=1}^{d} \int_{\mathbb{R}^d} \Big\{ \bar{a}_{jk}(f) \Big(G_\delta(t, D_v) \partial_{v_j} f(t,v) \Big) - G_\delta(t, D_v) \Big(\bar{a}_{jk}(f) \partial_{v_j} f(t,v) \Big) \Big\} \\ &\qquad\qquad\qquad\qquad \times \overline{\Big(\partial_{v_k} G_\delta(t, D_v) f(t,v) \Big)}\, dv \\ = \ & \int_{\mathbb{R}^d} \Big\{ \sum_{j,k=1}^{d} \xi_k \Big[\Big(\delta_{jk} \Delta_\xi - \partial_{\xi_k} \partial_{\xi_j} \Big) \, , \, G_\delta(t, |\xi|) \Big] \, \xi_j \hat{f}(t, \xi)) \Big\} \times G_\delta(t, |\xi|) \overline{\hat{f}(t, \xi)} d\xi. \end{aligned}$$

Noting, in polar coordinates on $\mathbb{R}^d_\xi$,

$$\Delta_\xi = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{j=1}^{d} \Omega_j^2 \, ,$$

we have, denoting by $\tilde{G}(r^2) = G_\delta(t,\, r)$,

$$\begin{aligned} &\sum_{j,k=1}^{d} \omega_k \Big[\Big(\delta_{jk} \Big\{ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} \Big\} \\ &\qquad\qquad - \Big\{ (\omega_k \partial / \partial r + r^{-1} \Omega_k)(\omega_j \partial / \partial r + r^{-1} \Omega_j) \Big\} \Big) \, , \, \tilde{G}(r^2) \Big] \, \omega_j \\ = \ & \Big[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} \, , \, \tilde{G}(r^2) \Big] \\ &\qquad - \Big[\Big(\sum_{k=1}^{d} (\omega_k^2 \partial / \partial r + r^{-1} \omega_k \Omega_k) \sum_{j=1}^{d} (\omega_j^2 \partial / \partial r + r^{-1} \Omega_j \omega_j) \Big) \, , \, \tilde{G}(r^2) \Big] \\ = \ & \Big[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} \, , \, \tilde{G}(r^2) \Big] - \Big[\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \frac{d-1}{r} \, , \, \tilde{G}(r^2) \Big] = 0, \end{aligned}$$

where we have used (2.8). Then we finish the proof of Proposition 2.2.

Remark 2.1. *In the above proof of Proposition 2.2, we have used the polar coordinates in the dual variable of v, which is essentially related to a form of the Landau operator with Maxwellian molecules. We notice that the same relation (in v variable) was described by Villani [19] and Desvillettes-Villani [10].*

End of proof of Theorem 1.1 :

From Propositions 2.1 and 2.2, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| G_\delta(t) f(t, \cdot) \|^2_{L^2(\mathbb{R}^d)} + (C_1 - \frac{1}{2} c_0 - 2c_0 t) \| \nabla_v G_\delta(t) f(t, \cdot) \|^2_{L^2(\mathbb{R}^d)} \\ \leq \frac{d}{2} \| G_\delta(t) f(t, \cdot) \|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

For any $0 < T_0 < T$, choose c_0 small enough such that $C_1 - \frac{1}{2}c_0 - 2c_0T_0 \geq 0$. Then we get

$$\frac{d}{dt}\|G_\delta(t)f(t,\,\cdot\,)\|_{L^2(\mathbb{R}^d)} \leq \frac{d}{2}\|G_\delta(t)f(t,\,\cdot\,)\|_{L^2(\mathbb{R}^d)}. \tag{2.9}$$

Integrating the inequality (2.9) on $]0,\,t[$, we obtain

$$\|G_\delta(t)f(t,\,\cdot\,)\|_{L^2(\mathbb{R}^d)} \leq e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^d)}. \tag{2.10}$$

Take limit $\delta \to 0$ in (2.10). Then we get

$$\|e^{-c_0 t \triangle_v} f(t,\,\cdot\,)\|_{L^2(\mathbb{R}^d)} \leq e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^d)} \tag{2.11}$$

for any $0 < t \leq T_0$. We have now proved $f(t,\,\cdot\,) \in \mathcal{A}^{1/2}(\mathbb{R}^d)$ and Theorem 1.1.

3. Linear Fokker-Planck equations

In the paper [19], there is an exact solution for spatially homogeneous linear Fokker-Planck equation. In the inhomogeneous case we can also obtain an exact solution of the Cauchy problem (1.4). Denote by

$$\hat{f}(t,\eta,\xi) = \mathcal{F}_{x,v}(f(t,x,v)),$$

the partial Fourier transformation of f with respect to (x,v) variable. Then, by Fourier transformation for (x,v) variables, the linear Fokker-Planck equation (1.4) becomes

$$\begin{cases} \frac{\partial}{\partial t}\hat{f}(t,\eta,\xi) - \eta\,\cdot\,\nabla_\xi \hat{f}(t,\eta,\xi) + \xi\,\cdot\,\nabla_\xi \hat{f}(t,\eta,\xi) = -|\xi|^2\hat{f}(t,\eta,\xi)\,; \\ \hat{f}|_{t=0} = \mathcal{F}(f_0)(\eta,\xi). \end{cases}$$

Therefore we obtain the exact solution

$$\hat{f}(t,\xi,\eta) = \hat{f}(0,\xi e^{-t} + \eta(1-e^{-t}),\eta)\ \exp\Big(-\int_0^t |\xi e^{\tau - t} + \eta(1-e^{\tau - t})|^2 d\tau\Big).$$

Note that

$$\begin{aligned} & \int_0^t |\xi e^{-\tau} + \eta(1-e^{-\tau})|^2 d\tau \\ = \ & \frac{1-e^{-2t}}{2}|\xi|^2 + (1-e^{-t})^2\xi\cdot\eta + \Big(t - \frac{3+e^{-2t}}{2} + 2e^{-t}\Big)|\eta|^2 \\ = \ & (X - \frac{X^2}{2})|\xi|^2 + X^2\xi\cdot\eta + (-\log(1-X) - X - \frac{X^2}{2})|\eta|^2, \end{aligned}$$

where $X = 1 - e^{-t} \sim t$. We have for $0 < K < 2/3$

$$\int_0^t |\xi e^{-\tau} + \eta(1-e^{-\tau})|^2 d\tau \geq X(1 - 1/(2K) - X/2)|\xi|^2 + (1/3 - K/2)X^3|\eta|^2.$$

Hence for $t \sim X < 2 - 1/K$, we get

$$f(t,\,\cdot,\,\cdot\,) \in \mathcal{A}^{1/2}(\mathbb{R}^{2d}),$$

so that the ultra analytic effect holds for any $t > 0$ by means of the semi-group property. But we can not get the uniform estimate (1.5).

We present now the proof of (1.5) which implies the ultra-analytic effect, by commutator estimates similarly as for homogeneous Landau equation. Setting

$$w(t,\eta,\xi)=\hat{f}(t,\eta,\xi-t\eta),$$

Then the Cauchy problem (1.4) is equivalent to

$$\begin{cases} \frac{\partial}{\partial t}w(t,\eta,\xi)=-|\xi-t\eta|^2 w(t,\eta,\xi)-(\xi-t\eta)\cdot\nabla_\xi w(t,\eta,\xi)\,; \\ w|_{t=0}=\mathcal{F}(f_0)(\eta,\xi). \end{cases} \tag{3.1}$$

Since we need to study the function $\int_0^t|\xi-s\eta|^2 ds$, we prove the following estimate.

Lemma 3.1. *For any $\alpha>0$, there exists a constant $c_\alpha>0$ such that*

$$\int_0^t|\xi-s\eta|^\alpha ds\ge c_\alpha(t|\xi|^\alpha+t^{\alpha+1}|\eta|^\alpha). \tag{3.2}$$

Remark : If $\alpha=2$, we can get above estimate by direct calculation. The following simple proof is due to Seiji Ukai.
Proof : Setting $s=t\tau$ and $\tilde{\eta}=t\eta$, we see that the estimate is equivalent to

$$\int_0^1|\xi-\tau\tilde{\eta}|^\alpha d\tau\ge c_\alpha(|\xi|^\alpha+|\tilde{\eta}|^\alpha).$$

Since this is trivial when $\tilde{\eta}=0$, we may assume $\tilde{\eta}\neq 0$. If $|\xi|<|\tilde{\eta}|$ then

$$\begin{aligned}
&\int_0^1|\xi-\tau\tilde{\eta}|^\alpha d\tau\ge|\tilde{\eta}|^\alpha\int_0^1\left|\tau-\frac{|\xi|}{|\tilde{\eta}|}\right|^\alpha d\tau\\
=\;&|\tilde{\eta}|^\alpha\left\{\int_0^{|\xi|/|\tilde{\eta}|}\Big(\frac{|\xi|}{|\tilde{\eta}|}-\tau\Big)^\alpha d\tau+\int_{|\xi|/|\tilde{\eta}|}^1\Big(\tau-\frac{|\xi|}{|\tilde{\eta}|}\Big)^\alpha d\tau\right\}\\
\ge\;&\frac{|\tilde{\eta}|^\alpha}{\alpha+1}\min_{0\le\theta\le1}(\theta^{\alpha+1}+(1-\theta)^{\alpha+1})=\frac{|\tilde{\eta}|^\alpha}{2^\alpha(\alpha+1)}\\
\ge\;&\frac{1}{2^{\alpha+1}(\alpha+1)}(|\xi|^\alpha+|\tilde{\eta}|^\alpha).
\end{aligned}$$

If $|\xi|\ge|\tilde{\eta}|$ then

$$\begin{aligned}
\int_0^1|\xi-\tau\tilde{\eta}|^\alpha d\tau\;&\ge\;|\xi|^\alpha\int_0^1\Big(1-\tau\frac{|\tilde{\eta}|}{|\xi|}\Big)^\alpha d\tau\ge|\xi|^\alpha\int_0^1\big(1-\tau\big)^\alpha d\tau\\
&=\;\frac{|\xi|^\alpha}{\alpha+1}\ge\frac{1}{2(\alpha+1)}(|\xi|^\alpha+|\tilde{\eta}|^\alpha).
\end{aligned}$$

Hence we obtain (3.2).

Set now

$$\phi(t,\eta,\xi)=c_0\left(\int_0^t|\xi-s\eta|^2 ds-\frac{c_2}{2}t^3|\eta|^2\right),$$

where $c_0 > 0$ is a small constant to choose later, and c_2 is the constant in (3.2) with $\alpha = 2$. Then (3.2) implies

$$\phi(t, \eta, \xi) \geq c_0 \frac{c_2}{2}(t|\xi|^2 + t^3|\eta|^2). \tag{3.3}$$

Let $N = (2d+1)/4$. For $0 < \delta < 1/4N^2$ and $t > 0$, set

$$G_\delta = G_\delta(t,\, \eta,\, \xi) = \frac{e^{\phi(t,\eta,\xi)}}{\big(1 + \delta e^{\phi(t,\eta,\xi)}\big)\,(1 + \delta(|\eta|^2 + |\xi|^2))^N}\,. \tag{3.4}$$

Since $G_\delta(t,\, \cdot\, \cdot) \in L^\infty(\mathbb{R}^{2d})$, we can use it as Fourier multiplier, denoted by

$$\big(G_\delta(t, D_x,\, D_v)u\big)(t,\, x,\, v) = \mathcal{F}^{-1}_{\eta,\,\xi}\Big(G_\delta(t,\, \eta,\, \xi)\hat{u}(t,\, \eta,\, \xi)\Big).$$

Lemma 3.2. *Assume that $f(t, \cdot) \in L^2(\mathbb{R}^{2d}_{x,\,v}) \cap L^1_1(\mathbb{R}^{2d}_{x,\,v})$ for any $t \in]0, T[$. Then $\nabla_\xi w(t,\, \eta,\, \xi\,) \in L^\infty(\mathbb{R}^{2d}_{\eta,\xi})$, and*

$$|\xi - t\eta| G_\delta(t, \eta,\, \xi)^2 \bar{w}(t,\, \eta,\, \xi\,),\ |\eta| G_\delta(t, \eta,\, \xi)^2 \bar{w}(t,\, \eta,\, \xi\,),\ \nabla_\xi\Big(G_\delta(t, \eta,\, \xi)^2 \bar{w}(t,\, \eta,\, \xi\,)\Big) \tag{3.5}$$

belong to $L^2(\mathbb{R}^{2d}_{\eta,\xi})$ for any $t \in]0, T[$.

Proof : Since $\partial_{\xi_j} w = -i\mathcal{F}(v_j f)$, it follows from $f \in L^1_1(\mathbb{R}^{2d}_{x,\,v})$ that $\nabla_\xi w(t,\, \eta,\, \xi\,) \in L^\infty(\mathbb{R}^{2d}_{\eta,\xi})$. Noting

$$|\xi - t\eta| G_\delta(t, \eta,\, \xi)^2, |\eta| G_\delta(t, \eta,\, \xi)^2 \in L^\infty(\mathbb{R}^{2d}_{\eta,\xi}),$$

we see that the first two terms of (3.5) are obvious. To check the last term in (3.5), note

$$\begin{aligned}\partial_{\xi_j} G_\delta(t, \eta, \xi) = 2c_0 t(\xi_j - \frac{t}{2}\eta_j) G_\delta(t, \eta, \xi) \frac{1}{\big(1 + \delta e^{\phi(t,\eta,\xi)}\big)} \\ - \frac{2N\delta\xi_j}{(1 + \delta(|\eta|^2 + |\xi|^2))} G_\delta(t, \eta, \xi).\end{aligned} \tag{3.6}$$

Then, we have

$$\begin{aligned}&\nabla_\xi\Big(G_\delta(t, \eta,\, \xi)^2 \bar{w}(t,\, \eta,\, \xi\,)\Big) = G_\delta(t, \eta,\, \xi)^2 \nabla_\xi \bar{w}(t,\, \eta,\, \xi\,) + \nabla_\xi\Big(G_\delta(t, \eta,\, \xi)^2\Big)\bar{w}(t,\, \eta,\, \xi\,) \\ &= G_\delta(t, \eta,\, \xi)^2 \nabla_\xi \bar{w}(t,\, \eta,\, \xi\,) + 4c_0 t(\xi - \frac{t}{2}\eta) \frac{1}{\big(1 + \delta e^{\phi(t,\eta,\xi)}\big)} G_\delta(t, \eta, \xi)^2 \bar{w}(t,\, \eta,\, \xi\,) \\ &- \frac{4N\delta\xi}{(1 + \delta(|\eta|^2 + |\xi|^2))} G_\delta(t, \eta, \xi)^2 \bar{w}(t,\, \eta,\, \xi\,).\end{aligned}$$

Since $G_\delta(t, \eta,\, \xi)^2 \in L^2(\mathbb{R}^{2d}_{x,\,v})$ we have

$$G_\delta(t, \eta,\, \xi)^2 \nabla_\xi \bar{w}(t,\, \eta,\, \xi\,) \in L^2(\mathbb{R}^{2d}).$$

Using

$$\left|\frac{1}{\big(1 + \delta e^{\phi(t,\eta,\xi)}\big)}\right| \leq 1, \quad \left|\frac{2N\delta\xi}{(1 + \delta(|\eta|^2 + |\xi|^2))}\right| \leq 1,$$

and

$$\Big|(\xi-\frac{t}{2}\eta)G_\delta(t,\eta,\xi)^2\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}\bar{w}(t,\,\eta,\,\xi\,)\Big|\le|\xi-\frac{t}{2}\eta|G_\delta(t,\eta,\xi)^2|\bar{w}(t,\,\eta,\,\xi\,)|$$
$$\le|\xi-t\eta|G_\delta(t,\eta,\xi)^2|\bar{w}(t,\,\eta,\,\xi\,)|+\frac{t}{2}|\eta|G_\delta(t,\eta,\xi)^2|\bar{w}(t,\,\eta,\,\xi\,)|\in L^2(\mathbb{R}^{2d}).$$

We have proved Lemma 3.2

We take now $G_\delta(t,\eta,\,\xi)^2\bar{w}(t,\,\eta,\,\xi\,)$ as test function in the equation of (3.1). Then we have

$$\frac{d}{dt}\|G_\delta(t,\cdot,\cdot)w(t,\,\cdot\,,\,\cdot\,)\|^2_{L^2(\mathbb{R}^{2d})}+2\int_{\mathbb{R}^{2d}}|(\xi-t\eta)G_\delta(t,\eta,\xi)w(t,\,\eta,\,\xi\,)|^2d\eta d\xi$$
$$=2\sum_{j=1}^d\int_{\mathbb{R}^{2d}}w(t,\eta,\xi)\overline{\Big(\partial_{\xi_j}(\xi_j-t\eta_j)G_\delta(t,\eta,\xi)^2w(t,\eta,\xi)\Big)}d\eta d\xi$$
$$+\Big(\Big(\partial_tG_\delta(t,\cdot,\,\cdot)\Big)w(t,\,\cdot\,,\,\cdot\,),G_\delta(t,\cdot,\,\cdot)w(t,\,\cdot\,,\,\cdot\,)\Big)_{L^2(\mathbb{R}^{2d})}. \tag{3.7}$$

We prove now the following;

Proposition 3.1. *We have*

$$\begin{aligned}&\Big(\Big(\partial_tG_\delta(t,\cdot,\cdot)\Big)w,G_\delta(t,\cdot,\cdot)w\Big)_{L^2(\mathbb{R}^{2d})}\\ =\;& c_0\int_{\mathbb{R}^{2d}}|(\xi-t\eta)G_\delta(t,\eta,\xi)w(t,\,\eta,\,\xi\,)|^2d\eta d\xi\\ &-\frac{3}{2}c_0c_2t^2\int_{\mathbb{R}^{2d}}|\eta|^2|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}\,d\eta d\xi.\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\mathrm{Re}\sum_{j=1}^d\int_{\mathbb{R}^{2d}}w(t,\eta,\xi)\overline{\partial_{\xi_j}\Big((\xi_j-t\eta_j)G_\delta(t,\eta,\xi)^2w(t,\eta,\xi)\Big)}d\eta d\xi\\ &\le\Big(2c_0t+\frac{c_0t^2}{3c_2}+c_0\Big)\int_{\mathbb{R}^{2d}}|(\xi-t\eta)G_\delta(t,\eta,\xi)w(t,\,\eta,\,\xi\,)|^2d\eta d\xi\\ &+\frac{d+2N^2\delta/c_0}{2}\|G_\delta(t,\cdot,\cdot)w(t,\,\cdot,\,\cdot)\|^2_{L^2(\mathbb{R}^{2d})}\\ &+\frac{3}{4}c_0c_2t^2\int_{\mathbb{R}^{2d}}|\eta|^2|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}\,d\eta d\xi\,.\end{aligned}\tag{3.9}$$

Proof of Proposition 3.1 : The estimate (3.8) is deduced from

$$\partial_tG_\delta(t,\eta,\xi)=c_0(|\xi-t\eta|^2-\frac{3}{2}c_2t^2|\eta|^2)G_\delta(t,\eta,\xi)\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}.$$

Since it follows from (3.6) that

$$\begin{aligned}
\mathcal{I} &= \mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} w(t,\eta,\xi)\overline{\partial_{\xi_j}\Big((\xi_j-t\eta_j)G_\delta(t,\eta,\xi)^2 w(t,\eta,\xi)\Big)}d\eta d\xi \\
&= \mathrm{Re}\ 2c_0 t\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}(\xi_j-t\eta_j)(\xi_j-\frac{t}{2}\eta_j)|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}\,d\eta d\xi \\
&\quad -\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\frac{2N\delta\xi_j(\xi_j-t\eta_j)}{(1+\delta(|\eta|^2+|\xi|^2))}|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\,d\eta d\xi \\
&\quad -\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}(\xi_j-t\eta_j)\Big(\partial_{\xi_j}G_\delta(t,\eta,\xi)w(t,\eta,\xi)\Big)\overline{G_\delta(t,\eta,\xi)w(t,\eta,\xi)}d\eta d\xi,
\end{aligned}$$

we get

$$\begin{aligned}
\mathcal{I} &= 2c_0 t\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}(\xi_j-t\eta_j)(\xi_j-\frac{t}{2}\eta_j)|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}\,d\eta d\xi \\
&- \sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\frac{2N\delta\xi_j(\xi_j-t\eta_j)}{(1+\delta(|\eta|^2+|\xi|^2))}|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\,d\eta d\xi \\
&\qquad\qquad\qquad +\frac{d}{2}\|G_\delta(t,\cdot,\cdot)w(t,\,\cdot,\,\cdot)\|^2_{L^2(\mathbb{R}^{2d})} \\
&= 2c_0 t\int_{\mathbb{R}^{2d}}|\xi-t\eta|^2|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}\,d\eta d\xi \\
&+ c_0t^2\int_{\mathbb{R}^{2d}}(\xi-t\eta)\,\cdot\,\eta|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}\,d\eta d\xi \\
&- \sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\frac{2N\delta\xi_j(\xi_j-t\eta_j)}{(1+\delta(|\eta|^2+|\xi|^2))}|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\,d\eta d\xi \\
&\qquad\qquad\qquad +\frac{d}{2}\|G_\delta(t,\cdot,\cdot)w(t,\,\cdot,\,\cdot)\|^2_{L^2(\mathbb{R}^{2d})}.
\end{aligned}$$

For the last term, noting

$$\sum_{j=1}^{d}\frac{2N\delta\xi_j(\xi_j-t\eta_j)}{(1+\delta(|\eta|^2+|\xi|^2))}\le\frac{(N^2/c_0)\delta^2|\xi|^2+c_0|\xi-t\eta|^2}{(1+\delta(|\eta|^2+|\xi|^2))}\le N^2\delta/c_0+c_0|\xi-t\eta|^2$$

we finally obtain

$$\begin{aligned}
\mathcal{I} \le\ & \Big(2c_0t+\frac{c_0t^2}{3c_2}+c_0\Big)\int_{\mathbb{R}^{2d}}|(\xi-t\eta)G_\delta(t,\eta,\xi)w(t,\,\eta,\,\xi\,)|^2d\eta d\xi \\
&+\frac{d+2N^2\delta/c_0}{2}\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^{2d})} \\
&+\frac{3}{4}c_0c_2t^2\int_{\mathbb{R}^{2d}}|\eta|^2|G_\delta(t,\eta,\xi)w(t,\eta,\xi)|^2\frac{1}{\big(1+\delta e^{\phi(t,\eta,\xi)}\big)}\,d\eta d\xi\,.
\end{aligned}$$

Thus we have proved Proposition 3.1.

End of proof of Theorem 1.2 :

Now the equation (3.7), the estimate (3.8) and (3.9) deduce

$$\begin{aligned}
&\frac{d}{dt}\|G_\delta(t,\cdot,\cdot)w(t,\cdot\cdot)\|^2_{L^2(\mathbb{R}^{2d})}\\
&\qquad+\Big(2-3c_0-4c_0t-\frac{2c_0t^2}{3c_2}\Big)\int_{\mathbb{R}^{2d}}|(\xi-t\eta)G_\delta(t,\eta,\xi)w(t,\,\eta,\,\xi\,)|^2d\eta d\xi\\
\leq\ &(d+2N^2\delta/c_0)\,\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^{2d})}.
\end{aligned}$$

Then for any $0<T_0<T$ choose $c_0>0$ (depends on T_0) small enough such that

$$2-3c_0-4c_0T_0-\frac{2c_0T_0^2}{3c_2}\geq 0,$$

then for any $0<t\leq T_0$,

$$\frac{d}{dt}\|G_\delta(t,\cdot,\cdot)w(t,\,\cdot,\,\cdot\,)\|_{L^2(\mathbb{R}^{2d})}\leq\frac{d+2N^2\delta/c_0}{2}\|G_\delta(t,\cdot,\cdot)w(t,\,\cdot,\,\cdot\,)\|_{L^2(\mathbb{R}^{2d})},$$

which gives

$$\|G_\delta(t,\cdot,\cdot)w(t,\,\cdot,\,\cdot\,)\|_{L^2(\mathbb{R}^{2d})}\leq e^{\frac{d+2N^2\delta/c_0}{2}t}\|f_0\|_{L^2(\mathbb{R}^{2d})}.$$

Take $\delta\to 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}}e^{c_0\int_0^t|\xi-s\eta|^2ds-c_1t^3|\eta|^2}|\hat f(t,\,\eta,\,\xi-t\eta\,)|^2d\eta d\xi\\
=\ &\int_{\mathbb{R}^{2d}}e^{c_0\int_0^t|\xi+(t-s)\eta|^2ds-c_1t^3|\eta|^2}|\hat f(t,\,\eta,\,\xi\,)|^2d\eta d\xi\leq e^{d\,t}\|f_0\|^2_{L^2(\mathbb{R}^{2d})}.
\end{aligned}$$

By using (3.3), we get finally

$$\|e^{-\tilde c_0(t\triangle_v+t^3\triangle_x)}f(t,\,\cdot,\,\cdot\,)\|_{L^2(\mathbb{R}^{2d})}\leq e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^{2d})}$$

for any $0<t\leq T_0$, where $\tilde c_0=\frac{c_0c_2}{2}>0$. This is the desired estimate (1.5), which implies

$$f(t,\,\cdot,\,\cdot\,)\in\mathcal{A}^{1/2}(\mathbb{R}^{2d}).$$

We have thus proved Theorem 1.2.

4. Linear model of inhomogeneous Landau equations

We prove now the Theorem 1.3 in this section. By the change of variables $(t,x,v)\longrightarrow(t,\,x+v\,t,\,v)$, the Cauchy problem (1.8) is reduced to

$$\left\{\begin{array}{l} f_t=(\nabla_v-t\nabla_x)\Big(\bar a(\mu)\cdot(\nabla_v-t\nabla_x)f-\bar b(\mu)f\Big),\\ f|_{t=0}=g_0(x,v),\end{array}\right. \tag{4.1}$$

where $f(t,\, x,\, v) = g(t,\, x + v\, t,\, v)$. Recall that

$$\bar{a}_{ij}(\mu) = a_{ij} \star \mu = \delta_{ij}(|v|^2 + 1) - v_i v_j \;;$$
$$\bar{b}_j(\mu) = \sum_{i=1}^{d} \big(\partial_{v_i} a_{ij}\big) \star \mu = -v_j \;; \quad i,\, j = 1, \cdots, d,$$

and

$$\sum_{ij=1}^{d} \bar{a}_{ij}(\mu)\xi_i\xi_j \geq |\xi|^2, \quad \text{for all } (v,\, \xi) \in \mathbb{R}^{2d}.$$

In view of this Cauchy problem , we set

$$\Psi(t, \eta, \xi) = c_0 \int_0^t |\xi - s\eta| ds,$$

for a sufficiently small $c_0 > 0$ which will be chosen later on. Then we can use the (3.2) with $\alpha = 1$ to estimate Ψ. Setting

$$F_\delta(t,\, \eta,\, \xi) = \frac{e^{\Psi}}{(1 + \delta e^{\Psi})(1 + \delta\Psi)^N}$$

for $N = d + 1, 0 < \delta \leq \frac{1}{N}$. If A is a first order differential operator of (t, η, ξ) variables then we have

$$AF_\delta = \left(\frac{1}{1 + \delta e^{\Psi}} - \frac{N\delta}{1 + \delta\Psi}\right)(A\Psi)F_\delta, \tag{4.2}$$

and

$$\left|\frac{1}{1 + \delta e^{\Psi}} - \frac{N\delta}{1 + \delta\Psi}\right| \leq 1.$$

Taking

$$F_\delta(t, D_x, D_v)^2 f = F_\delta(t)^2 f \in H^{2N}(\mathbb{R}^{2d})$$

as a test function in the weak solution formula of (4.1), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|F_\delta(t)f\|^2_{L^2(\mathbb{R}^{2d})} + \Big(\bar{a}(\mu)\Big((\nabla_v - t\nabla_x)F_\delta(t)f\Big),\, \Big((\nabla_v - t\nabla_x)F_\delta(t)f\Big)\Big)_{L^2(\mathbb{R}^{2d})} \\
= \quad & -\sum_{j=1}^{d} \int_{\mathbb{R}^{2d}} v_j f\, \overline{\Big((\partial_{v_j} - t\partial_{x_j})F_\delta(t)^2 f\Big)} dxdv + \frac{1}{2}\Big(\big(\partial_t F_\delta\big)f,\, F_\delta(t)f\Big)_{L^2(\mathbb{R}^{2d})} \\
& + \sum_{j,k=1}^{d} \int_{\mathbb{R}^{2d}} \Big\{\bar{a}_{jk}(\mu)\Big(F_\delta(t)(\partial_{v_j} - t\partial_{x_j})\Big) f - F_\delta(t)\Big(\bar{a}_{jk}(\mu)(\partial_{v_j} - t\partial_{x_j})f\Big)\Big\} \\
& \qquad\qquad\qquad \times \overline{\Big((\partial_{v_k} - t\partial_{x_k})F_\delta(t)f\Big)} dxdv.
\end{aligned}$$

We prove now the following results.

Proposition 4.1. *We have*

$$\|(\nabla_v - t\nabla_x)F_\delta(t)f\|^2_{L^2(\mathbb{R}^{2d})}$$
$$\leq \Big(\bar{a}(\mu)\Big((\nabla_v - t\nabla_x)F_\delta(t)f\Big),\ \Big((\nabla_v - t\nabla_x)F_\delta(t)f\Big)\Big)_{L^2(\mathbb{R}^{2d})}. \qquad (4.3)$$

$$\Big|\Big(\Big(\partial_t F_\delta(t)\Big)f,\ F_\delta(t)f\Big)_{L^2}\Big| \leq c_0\|(\nabla_v - t\nabla_x)F_\delta(t)f\|_{L^2}\|F_\delta(t)f\|_{L^2}. \qquad (4.4)$$

$$-\operatorname{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} v_j f\ \overline{\Big((\partial_{v_j} - t\partial_{x_j})F_\delta(t)^2 f\Big)} \leq \frac{d}{2}\|F_\delta(t)f\|^2_{L^2}$$
$$+c_0 t\|(\nabla_v - t\nabla_x)F_\delta f(t)\|_{L^2}\|F_\delta f(t)\|_{L^2}. \qquad (4.5)$$

Proof : The estimate (4.3) is a direct consequence of the elliptic condition (1.7). Using the Fourier transformation and noting (4.2), we see that (4.4) is derived from

$$\partial_t F_\delta(t,\eta,\xi) = \left(\frac{1}{1+\delta e^{\Psi}} - \frac{N\delta}{1+\delta\Psi}\right)(\partial_t\Psi)F_\delta\,,\quad \partial_t\Psi = c_0|\xi - t\eta|.$$

For (4.5), we have firstly

$$-\operatorname{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^6} v_j F_\delta(t)f\ \overline{\Big((\partial_{v_j} - t\partial_{x_j})F_\delta(t)f\Big)} = \frac{d}{2}\|F_\delta(t)f\|^2_{L^2}.$$

For the commutators $[v_j,\ F_\delta(t)]$, using Fourier transformation, we have that for $t > 0$ and $\hat{f} = \hat{f}(t,\eta,\xi)$

$$-\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\Big([F_\delta(t,D_x,D_v),v_j]f(t,x,v)\Big)\overline{\Big((\partial_{v_j} - t\partial_{x_j})F_\delta(t,D_x,D_v)f(t,x,v)\Big)}dxdv$$
$$= -\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\Big(F_\delta(t,D_x,D_v)v_j f(t) - v_j F_\delta(t,D_x,D_v)f(t)\Big)$$
$$\times\ \overline{\Big((\partial_{v_j} - t\partial_{x_j})F_\delta(t,D_v)f(t)\Big)}dxdv$$
$$= \sum_{j=1}^{3}\int_{\mathbb{R}^{2d}}\Big\{i\partial_{\xi_j}\big(F_\delta(t,\eta,\xi)\hat{f}(t)\big) - F_\delta(t,\eta,\xi)\big(i\partial_{\xi_j}\hat{f}(t)\big)\Big\}F_\delta(t,\eta,\xi)$$
$$\times\ \overline{i(\xi_j - t\eta_j)\hat{f}(t)}d\eta d\xi$$
$$= \sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\Big(\partial_{\xi_j}F_\delta(t,\eta,\xi)\Big)\hat{f}(t)(\xi_j - t\eta_j)F_\delta(t,\eta,\xi)\overline{\hat{f}(t)}d\eta d\xi$$
$$\leq c_0 t\int_{\mathbb{R}^{2d}}|\xi - t\eta||F_\delta(t,\eta,\xi)\hat{f}(t)|^2 d\eta d\xi \leq c_0 t\|(\nabla_v - t\nabla_x)F_\delta f(t)\|_{L^2}\|F_\delta f(t)\|_{L^2},$$

where, in view of (4.2), we have used the fact that

$$\left|\sum_{j=1}^{d}(\partial_{\xi_j}\Psi)(t,\eta,\xi)\times(\xi_j - t\eta_j)\right| \leq c_0\int_0^1\left|\sum_{j=1}^{3}\frac{\xi_j - s\eta_j}{|\xi - s\eta|}(\xi_j - t\eta_j)\right|ds \leq c_0 t|\xi - t\eta|.$$

Thus (4.5) has been proved.

For the commutator terms, we have

Proposition 4.2. *There exists a constant $C_1 > 0$ independent on $\delta > 0$ such that*

$$\Big| \sum_{j,k=1}^{d} \int_{\mathbb{R}^{2d}} \Big\{ \bar{a}_{jk}(\mu)\Big(F_\delta(t)(\partial_{v_j} - t\partial_{x_j})\Big) f - F_\delta(t)\Big(\bar{a}_{jk}(\mu)(\partial_{v_j} - t\partial_{x_j})f\Big)\Big\} \times \overline{\Big((\partial_{v_k} - t\partial_{x_k})F_\delta(t)f\Big)} \Big|$$
$$\le C_1 \left\{(c_0 t)^2 \|(\nabla_v - t\nabla_x)F_\delta(t)f\|_{L^2}^2 + \|F_\delta(t)f\|_{L^2}^2\right\}. \tag{4.6}$$

Proof : In order to prove (4.6), we introduce the polar coordinates of ξ centered at $t\eta$, that is ,

$$r = |\xi - t\eta| \quad \text{and} \quad \omega = \frac{\xi - t\eta}{|\xi - t\eta|} \in \mathbb{S}^{d-1}.$$

Note again that $\partial/\partial\xi_j = \omega_j \partial/\partial r + r^{-1}\Omega_j$ where Ω_j is a vector field on $\mathbb{S}^{d-1}$. We have again

$$\sum_{j=1}^{d} \omega_j \Omega_j = 0, \ \sum_{j=1}^{d} \Omega_j \omega_j = d-1, \ .$$

By means of Plancherel formula, we have

$$\begin{aligned}
&\sum_{j,k=1}^{d} \int_{\mathbb{R}^{2d}} \Big\{ \bar{a}_{jk}(\mu)\Big(F_\delta(t)(\partial_{v_j} - t\partial_{x_j})\Big) - F_\delta(t)\Big(\bar{a}_{jk}(\mu)(\partial_{v_j} - t\partial_{x_j})f\Big)\Big\} \times \overline{\Big((\partial_{v_k} - t\partial_{x_k})F_\delta(t)f\Big)} \\
&= -\int_{\mathbb{R}^{2d}} \Big\{ \sum_{j,k=1}^{d} (\xi_k - t\eta_k)\Big[\Big(\delta_{jk}\Delta_\xi - \partial_{\xi_k}\partial_{\xi_j}\Big), F_\delta(t,\eta,\xi)\Big] (\xi_j - t\eta_j)\hat{f}(t)\Big\} \times \overline{F_\delta(t,\eta,\xi)\hat{f}(t)} d\xi d\eta \\
&= J\,.
\end{aligned}$$

Noting again

$$\Delta_\xi = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\sum_{l=1}^{d} \Omega_l^2\,,$$

we have with $\tilde{F}_\delta(t,\eta,r,\omega) = F_\delta(t,\eta, r\cdot\omega + t\eta) = F_\delta(t,\eta,\xi)$

$$\begin{aligned}
&-\sum_{j,k=1}^{d} \omega_k \Bigg[\Big(\delta_{jk}\Big\{\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\sum_{l=1}^{d}\Omega_l^2\Big\} \\
&\qquad - \Big\{(\omega_k \frac{\partial}{\partial r} + r^{-1}\Omega_k)(\omega_j\frac{\partial}{\partial r} + r^{-1}\Omega_j)\Big\}\Big),\ \tilde{F}_\delta \Bigg] \omega_j \\
&= -\Big[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r},\ \tilde{F}_\delta\Big]
\end{aligned}$$

$$+\Big[\Big(\sum_{k=1}^{d}(\omega_k^2\frac{\partial}{\partial r}+r^{-1}\omega_k\Omega_k)\sum_{j=1}^{d}(\omega_j^2\frac{\partial}{\partial r}+r^{-1}\Omega_j\omega_j)\Big),\ \tilde{F}_\delta\Big]$$

$$-\frac{1}{r^2}\sum_{j=1}^{d}\omega_j\Big[\sum_{l=1}^{d}\Omega_l^2,\ \tilde{F}_\delta\Big]\omega_j=A_1+A_2+A_3.$$

Note again that

$$A_1+A_2=-\Big[\frac{\partial^2}{\partial r^2}+\frac{d-1}{r}\frac{\partial}{\partial r}\,,\,\tilde{F}_\delta\Big]+\Big[\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial r}\frac{d-1}{r}\,,\,\tilde{F}_\delta\Big]=0.$$

On the other hand, we have in view of (4.2)

$$\begin{aligned}A_3&=-\frac{1}{r^2}\sum_{j,l=1}^{d}\omega_j\Big(2\Omega_l[\Omega_l,\ \tilde{F}_\delta]-\Big[\Omega_l\,,\ [\Omega_l,\ \tilde{F}_\delta]\Big]\Big)\omega_j\\&=-\frac{1}{r^2}\sum_{j,l=1}^{d}\omega_j\left(2\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\tilde{F}_\delta\right.\\&\left.-\left(\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)^2+\left(\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\right)\right)\tilde{F}_\delta\right)\omega_j.\end{aligned}$$

Putting $w_j=\omega_j\tilde{F}_\delta\ w$ with $w(t,\eta,r,\omega)=\hat{f}(t,\eta,r\cdot\omega+t\eta)$, we have

$$\begin{aligned}J\ &=\mathrm{Re}\ J=\mathrm{Re}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}r^2(A_3w)\ \overline{\tilde{F}_\delta w}\ r^{d-1}drd\omega d\eta\\&=\ -\sum_{j,l=1}^{d}\mathrm{Re}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\left\{2\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)w_j\right\}\overline{w_j}\ r^{d-1}drd\omega d\eta\\&+\ \sum_{j,l=1}^{d}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\left(\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)^2\right.\\&\qquad\left.+\left(\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\right)\right)|w_j|^2r^{d-1}drd\omega d\eta\\&=\ J_1+J_2.\end{aligned}$$

Since $\Omega_l^*=-\Omega_l+(d-1)\omega_l$, the integration by parts gives

$$\begin{aligned}J_1=\ &-\textstyle\sum_{j,l=1}^{d}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\Big\{\Big(\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\Big)\\&+(d-1)\omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\Big\}|w_j|^2r^{d-1}drd\omega d\eta.\end{aligned}$$

Hence we obtain

$$J\ =\ \sum_{j,l=1}^{d}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\left\{\left(\frac{1}{1+\delta e^\Psi}-\frac{N\delta}{1+\delta\Psi}\right)^2(\Omega_l\Psi)^2\right.\tag{4.7}$$

$$-(d-1)\omega_l\left(\frac{1}{1+\delta e^{\Psi}}-\frac{N\delta}{1+\delta\Psi}\right)(\Omega_l\Psi)\Bigg\}|w_j|^2r^{d-1}drd\omega d\eta$$

$$= \int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\left\{\left(\frac{1}{1+\delta e^{\Psi}}-\frac{N\delta}{1+\delta\Psi}\right)^2\left(\sum_{l=1}^d(\Omega_l\Psi)^2\right)\right.$$

$$\left.-(d-1)\left(\frac{1}{1+\delta e^{\Psi}}-\frac{N\delta}{1+\delta\Psi}\right)\left(\sum_{l=1}^d\omega_l(\Omega_l\Psi)\right)\right\}|\tilde{F}_\delta w|^2r^{d-1}drd\omega d\eta.$$

Since there exists a constant $C_d > 0$ such that

$$|\Omega_l\Psi| = c_0r\left|\sum_{j=1}^d\int_0^t\frac{\xi_j-s\eta_j}{|\xi-s\eta|}ds(\Omega_l\omega_j)\right| \le c_0C_dtr, \tag{4.8}$$

we have

$$|J| \le C'_d\left\{(c_0t)^2\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}r^2|\tilde{F}_\delta w|^2r^{d-1}drd\omega d\eta\right.$$

$$\left.+\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}|\tilde{F}_\delta w|^2r^{d-1}drd\omega d\eta\right\},$$

which yields (4.6). The proof of Proposition 4.2 is now complete.

End of proof of Theorem 1.3 :

From Proposition 4.1 and Proposition 4.2, there exist constants C_2, $C_3 > 0$ independent of $\delta > 0$ and $t > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\|(F_\delta f)(t)\|^2_{L^2(\mathbb{R}^{2d})}+\left(\frac{1}{2}-(c_0t)^2C_2\right)\|(\nabla_v-t\nabla_x)(F_\delta f)(t)\|^2_{L^2(\mathbb{R}^{2d})}$$

$$\le C_3\|(F_\delta f)(t)\|^2_{L^2(\mathbb{R}^{2d})}.$$

So that if $\frac{1}{2}-(c_0t)^2C_2 \ge 0$, we have,

$$\frac{d}{dt}\|(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})} \le C_3\|(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})}. \tag{4.9}$$

Using the fact $(F_\delta f)(0) = \frac{1}{1+\delta}\,g_0$, we get

$$\|(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})} \le e^{C_3t}\|g_0\|_{L^2(\mathbb{R}^{2d})}.$$

Take the limit $\delta \to 0$. Then we have

$$\int_{\mathbb{R}^{2d}}e^{2\Psi(t,\eta,\xi)}|\hat{f}(t,\eta,\xi)|^2d\eta\xi \le e^{2C_3t}\|g_0\|^2_{L^2(\mathbb{R}^{2d})}. \tag{4.10}$$

On the other hand, by Lemma 3.1, there exists a $c_1 > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^{2d}}e^{2\Psi(t,\eta,\xi)}|\hat{f}(t,\eta,\xi)|^2d\eta d\xi &= \int_{\mathbb{R}^{2d}}e^{2c_0\int_0^t|\xi-s\eta|ds}|\hat{g}(t,\eta,\xi-t\eta)|^2d\eta d\xi\\ &= \int_{\mathbb{R}^{2d}}e^{2c_0\int_0^t|\xi+(t-s)\eta|ds}|\hat{g}(t,\eta,\xi)|^2d\eta d\xi\\ &\ge \int_{\mathbb{R}^{2d}}e^{2c_0c_1(t|\xi|+t^2|\eta|)}|\hat{g}(t,\eta,\xi)|^2d\eta d\xi.\end{aligned}$$

Finally, for any $0 < T_0 < T$, choosing $c_0 > 0$ small enough such that $\frac{1}{2} - (c_0 T_0)^2 C_2 \geq 0$, we have proved,

$$\int_{\mathbb{R}^{2d}} \left| e^{c_0 c_1 (t(-\triangle_v)^{1/2} + t^2(-\triangle_x)^{1/2}} g(t,\, x,\, v) \right|^2 dx\, dv \leq e^{2C_3 t} \|g_0\|^2_{L^2(\mathbb{R}^{2d})} \quad \text{for any } \ 0 < t \leq T_0\,,$$

which completes the proof of Theorem 1.3 with $C = 2C_3$ depending only on d.

Remark 4.1. *The formulas (4.7) and (4.8) show that we can not get the ultra-analytic effect of order* $1/2$ *as in Theorem 1.2. It is the same reason why we do not consider the symmetric term* $Q(g, \mu)$ *in the equation (1.8) as in [1].*

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