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REGULARITY OF SOLUTIONS TO THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF

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Abstract. Most of the work on the Boltzmann equation is based on the Grad's angular cutoff assumption. Even though the smoothing effect from the singular cross-section without the angular cutoff corresponding to the grazing collision is expected, there is no general mathematical theory especially for the spatially inhomogeneous case. As a further study on the problem in the spatially homogeneous situation, in this paper, we will prove the Gevrey smoothing property of the solutions to the Cauchy problem for Maxwellian molecules without angular cutoff by using pseudo-differential calculus. Furthermore, we apply similar analytic techniques for the Sobolev space regularity to the nonlinear equation, and prove the smoothing property of solutions for the spatially homogeneous nonlinear Boltzmann equation with the Debye-Yukawa potential.

1. **Introduction.** Among the extensive studies on the Boltzmann equation, most of them are based on the Grad's angular cutoff assumption to avoid the mathematical difficulty due to the grazing effects in the elastic collisions between particles. Recently, a lot of progress has been made on the study of the non-cutoff problems, cf. [1, 2, 9, 3, 5, 10, 12, 13, 14, 19] and references therein, which shows that the singularity of collision cross-section yields some gain of regularity on weak solutions. In some sense, this gives the hypoellipticity property of the Boltzmann operator

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without angular cutoff. However, so far the study in this direction is still not satisfactory because there is no general theory especially for the spatially inhomogeneous problems.

This paper is concerned with the smoothing effect of the singular integral kernel in the collision operator coming from the non-cutoff cross-sections in the Boltzmann equation. There are two main results in this paper. One is about the smoothing effect of the non-cutoff Debye-Yukawa potential which gives the gain of a fraction of the logarithm of Laplacian regularity. This is different from the non-cutoff inverse power laws which give the gain of a fraction of Laplacian regularity. Another problem is concerned with the Gevrey regularity of the non-cutoff inverse power laws. Even though both results are about the spatially homogeneous problem, it provides some new aspects of the regularity for the singular cross-sections.

Consider the Cauchy problem of spatially homogeneous nonlinear Boltzmann equation

$$\frac{\partial f}{\partial t} = Q(f,f),\ \ v \in \mathbb{R}^3,\ \ t > 0\,; \qquad f|_{t=0} = f_0, \tag{1.1}$$

where $f = f(t,v) \geq 0$ on $[0,\infty) \times \mathbb{R}^3_v$ represents the particle distribution function. In the following, we assume that the initial datum $f_0 \not\equiv 0$ satisfies the natural boundedness on the mass, energy and entropy, that is,

$$f_0 \geq 0, \quad \int_{\mathbb{R}^3} f_0(v)(1+|v|^2+\log(1+f_0(v)))dv < +\infty. \tag{1.2}$$

The Boltzmann quadratic operator $Q(g,f)$ is a bi-linear functional representing the change rate of the particle distribution through the elastic binary collisions which takes the form

$$Q(g,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left(v - v_*, \sigma\right)\left\{g(v'_*)f(v') - g(v_*)f(v)\right\} d\sigma dv_*\,, \tag{1.3}$$

where for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\ \ v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \tag{1.4}$$

are the relations between the post and pre collision velocities. The non-negative function $B(z,\sigma)$ called the Boltzmann collision kernel depends only on $|z|$ and the scalar product $< \frac{z}{|z|}, \sigma >$. In most of the cases, the collision kernel B can not be expressed explicitly. However, to capture its main property, it can be assumed to be in the form

$$B(|v-v_*|, \cos\theta) = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \Big\langle \frac{v-v_*}{|v-v_*|}\,,\,\sigma\Big\rangle,\ \ 0 \leq \theta \leq \frac{\pi}{2}.$$

In this paper, we consider only the mathematical Maxwellian case, that is, we take $\Phi \equiv 1$. Except for hard sphere model, the function $b(\cos(\cdot))$ has a singularity at $\theta = 0$. For example, if the inter-molecule potential satisfies the inverse-power law potential $U(\rho) = \rho^{-(\gamma-1)}, \gamma > 2$, then

$$\sin\theta\, b(\cos\theta) \ \approx\ K\theta^{-1-2\alpha} \ \text{when}\ \theta \to 0, \tag{1.5}$$

where $K > 0, 0 < \alpha = \frac{1}{\gamma-1} < 1$. The Maxwellian molecule case corresponds to $\gamma = 5$ and $\Phi = 1$. Notice that the Boltzmann collision operator is not well defined for the case when $\gamma = 2$ corresponding to the Coulomb potential. In the great majority of works on the Boltzmann equation, the angular singularity at $\theta = 0$ is removed by using the Grad's angular cutoff assumption so that B is locally integrable in σ variable.

We will first consider a family of Debye-Yukawa type potentials where the potential function is given by

$$U(\rho) = \rho^{-1} e^{-\rho^s}, \ \text{ with } s > 0. \tag{1.6}$$

In some sense, it is a model between the Coulomb potential corresponding to $s = 0$ and the potential satisfying the inverse power law. In fact, the classical Debye-Yukawa potential is the case when $s = 1$.

In the appendix, we will show that the collision cross-section of this kind of potentials has the singularity in the following form, cf. (5.2),

$$b(\cos\theta) \ \approx \ K\theta^{-2}\left(\log\theta^{-1}\right)^{\frac{2}{s}-1} \ \text{ when } \theta \to 0.$$

where $K > 0$ is constant. This singularity endows the collision operator Q with the logarithmic regularity property, see Proposition 2.1. We mention that the logarithmic regularity theory was first introduced in [15] on the hypoellipticity of the infinitely degenerate elliptic operator and was developed in [16, 17] on the logarithmic Sobolev estimates.

Suppose that there exists a weak solution to the Cauchy problem (1.1) with the following natural bound for some time $T > 0$, see the Definition 3.1 of Section 3 and also [19],

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} f(t,v)(1 + |v|^2 + \log(1 + f(t,v)))dv < +\infty. \tag{1.7}$$

In Section 3, we are going to prove the following theorem on the regularity of weak solutions.

Theorem 1.1. *Assume that the initial datum f_0 satisfies (1.2) and the collision cross-section satisfies*

$$B(|v - v_*|, \cos\theta) = b(\cos\theta) \ \approx \ K\theta^{-2}\left(\log\theta^{-1}\right)^m \ \textit{ when } \theta \to 0, \tag{1.8}$$

with $K > 0, m > 0$. Let f be a weak solution of the Cauchy problem (1.1) satisfying (1.7). Then for any $0 < t \le T$, we have $f(t,\cdot) \in H^{+\infty}(\mathbb{R}^3)$.

Remark 1.1. Note that $m > 0$ corresponds to $0 < s < 2$ in (1.6). In [2, 10], the $H^{+\infty}(\mathbb{R}^3)$ regularity of weak solutions was proved under the condition (1.5). Notice that the condition (1.8) is much weaker than (1.5) and the theorem shows that it still leads to $H^{+\infty}(\mathbb{R}^3)$ regularity on the weak solutions. Moreover, the following proof on the regularity of weak solutions is more straightforward and illustrative than the previous methods. Even though the assumption (1.8) on the cross-section is mathematical because the exact cross-section depends also on the relative velocity as given in (5.2), the following analysis reveals the smoothing effect of the singularity in the collision operator on the weak solution to the Boltzmann equation.

To have a more precise description on the regularity, in the second part of the paper, we consider the Gevrey regularity of solutions with cross-section satisfying (1.5). Notice that while local solutions having the Gevrey regularity have been constructed in [18] for initial data having higher Gevrey regularity, the result given here is concerned with the production of the Gevrey regularity for weak solutions whose initial data have no regularity.

Before stating the result, we now recall the definition of Gevrey regularity. For $s \ge 1$, $u \in G^s(\mathbb{R}^3)$ which is the Gevrey class function space with index s, if there exists $C > 0$ such that for any $k \in \mathbb{N}$,

$$\|D^k u\|_{L^2} \le C^{k+1}(k!)^s,$$

or equivalently, there exist $\varepsilon_0 > 0$ such that $e^{\varepsilon_0 \langle D\rangle^{1/s}} u \in L^2$, where $L^2 = L^2(\mathbb{R}^3)$ and

$$\langle D\rangle = (1 + |D_v|^2)^{1/2}, \qquad \|D^k u\|^2_{L^2} = \sum_{|\beta|=k} \|D^\beta u\|^2_{L^2}.$$

Note that $G^1(\mathbb{R}^3)$ is usual analytic function space. In the following discussion, we also adopt the following notations,

$$\|f\|_{L^k_\ell} = \left(\int_{\mathbb{R}^3} |f(v)|^k (1+|v|)^{k\ell} dv \right)^{\frac{1}{k}} ; \; \|f\|_{L\log L} = \int_{\mathbb{R}^3} |f(v)| \log(1+|f(v)|) dv.$$

What we are going to show in Section 4 is that the weak solutions to the linearized Boltzmann equation with the cross-section satisfying (1.5) are in the Gevrey class with index $\frac{1}{\alpha}$ for $t > 0$. For this, we first linearize the Boltzmann equation near the absolute Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}. \tag{1.9}$$

Since $Q(\mu, \mu) = 0$, we have

$$Q(\mu + g, \mu + g) = Q(\mu, g) + Q(g, \mu) + Q(g, g).$$

Set

$$Lg = Q(\mu, g) + Q(g, \mu),$$

where L is the usual linearized collision operator. Then, consider the linear Cauchy problem

$$\frac{\partial g}{\partial t} = Lg, \;\; v \in \mathbb{R}^3, \;\; t > 0\,; \qquad g|_{t=0} = g_0. \tag{1.10}$$

The definition of weak solution for this linear equation is standard which is given precisely in Section 4. The result on Gevrey regularity can be stated as follows.

Theorem 1.2. *Assume that the initial perturbation in (1.10) satisfies* $g_0 \in L^1_2(\mathbb{R}^3)$, *and* Q *is defined by the Maxwellian collision cross-section* B *satisfying (1.5) with* $0 < \alpha < 1$. *For* $T_0 > 0$, *if* $g \in L^1([0, T_0]; L^1_2(\mathbb{R}^3)) \cap L^\infty([0, T_0]; L^1(\mathbb{R}^3))$ *is a weak solution of the Cauchy problem (1.10), then* $g(t, \cdot) \in G^{1/\alpha}(\mathbb{R}^3) \cap L^1_2(\mathbb{R}^3)$ *for any* $0 < t \leq T_0$.

The assumption in the above theorem may not be strong enough for the existence of weak solutions which meet the requirement of the theorem, but such weak solutions can be constructed under some additional assumptions. In Section 4, we will prove the

Proposition 1.1. *Suppose that* $0 < \alpha < 1/2$ *and* $g_0 \in L^2_\ell$ *for some* $\ell > 5/2$. *Then, (1.10) possesses a weak solution* $g \in L^\infty(0, T_0;\; L^2_\ell(\mathbb{R}^3))$ *for any* $T_0 > 0$.

Remark 1.2. (1) Evidently, $L^2_\ell(\mathbb{R}^3) \subset L^1_2(\mathbb{R}^3)$ when $\ell > 7/2$.
(2) Even though the above results are given when the space dimension equals to three, they hold for any space dimensions with due modification.

2. **Logarithmic regularity estimate.** We prove firstly the following logarithmic regularity estimate for the collision operator.

Proposition 2.1. *Assume that the collision kernel B satisfies the assumption (1.8) and $g \geq 0$, $g \not\equiv 0$, $g \in L^1_2 \bigcap L\log L$. Then there exists a constant $C_g > 0$ depending only on B, $\|g\|_{L^1_2}$ and $\|g\|_{L\log L}$ such that for any smooth function $f \in H^2(\mathbb{R}^3)$,*

$$\| \left(\log \Lambda\right)^{\frac{m+1}{2}} f\|^2_{L^2(\mathbb{R}^3)} \leq C_g \left\{ (-Q(g,f),f)_{L^2(\mathbb{R}^3)} + \|f\|^2_{L^2(\mathbb{R}^3)} \right\}, \tag{2.1}$$

where $\Lambda = (e + |D_v|^2)^{1/2}$.

Remark 2.1. With hypothesis (1.5), we have the following sub-elliptic estimate (see [2, 7, 10])

$$\|\Lambda^\alpha f\|^2_{L^2(\mathbb{R}^3)} \leq C_g \left\{ (-Q(g,f),f)_{L^2(\mathbb{R}^3)} + \|f\|^2_{L^2(\mathbb{R}^3)} \right\}. \tag{2.2}$$

And we will use this estimate to prove the Gevrey regularity stated in the Theorem 1.2.

Proof of Proposition 2.1. For $f \in H^2(\mathbb{R}^3)$, we have

$$\begin{aligned} (-Q(g,f),f)_{L^2(\mathbb{R}^3)} &= -\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma) g(v_*) f(v)\Big(f(v') - f(v)\Big) d\sigma dv_* dv \\ &= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma) g(v_*) \Big(f(v') - f(v)\Big)^2 d\sigma dv_* dv \\ &- \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma) g(v_*) \Big(f(v')^2 - f(v)^2\Big) d\sigma dv_* dv. \end{aligned}$$

According to cancellation lemma (Corollary 2 of [1]), we have

$$\left| \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma) g(v_*) \Big(f(v')^2 - f(v)^2\Big) d\sigma dv_* dv \right| \leq C\|g\|_{L^1}\|f\|^2_{L^2}.$$

Notice that there is no weight in the norm of f in L^2 on the right hand side of the above inequality because we consider the Maxwellian molecule type of cross-sections so that it is a direct consequence of

$$\int_{-\pi/2}^{\pi/2} \sin\theta b(\cos\theta)\left(\frac{1}{\cos^3(\theta/2)} - 1\right) d\theta < \infty.$$

Now the proof of Proposition 2.1 is reduced to the proof of the following lemma. $\square$

Lemma 2.1. *There exists a constant $C_g > 0$, depending only on b, $\|g\|_{L^1_1}$ and $\|g\|_{L\log L}$ such that*

$$\| \left(\log \Lambda\right)^{\frac{m+1}{2}} f\|^2_{L^2} \leq C_g \Big\{ \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma) g(v_*) \Big(f(v') - f(v)\Big)^2 d\sigma dv_* dv + \|f\|^2_{L^2} \Big\}.$$

Proof. The proof of this lemma is similar to that of Theorem 1 in [1]. In fact, by taking the Fourier transform on the collision operator and applying the Bobylev identity, we have

$$\begin{aligned}&\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma)g(v_*)\Big(f(v)-f(v')\Big)^2 d\sigma dv_* dv\\&=(2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{\hat g(0)|\hat f(\xi)|^2+\hat g(0)|\hat f(\xi^+)|^2\\&\qquad\qquad -2Re\,\hat g(\xi^-)\hat f(\xi^+)\bar{\hat f}(\xi)\Big\}d\sigma d\xi\\&\geq\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}|\hat f(\xi)|^2\left\{\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(\hat g(0)-|\hat g(\xi^-)|)d\sigma\right\}d\xi,\end{aligned}$$

where

$$\xi^+=\frac{\xi+|\xi|\sigma}{2},\quad \xi^-=\frac{\xi-|\xi|\sigma}{2}.$$

By using the condition that $g\geq 0, g\in L^1_1\cap L\log L$ and the assumption (1.8), similar to the argument in [1], we can show that there exists a positive constant C_g depending only on $\|g\|_{L^1_1}$ and $\|g\|_{L\log L}$ such that

$$\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(\hat g(0)-|\hat g(\xi^-)|)d\sigma\geq C_g^{-1}\big(\log<\xi>\big)^{m+1}-C_g.$$

And this completes the proof of lemma 2.1. □

3. **Smoothing effect for the nonlinear Cauchy problem.**

We will give the proof of Theorem 1.1 on the smoothing effect of the collision operator for the Debye-Yukawa type potentials in this section. Before that, let us recall the definition of weak solution for the Cauchy problem (1.1), cf. [19].

Definition 3.1. *Let $f_0(v)\geq 0$ be a function defined on $\mathbb{R}^3$ with finite mass, energy and entropy. $f(t,v)$ is called a weak solution of the Cauchy problem (1.1), if it satisfies the following conditions:*

$$f(t,v)\geq 0,\quad f(t,v)\in C(\mathbb{R}^+;\mathcal{D}'(\mathbb{R}^3))\cap L^1([0,T];L^1_2(\mathbb{R}^3)),\quad f(0,v)=f_0(v);$$

$$\int_{\mathbb{R}^3} f(t,v)\psi(v)dv=\int_{\mathbb{R}^3} f_0(v)\psi(v)dv\ \ \textit{for}\ \ \psi=1,v_j,|v|^2;$$

$$f(t,v)\in L^1(\mathbb{R}^3)\log L^1(\mathbb{R}^3),\ \int_{\mathbb{R}^3} f(t,v)\log f(t,v)dv\leq\int_{\mathbb{R}^3} f_0\log f_0 dv,\ \forall t\geq 0;$$

$$\begin{aligned}\int_{\mathbb{R}^3} f(t,v)\varphi(t,v)dv-\int_{\mathbb{R}^3} f_0\varphi(0,v)dv-\int_0^t d\tau\int_{\mathbb{R}^3} f(\tau,v)\partial_\tau\varphi(\tau,v)dv\\=\int_0^t d\tau\int_{\mathbb{R}^3} Q(f,f)(\tau,v)\varphi(\tau,v)dv,\end{aligned}$$

where $\varphi(t,v)\in C^1(\mathbb{R}^+;C_0^\infty(\mathbb{R}^3))$. Here, the right hand side of the last integral given above is defined by

$$\begin{aligned}&\int_{\mathbb{R}^3} Q(f,f)(v)\varphi(v)dv\\&\quad=\frac12\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} Bf(v_*)f(v)(\varphi(v')+\varphi(v'_*)-\varphi(v)-\varphi(v_*))dvdv_*d\sigma.\end{aligned}$$

Hence, this integral is well defined for any test function $\varphi\in L^\infty([0,T];W^{2,\infty}(\mathbb{R}^3))$ (see p. 291 of [19]).

Since the existence of weak solution was already proved in [19], we will then only need to show the regularity of the weak solution. Let f be a weak solution of the Cauchy problem (1.1). For any fixed $T_0 > 0$, we know that $f(t) \in L^1(\mathbb{R}^3) \subset H^{-2}(\mathbb{R}^3)$ for all $t \in [0, T_0]$. For $t \in [0, T_0]$, $N > 0$ and $0 < \delta < 1$, set

$$M_\delta(t, \xi) = \Big(1 + |\xi|^2\Big)^{\frac{Nt-4}{2}} \times \Big(1 + \delta|\xi|^2\Big)^{-N_0},$$

with $N_0 = \frac{NT_0}{2} + 2$. Then, for any $\delta \in]0, 1[$

$$M_\delta(t, D_v)f \in L^\infty([0, T_0]; W^{2,\infty}(\mathbb{R}^3)),$$

whose norm is bounded above from $C_\delta\|f_0\|_{L^1}$.

By using Proposition 2.1, we have

$$\|\,(\log\Lambda)^{\frac{m+1}{2}}\, M_\delta(t, D_v)f\|^2_{L^2} \le C_f\left\{(-Q(f, M_\delta f), M_\delta f)_{L^2} + \|M_\delta f\|^2_{L^2}\right\}, \tag{3.1}$$

where the constant C_f is independent of $\delta \in]0, 1[$.

To apply this logarithmic regularity estimate to the nonlinear Boltzmann equation, we need to estimate the commutators of the pseudo-differential operator $M_\delta(t, D_v)$ and the nonlinear operator $Q(f, \cdot)$ which is given in the following lemma.

Lemma 3.1. *Under the hypothesis of Theorem 1.1, we have that*

$$|(Q(f, M_\delta f), M_\delta f)_{L^2} - (Q(f, f), M_\delta^2 f)_{L^2}| \le C_f\|M_\delta f\|^2_{L^2} \tag{3.2}$$

with a constant $C_f > 0$ *independent of* $0 < \delta < 1$.

Proof. By applying Proposition 2.1 to the function $M_\delta f \in H^2$, we have

$$\begin{aligned}
&(-Q(f, M_\delta f), M_\delta f)_{L^2(\mathbb{R}^3)} + O(\|M_\delta f\|^2_{L^2})\\
= \;&\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{\hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2 + \hat f(0)M_\delta^2(t,\xi^+)|\hat f(\xi^+)|^2\\
- \;&2Re\,\hat f(\xi^-)M_\delta(t,\xi^+)\hat f(\xi^+)M_\delta(t,\xi)\bar{\hat f}(\xi)\Big\}d\sigma d\xi,
\end{aligned}$$

By the Bobylev identity, we also have

$$\begin{aligned}
&\Big(Q(f,f), M_\delta^2 f\Big)_{L^2} = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma)f(v_*)f(v)\Big(M_\delta^2 f(v') - M_\delta^2 f(v)\Big)dv_* d\sigma dv\\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{\hat f(\xi^-)M_\delta^2(t,\xi)\hat f(\xi^+)\bar{\hat f}(\xi) - \hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2\Big\}\,d\sigma d\xi.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Big(Q(f,f), M_\delta^2 f\Big)_{L^2} &= -\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{\hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2\\
&\quad + \hat f(0)M_\delta^2(t,\xi^+)|\hat f(\xi^+)|^2 - 2Re\,\hat f(\xi^-)M_\delta(t,\xi^+)\hat f(\xi^+)M_\delta(t,\xi)\bar{\hat f}(\xi)\Big\}d\sigma d\xi\\
&\quad + \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{\hat f(0)M_\delta^2(t,\xi^+)|\hat f(\xi^+)|^2 - \hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2\\
&\quad + 2Re\,\hat f(\xi^-)M_\delta(t,\xi)\hat f(\xi^+)\bar{\hat f}(\xi)\big[M_\delta(t,\xi) - M_\delta(t,\xi^+)\big]\Big\}d\sigma d\xi.\\
&= \Big(Q(f, M_\delta f), M_\delta f\Big)_{L^2} + O(\|M_\delta f\|^2_{L^2})\\
&\quad + \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{\hat f(0)M_\delta^2(t,\xi^+)|\hat f(\xi^+)|^2 - \hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2\\
&\quad + 2Re\,\hat f(\xi^-)M_\delta(t,\xi)\hat f(\xi^+)\bar{\hat f}(\xi)\big[M_\delta(t,\xi) - M_\delta(t,\xi^+)\big]\Big\}d\sigma d\xi.
\end{aligned}$$

Hence, it remains to show that

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big\{\hat f(0)M_\delta^2(t,\xi^+)|\hat f(\xi^+)|^2-\hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2 d\sigma d\xi\right| \le C_f\|M_\delta f\|^2_{L^2}\,,$$

and

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big\{Re\,\hat f(\xi^-)M_\delta(t,\xi)\hat f(\xi^+)\bar{\hat f}(\xi)\big[M_\delta(t,\xi)-M_\delta(t,\xi^+)\big]\Big\}d\sigma d\xi\right| \\ \le C_f\|M_\delta f\|^2_{L^2}\,. \tag{3.3}$$

The first estimate can be obtained by using the argument for the cancellation lemma given in [1] because

$$\begin{aligned}&|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big\{\hat f(0)M_\delta^2(t,\xi^+)|\hat f(\xi^+)|^2-\hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2|\\ &=(2\pi)|\int_{\mathbb{R}^3}\int_{-\pi/2}^{\pi/2}\sin\theta\, b(\cos\theta)\hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2\big[\frac{1}{\cos^3(\theta/2)}-1\big]d\theta d\xi|\\ &\le C_0\|f\|_{L^1}\|M_\delta f\|^2_{L^2}.\end{aligned}$$

To prove the second estimate, we need to show that

$$|M_\delta(t,\xi^+)-M_\delta(t,\xi)|\le N_0 2^{(NT_0+4)/2}\sin^2\frac{\theta}{2}M_\delta(t,\xi^+). \tag{3.4}$$

For this, recall

$$\xi^+=\frac{\xi+|\xi|\sigma}{2},\ \ |\xi^+|^2=|\xi|^2\cos^2\frac{\theta}{2},\ \ \frac{\xi}{|\xi|}\cdot\sigma=\cos\theta,$$

and the collision kernel is supported in $|\theta|\le\pi/2$. Then

$$\frac{|\xi|^2}{2}\le|\xi^+|^2\le|\xi|^2,\quad |\xi|^2-|\xi^+|^2=|\xi^-|^2=\sin^2\frac{\theta}{2}|\xi|^2.$$

Denote

$$\tilde M_\delta(t,s)=(1+s)^{\frac{Nt-4}{2}}\times(1+\delta s)^{-N_0},\quad s=|\xi|^2,\quad s_+=|\xi^+|^2,$$

so that

$$M_\delta(t,\xi)=\tilde M_\delta(t,|\xi|^2).$$

Then, there exists $s^+<\tilde s<s$ such that

$$\tilde M_\delta(t,s)-\tilde M_\delta(t,s_+)=\frac{\partial\tilde M_\delta}{\partial s}(t,\tilde s)(s-s_+).$$

Note that $s-s_+=s\sin^2\frac{\theta}{2}$ and

$$\frac{\partial\tilde M_\delta}{\partial s}(t,s)=\left\{(Nt-4)\frac{1}{2(1+s)}-N_0\frac{\delta}{1+\delta s}\right\}\tilde M_\delta(t,s).$$

By using

$$\frac{s}{1+s},\ \ \frac{\delta s}{1+\delta s}\le 1,$$

and

$$\left|\frac{\tilde M_\delta(t,\tilde s)}{\tilde M_\delta(t,s_+)}\right|\le 2^{(NT_0+4)/2},$$

we have

$$|\tilde M_\delta(t,s)-\tilde M_\delta(t,s_+)|\le N_0 2^{(NT_0+4)/2}\sin^2\frac{\theta}{2}\tilde M_\delta(t,s_+),$$

which gives (3.4). Now the second estimate in (3.3) can be proved as follows,

$$\begin{aligned}
&\Big|\int b\Big\{Re\,\hat f(\xi^-)M_\delta(t,\xi)\hat f(\xi^+)\bar{\hat f}(\xi)\big[M_\delta(t,\xi)-M_\delta(t,\xi^+)\big]\Big\}d\sigma d\xi\Big|\\
&\le C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\sin^2\frac{\theta}{2}|\hat f(\xi^-)|M_\delta(t,\xi^+)|\hat f(\xi^+)|M_\delta(t,\xi)|\hat f(\xi)|d\sigma d\xi\\
&\le C\int_{\mathbb{R}^3}\int_{-\pi/2}^{\pi/2} b(\cos\theta)\sin^2\frac{\theta}{2}\sin\theta|\hat f(\xi^-)|M_\delta(t,\xi^+)|\hat f(\xi^+)|M_\delta(t,\xi)|\hat f(\xi)|d\theta d\xi\\
&\le C\|f\|_{L^1}\|M_\delta f\|^2_{L^2}.
\end{aligned}$$

□

If $M_\delta(t,D_v)$ is replaced by the differential operator D_v^k, then the commutator is given by Leibniz formula. Therefore, in some sense, this lemma is a microlocal version of the computation given in [10]. We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Firstly, we note from Definition 3.1 that any weak solution f has the following properties. That is, $M_\delta^2 f\in L^\infty([0,T_0];W^{2,\infty}(\mathbb{R}^3))$,

$$M_\delta f\in C([0,T_0];L^2(\mathbb{R}^3)),\tag{3.5}$$

and for any $t\in]0,T_0]$, we have

$$\begin{aligned}
&\frac12\int_{\mathbb{R}^3} f(t)M_\delta^2(t)f(t)dv-\frac12\int_0^t\int_{\mathbb{R}^3} f(\tau)\Big(\partial_t M_\delta^2(\tau)\Big)f(\tau)dvd\tau\\
&=\frac12\int_{\mathbb{R}^3} f_0M_\delta^2(0)f_0dv+\int_0^t\Big(Q(f,f)(\tau),M_\delta^2(\tau)f(\tau)\Big)_{L^2}d\tau.
\end{aligned}\tag{3.6}$$

The proof of (3.5) and (3.6) will be given at the end of this section.

On the other hand, it follows from (3.1) and (3.2) that

$$\|\left(\log\Lambda\right)^{\frac{m+1}{2}}M_\delta f\|^2_{L^2}\le C_f\left\{(-Q(f,f),M_\delta^2 f)_{L^2}+\|M_\delta f\|^2_{L^2}\right\}.\tag{3.7}$$

Since

$$(\partial_t M_\delta)(t,\xi)=N\log<\xi>M_\delta(t,\xi),$$

we obtain

$$\Big|\int_0^t\int_{\mathbb{R}^3} f(\tau)\Big(\partial_t M_\delta^2(\tau)\Big)f(\tau)dvd\tau\Big|\le 2N\int_0^t\|\left(\log\Lambda\right)^{\frac12}(M_\delta f)(\tau)\|^2_{L^2}d\tau.$$

This together with (3.6) and (3.7), imply

$$\begin{gathered}
\|(M_\delta f)(t)\|^2_{L^2}+\tfrac{1}{2C_f}\int_0^t\|\left(\log\Lambda\right)^{\frac{m+1}{2}}(M_\delta f)(\tau)\|^2_{L^2}d\tau\le\\
\|M_\delta(0)f_0\|^2_{L^2}+2N\int_0^t\|\left(\log\Lambda\right)^{\frac12}(M_\delta f)(\tau)\|^2_{L^2}d\tau+\int_0^t\|(M_\delta f)(\tau)\|^2_{L^2}d\tau.
\end{gathered}$$

For $m>0$, by interpolation, we have for any $\varepsilon>0$,

$$\begin{aligned}
&\|(M_\delta f)(t)\|^2_{L^2}+\Big(\frac{1}{2C_f}-\varepsilon\Big)\int_0^t\|\left(\log\Lambda\right)^{\frac{m+1}{2}}(M_\delta f)(\tau)\|^2_{L^2}d\tau\\
&\qquad\le\|M_\delta(0)f_0\|^2_{L^2}+C_{\varepsilon,N}\int_0^t\|(M_\delta f)(\tau)\|^2_{L^2}d\tau.
\end{aligned}$$

By choosing $\varepsilon = \frac{1}{4C_f} > 0$, there exists $C_{f,N} > 0$ depending only on C_f, N, T_0 and being independent of $\delta \in]0,1[$, such that for any $t \in]0, T_0]$,

$$\|M_\delta(t)f(t)\|^2_{L^2} \le \|M_\delta(0)f_0\|^2_{L^2} + C_{f,N}\int_0^t \|M_\delta(\tau)f(\tau)\|^2_{L^2}d\tau.$$

Then Gronwall inequality yields

$$\|(M_\delta f)(t)\|^2_{L^2} \le e^{C_{f,N}t}\|M_\delta(0)f_0\|^2_{L^2}.$$

Since $\|M_\delta(t)f(t)\|^2_{L^2} = \|(1-\delta\triangle)^{-N_0}f(t)\|^2_{H^{Nt-4}(\mathbb{R}^3)}$, and

$$\|M_\delta(0)f_0\|^2_{L^2} = \|(1-\delta\triangle)^{-N_0}f_0\|^2_{H^{-4}(\mathbb{R}^3)} \le \|f_0\|^2_{H^{-4}(\mathbb{R}^3)} \le C_0\|f_0\|^2_{L^1},$$

we obtain

$$\|(1-\delta\triangle)^{-N_0}f(t)\|^2_{H^{Nt-4}(\mathbb{R}^3)} \le \tilde{C}e^{C_{f,N}t}\|f_0\|^2_{L^1},$$

where the constant $\tilde{C} > 0$ is independent of δ. Finally, for any given $t > 0$, since N can be arbitrarily large, by letting $\delta \to 0$, we have

$$f(t) \in H^{+\infty}(\mathbb{R}^3).$$

And this completes the proof of Theorem 1.1. □

The rest of this section is devoted to the proof of (3.5) and (3.6). In Definition 3.1, taking $\varphi(t,v) = \psi(v) \in C_0^\infty(\mathbb{R}^3)$, we get

$$\int_{\mathbb{R}^3} f(t)\psi dv - \int_{\mathbb{R}^3} f(s)\psi dv = \int_s^t d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau))\psi dv\,, \quad 0 \le s \le t \le T_0\,.$$

We can set $\psi = M_\delta^2 f(t), M_\delta^2 f(s)$ because they belong to $L^\infty([0,T]; W^{2,\infty}(\mathbb{R}^3))$. By taking the sum, we obtain

$$\int_{\mathbb{R}^3} f(t)M_\delta^2 f(t)dv - \int_{\mathbb{R}^3} f(s)M_\delta^2 f(s)dv = \int_{\mathbb{R}^3} f(t)\left(M_\delta^2(t) - M_\delta^2(s)\right)f(s)dv$$
$$+ \int_s^t d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau))\left(M_\delta^2 f(t) + M_\delta^2 f(s)\right)dv\,.$$

Since the integrand of the first term on the right is estimated by $|t-s|C'_\delta||f_0||_{L^1}f(t)$ and in addition, the collision integral term is bounded by $C''_\delta||f_0||_{L^1}||f||^2_{L^1_2}$, we obtain (3.5), namely $M_\delta f \in C([0,T_0]; L^2(\mathbb{R}^3))$. In Definition 3.1 , we can rewrite the term

$$\int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau\varphi(\tau,v)dv$$
$$= \lim_{h\to 0}\int_0^t d\tau \int_{\mathbb{R}^3} (f(\tau,v) + f(\tau+h,v))\frac{\varphi(\tau+h,v) - \varphi(\tau,v)}{2h}dv$$

for $\varphi(t,v) \in C^1(\mathbb{R}^+; C_0^\infty(\mathbb{R}^3))$, by noting $f \in C(\mathbb{R}^+; \mathcal{D}')$. Let

$$\varphi(t) \equiv M_\delta^2(t)f(t),$$

in the above equation, then its right hand side equals to

$$\lim_{h\to 0}\left\{\int_0^t d\tau \int_{\mathbb{R}^3} \frac{(M_\delta f)^2(\tau+h) - (M_\delta f)^2(\tau)}{2h}dv\right.$$
$$\left. + \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau)f(\tau+h)\frac{(M_\delta)^2(\tau+h) - (M_\delta)^2(\tau)}{2h}dv\right\}.$$

It follows from (3.5) that

$$\begin{aligned}&\lim_{h\to 0}\int_0^t d\tau\int_{\mathbb{R}^3}\frac{(M_\delta f)^2(\tau+h)-(M_\delta f)^2(\tau)}{2h}dv\\&=\lim_{h\to 0}\frac{1}{2h}\left\{\int_t^{t+h}d\tau-\int_0^h d\tau\right\}\int_{\mathbb{R}^3}(M_\delta f)^2(\tau)dv\\&=\frac{1}{2}\int_{\mathbb{R}^3}(M_\delta f)^2(t)dv-\frac{1}{2}\int_{\mathbb{R}^3}(M_\delta f)^2(0)dv.\end{aligned}$$

Hence, we obtain (3.6) because the Lebesgue convergence theorem shows that

$$\begin{aligned}&\lim_{h\to 0}\int_0^t d\tau\int_{\mathbb{R}^3}f(\tau)\frac{(M_\delta)^2(\tau+h)-(M_\delta)^2(\tau)}{2h}f(\tau+h)dv\\&=\frac{1}{2}\int_0^t\int_{\mathbb{R}^3}f(\tau)\Big(\partial_t M_\delta^2(\tau)\Big)f(\tau)dvd\tau.\end{aligned}$$

4. **Gevrey regularity for the linear Cauchy problem.**

In this section, we will consider the Gevrey property of the solutions to the Boltzmann equation for potentials satisfying the inverse power laws. The following analysis only applies to the linearized problem and the nonlinear problem will be pursued in the future. Consider the Cauchy problem for the linearized Boltzmann equation

$$\frac{\partial g}{\partial t}=Lg=Q(\mu,g)+Q(g,\mu),\ \ v\in\mathbb{R}^3,\ \ t>0\,;\qquad g|_{t=0}=g_0,\tag{4.1}$$

where μ is the normalized Maxwellian distribution given in the introduction. The definition of the weak solutions is similar to that in Definition 3.1.

Definition 4.1. *For an initial datum $g_0(v)\in L^1_2(\mathbb{R}^3)$, $g(t,v)$ is called a weak solution of the Cauchy problem (4.1) if it satisfies:*

$$\begin{gathered}g(t,v)\in C(\mathbb{R}^+;\mathcal{D}'(\mathbb{R}^3))\cap L^1([0,T_0];L^1_2(\mathbb{R}^3))\cap L^\infty([0,T_0];L^1(\mathbb{R}^3));\\ g(0,v)=g_0;\\ \textstyle\int_{\mathbb{R}^3}g(t,v)\varphi(t,v)dv-\int_{\mathbb{R}^3}g_0(v)\varphi(0,v)dv-\int_0^t d\tau\int_{\mathbb{R}^3}g(\tau,v)\partial_\tau\varphi(\tau,v)dv\\ \textstyle=\int_0^t d\tau\int_{\mathbb{R}^3}L(g)(\tau,v)\varphi(\tau,v)dv,\end{gathered}$$

for any test function $\varphi(t,v)\in C^1(\mathbb{R}^+;C_0^\infty(\mathbb{R}^3))$. The right hand side of the last integral given above is defined as the one in Definition 3.1 which makes sense for any $\varphi\in L^\infty([0,T_0];W^{2,\infty}(\mathbb{R}^3))$.

Notice that in the linear case, the non-negativity of $g\geq 0$ cannot be assumed because it represents the perturbation. Thus, even though the mass and energy conservation laws hold, they do not imply $g(t,\cdot)\in L^1_2$ from the same initial bounds.

From now on, we are going to show that the weak solution $g(t,\cdot)$ is $\in G^{\frac{1}{\alpha}}(\mathbb{R}^3)$ for $0<t\leq T_0$. The existence of weak solutions in this class will be discussed at the end of this section.

Under the assumption (1.5) on the cross-section, the following sub-elliptic estimate is known, cf. [1]:

$$\|\Lambda^\alpha f\|^2_{L^2}\leq C_h\left\{(-Q(h,f),f)_{L^2}+\|f\|^2_{L^2}\right\},\tag{4.2}$$

for any $f \in H^2$ and $h \geq 0$, $h \not\equiv 0$, $h \in L^1_1 \cap L\log L$. Here, the constant $C_h > 0$ depends only on $\|h\|_{L^1_1}$ and $\|h\|_{L\log L}$. For $0 < \delta < 1$, set

$$G_\delta(t,\xi) = \frac{e^{t\langle\xi\rangle^\alpha}}{1+\delta e^{t\langle\xi\rangle^\alpha}}.$$

Then if $g \in L^1([0,T_0]; L^1_2(\mathbb{R}^3))$, we have

$$\begin{gathered} G_\delta(t,D_v)\left\langle D_v\right\rangle^{-4} g \in L^1([0,T_0]; H^2(\mathbb{R}^3)); \\ G^2_\delta(t,D_v)\left\langle D_v\right\rangle^{-8} g \in L^1([0,T_0]; W^{2,\infty}(\mathbb{R}^3)), \end{gathered}$$

and

$$\begin{aligned} \|\Lambda^\alpha G_\delta(t,D_v)\left\langle D\right\rangle^{-4} g\|^2_{L^2} \quad &\leq \quad C_\mu\Big\{(-Q(\mu, G_\delta \left\langle D\right\rangle^{-4} g), G_\delta \left\langle D\right\rangle^{-4} g)_{L^2} \\ &\qquad + \|G_\delta(t,D_v)\left\langle D\right\rangle^{-4} g\|^2_{L^2}\Big\}, \end{aligned} \tag{4.3}$$

where the constant $C_\mu > 0$ is independent on δ.

As in the previous section, the following lemma gives the estimate on the commutator of the pseudo-differential operator $G_\delta(t,D_v)\left\langle D_v\right\rangle^{-4}$ and the collision operator $Q(\mu,\cdot)$.

Lemma 4.1. *For the function g and with the notations given above, we have*

$$\begin{aligned} &|(Q(\mu, G_\delta \left\langle D\right\rangle^{-4} g), G_\delta \left\langle D\right\rangle^{-4} g)_{L^2} - (Q(\mu,g), G^2_\delta \left\langle D\right\rangle^{-8} g)_{L^2}| \\ &\leq C_\mu \|G_\delta \left\langle D\right\rangle^{-4} g\|_{L^2} \|\Lambda^\alpha G_\delta \left\langle D\right\rangle^{-4} g\|_{L^2}, \end{aligned} \tag{4.4}$$

and

$$|(Q(g,\mu), G^2_\delta \left\langle D\right\rangle^{-8} g)_{L^2}| \leq C_\mu \|g\|_{L^1_2} \|G_\delta \left\langle D\right\rangle^{-4} g\|_{L^2}, \tag{4.5}$$

where $C_\mu > 0$ is independent of $0 < \delta < 1$.

Proof. For (4.4), similar to the proof of Lemma 3.1, we choose $G_\delta(t,D_v)\left\langle D\right\rangle^{-4} g \in H^2(\mathbb{R}^3)$ as the test function. Without loss of generality and for simplicity of notations, we drop the regularized operator $\left\langle D\right\rangle^{-4}$ in the following calculation because it does not create extra difficulty. In fact, the main problem is to estimate following term,

$$\left| \int b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{Re\, \hat\mu(\xi^-)\hat g(\xi^+) G_\delta(t,\xi)\bar{\hat g}(\xi)\big[G_\delta(t,\xi) - G_\delta(t,\xi^+)\big]\Big\} d\sigma d\xi \right|.$$

Notice that the weight $G_\delta(t,\xi)$ is an exponential function so that an estimate like (3.4) fails. Instead, we will show the following estimate

$$|G_\delta(t,\xi^+) - G_\delta(t,\xi)| \leq C \sin^2\frac{\theta}{2}\left\langle \xi\right\rangle^\alpha G_\delta(t,\xi^-) G_\delta(t,\xi^+), \tag{4.6}$$

where the constant $C > 0$ depends only on α and T_0. For this, set

$$\tilde G_\delta(s) = \frac{s}{1+\delta s}.$$

Note that $\frac{d}{ds}\tilde G_\delta(s) > 0$ and

$$G_\delta(t,\xi) = \tilde G_\delta\left(e^{t(1+|\xi|^2)^{\alpha/2}}\right).$$

By recalling $|\xi|^2 = |\xi^+|^2 + |\xi^-|^2$ and $|\xi^-|^2 = |\xi|^2 \sin^2 \frac{\theta}{2}$, we have

$$
\begin{aligned}
|G_\delta(t,\xi^+) - G_\delta(t,\xi)| = \Big| \int_0^1 & \frac{\exp t(1+|\xi|^2+\tau(|\xi^+|^2-|\xi|^2))^{\alpha/2}}{\big(1+\delta \exp t(1+|\xi|^2+\tau(|\xi^+|^2-|\xi|^2))^{\alpha/2}\big)^2} \\
\times \ & \frac{t\alpha}{2}(1+|\xi|^2+\tau(|\xi^+|^2-|\xi|^2))^{\alpha/2-1} d\tau \Big| \, |\xi^-|^2 \\
\leq & C G_\delta(t,\xi)(1+|\xi|^2)^{\alpha/2} \sin^2 \frac{\theta}{2},
\end{aligned}
$$

where we have used $\frac{1}{2}|\xi|^2 \leq |\xi|^2 + \tau(|\xi^+|^2 - |\xi|^2) \leq |\xi|^2$. Notice that for $0 < \alpha < 1$, $0 < \delta < 1$, and for any $a, b \geq 0$, we have

$$
(1+a+b)^\alpha \leq (1+a)^\alpha + (1+b)^\alpha \,, \quad (1+\delta e^a)(1+\delta e^b) \leq 3(1+\delta e^{a+b}).
$$

Then,

$$
\tilde{G}_\delta\left(e^{t(1+|\xi^+|^2+|\xi^-|^2)^{\alpha/2}}\right) \leq \tilde{G}_\delta\left(e^{t(1+|\xi^+|^2)^{\alpha/2}+t(1+|\xi^-|^2)^{\alpha/2}}\right) \leq 3 G_\delta(t,\xi^+) G_\delta(t,\xi^-),
$$

which gives (4.6). Therefore, we have

$$
\begin{aligned}
& \left| \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\big\{ Re\, \hat{\mu}(\xi^-)\hat{g}(\xi^+) G_\delta(t,\xi)\bar{\hat{g}}(\xi)\big[G_\delta(t,\xi) - G_\delta(t,\xi^+)\big]\big\} d\sigma d\xi \right| \\
\leq \ & C \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b \, \sin^2 \frac{\theta}{2} |G_\delta(t,\xi^-)\hat{\mu}(\xi^-)| G_\delta(t,\xi^+)|\hat{g}(\xi^+)| \left\langle \xi \right\rangle^\alpha G_\delta(t,\xi)|\hat{g}(\xi)| d\sigma d\xi \\
\leq \ & C \|G_\delta \mu\|_{L^1} \|G_\delta g\|_{L^2} \|\Lambda^\alpha G_\delta g\|_{L^2},
\end{aligned}
$$

so that (4.4) follows.

We now turn to prove (4.5). By using the Bobylev identity, and noting that $\hat{\mu}(\xi) = \hat{\mu}(\xi^+)\hat{\mu}(\xi^-)$, $\hat{\mu}(0) = 1$, we have

$$
\begin{aligned}
|(Q(g,\mu), G_\delta^2 g)_{L^2}| = & \left| \int b\left(\hat{g}(\xi^-)\hat{\mu}(\xi^+) - \hat{g}(0)\hat{\mu}(\xi)\right) G_\delta^2(t,\xi)\bar{\hat{g}}(\xi) d\sigma d\xi \right| \\
= \ & \left| \int b\left(\hat{g}(\xi^-) - \hat{g}(0)\hat{\mu}(\xi^-)\right) G_\delta(t,\xi)\hat{\mu}(\xi^+) G_\delta(t,\xi)\bar{\hat{g}}(\xi) d\sigma d\xi \right| \\
\leq \ & \left| \int b\, \hat{g}(0)\big(\hat{\mu}(\xi^-) - \hat{\mu}(0)\big) G_\delta(t,\xi)\hat{\mu}(\xi^+) G_\delta(t,\xi)\bar{\hat{g}}(\xi) d\sigma d\xi \right| \\
& + \left| \int b\left(\hat{g}(\xi^-) - \hat{g}(0)\right) G_\delta(t,\xi)\hat{\mu}(\xi^+) G_\delta(t,\xi)\bar{\hat{g}}(\xi) d\sigma d\xi \right| .
\end{aligned}
$$

For the first term in the last inequality, since

$$
|\hat{\mu}(\xi^-) - \hat{\mu}(0)| \leq |\xi^-|^2 \leq |\xi|^2 \sin^2 \frac{\theta}{2},
$$

we have

$$
\begin{aligned}
& \left| \int b\, \hat{g}(0)\big(\hat{\mu}(\xi^-) - \hat{\mu}(0)\big) G_\delta(t,\xi)\hat{\mu}(\xi^+) G_\delta(t,\xi)\bar{\hat{g}}(\xi) d\sigma d\xi \right| \\
\leq \ & \|g\|_{L^1} \left| \int b(\cos\theta) \sin^2 \frac{\theta}{2} G_\delta(t,\xi)|\xi|^2 \hat{\mu}(\xi^+) G_\delta(t,\xi)\bar{\hat{g}}(\xi) d\theta d\xi \right| \\
\leq \ & C_{T_0} \|g\|_{L^1} \|G_\delta g\|_{L^2},
\end{aligned}
$$

where $C_{T_0} = 4\|G_\delta(2T_0, D)|D|^2\mu\|_{L^2}$, while for the second term, when $0 < \alpha < 1/2$, the estimate

$$
|\hat{g}(\xi^-) - \hat{g}(0)| \leq \| \bigtriangledown \hat{g}\|_{L^\infty} |\xi^-| \leq \|g\|_{L^1_1} |\xi| \sin \frac{\theta}{2},
$$

gives

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\,\big(\hat g(\xi^-)-\hat g(0)\big)G_\delta(t,\xi)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\sigma d\xi\right|\\
\le\ & \|g\|_{L^1_1}\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\big|\sin\frac{\theta}{2}\big|G_\delta(t,\xi)|\xi|\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\theta d\xi\right|\\
\le\ & C_{T_0}\|g\|_{L^1_1}\|G_\delta g\|_{L^2}.
\end{aligned}
$$

On the other hand, when $1/2\le\alpha<1$, the above simple calculation does not work. Instead, we need to use the symmetry in the integral according to the geometric structure of

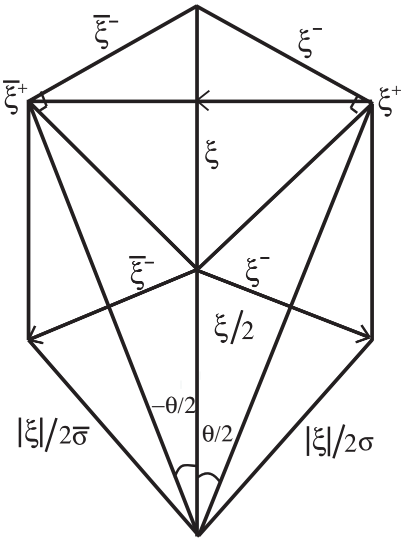
$$
\xi^+=\frac{\xi+|\xi|\sigma}{2},\quad \xi^-=\frac{\xi-|\xi|\sigma}{2},\quad \cos\theta=|\xi|^{-1}\left<\xi,\ \sigma\right>.
$$

For a fixed $\xi\neq 0$, denote the unit vector $\sigma=R_\theta\big(\frac{\xi}{|\xi|}\big)$ as a rotation of the unit vector $\frac{\xi}{|\xi|}$ by an angle θ. Moreover, denote $\bar\sigma=R_{-\theta}\big(\frac{\xi}{|\xi|}\big)$ and

$$
\bar{\xi}^+=\frac{\xi+|\xi|\bar\sigma}{2},\quad \bar{\xi}^-=\frac{\xi-|\xi|\bar\sigma}{2}.
$$

Then we have, cf. Figure 1,

$$
|\xi^+|=|\bar{\xi}^+|,\quad |\xi^-|=|\bar{\xi}^-|,\quad |\xi|^{-1}\left<\xi,\ \bar\sigma\right>=\cos\theta.
$$



With these notations, the integral can be estimated as follows,

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\big(\hat g(\xi^-)-\hat g(0)\big)G_\delta(t,\xi)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\sigma d\xi\\
=\ &\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\bar\sigma\Big)\big(\hat g(\bar{\xi}^-)-\hat g(0)\big)G_\delta(t,\xi)\hat\mu(\bar{\xi}^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\bar\sigma d\xi\\
=\ &\frac12\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\big(\hat g(\xi^-)+\hat g(\bar{\xi}^-)-2\hat g(0)\big)G_\delta(t,\xi)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\sigma d\xi.
\end{aligned}
$$

Here we have used the fact that $d\bar\sigma = d\sigma$ and $\mu(\xi^+) = \mu(\bar\xi^+)$. Notice that ξ^- and $\bar\xi^-$ are symmetric with respect to ξ so that we can denote them by

$$\xi^- = \vec a + \vec b, \ \ \bar\xi^- = \vec a - \vec b,$$

with

$$|\vec a| = \sin\frac{\theta}{2}|\xi^-| = \sin^2\frac{\theta}{2}|\xi|, \quad |\vec b| = \sin\frac{\theta}{2}|\xi^+| = \sin\frac{\theta}{2}\cos\frac{\theta}{2}|\xi|.$$

Thus,

$$\begin{aligned}&|\hat g(\xi^-) + \hat g(\bar\xi^-) - 2\hat g(0)| = |\hat g(\vec a + \vec b) - 2\hat g(\vec a) + \hat g(\vec a - \vec b) + 2(\hat g(\vec a) - \hat g(0))|\\ &\le \|g\|_{L^1_2}|\vec b|^2 + 2\|g\|_{L^1_1}|\vec a|.\end{aligned}$$

Finally, for $1/2 \le \alpha < 1$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\,\big(\hat g(\xi^-) - \hat g(0)\big)G_\delta(t,\xi)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\sigma d\xi\right|\\ &\quad \le (\|g\|_{L^1_1} + \|g\|_{L^1_2})\\ &\qquad \times\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\sin^2\frac{\theta}{2}G_\delta(t,\xi)(|\xi| + |\xi|^2)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\theta d\xi\right|\\ &\quad \le C_{T_0}(\|g\|_{L^1_1} + \|g\|_{L^1_2})\|G_\delta g\|_{L^2}.\end{aligned}$$

Therefore, we have obtained (4.5) and then completes the proof of the lemma. ☐

We are now ready to prove the second main result in this paper.

Proof of Theorem 1.2. By the same argument as given in Section 3, we see that

$$G^2_\delta(t, D_v)\left\langle D_v\right\rangle^{-8} g \in L^\infty([0,T_0]; W^{2,+\infty}(\mathbb{R}^3))$$

whose norm is bounded by $C_\delta \sup_{[0,T_0]}\|g(t)\|_{L^1}$, and moreover,

$$G_\delta(t, D_v)\left\langle D_v\right\rangle^{-4} g \in C([0,T_0]; L^2(\mathbb{R}^3)),$$

by using $g \in L^1([0,T_0], L^1_2)$. Hence, set

$$\varphi(t) = G^2_\delta(t, D_v)\left\langle D_v\right\rangle^{-8} g(t,v)$$

in the last equation of Definition 4.1. By a similar argument as the one for (3.6), we have

$$\begin{aligned}\frac12\int_{\mathbb{R}^3}|G_\delta(t)\left\langle D\right\rangle^{-4} g(t)|^2 dv - \frac12\int_{\mathbb{R}^3}|G_\delta(0)\left\langle D\right\rangle^{-4} g_0|^2 dv&\\ -\frac12\int_0^t\int_{\mathbb{R}^3} g(\tau)\big(\partial_t G^2_\delta(\tau)\big)\left\langle D\right\rangle^{-8} g(\tau)dvd\tau&\\ = \int_0^t \big(Lg(\tau), G^2_\delta(\tau)\left\langle D\right\rangle^{-8} g(\tau)\big)_{L^2}d\tau,&\end{aligned} \tag{4.7}$$

for any $t \in [0,T_0]$. On the other hand, it follows from (4.3), (4.4) and (4.5) that

$$\|\Lambda^\alpha G_\delta\left\langle D\right\rangle^{-4} g\|^2_{L^2} \le C_\mu\left\{(-Lg, G^2_\delta\left\langle D\right\rangle^{-8} g)_{L^2} + \|G_\delta\left\langle D\right\rangle^{-4} g\|^2_{L^2} + \|g\|^2_{L^1_2}\right\}. \tag{4.8}$$

Combining (4.7) and (4.8) implies

$$\begin{aligned}&\|G_\delta(t)\langle D\rangle^{-4}g(t)\|^2_{L^2}+\frac{1}{2C_\mu}\int_0^t\|\Lambda^\alpha G_\delta(\tau)\langle D\rangle^{-4}g(\tau)\|^2_{L^2}d\tau\le\\&\|G_\delta(0)\langle D\rangle^{-4}g_0\|^2_{L^2}+\left|\int_0^t\int_{\mathbb{R}^3}g(\tau)\big(\partial_tG^2_\delta(\tau)\big)\langle D\rangle^{-8}g(\tau)dvd\tau\right|\\&+\int_0^t\|G_\delta(\tau)\langle D\rangle^{-4}g(\tau)\|^2_{L^2}d\tau+\int_0^t\|g(\tau)\|^2_{L^1_2}d\tau.\end{aligned}$$

Since

$$|\partial_tG_\delta(t,\xi)|\le G_\delta(t,\xi)\langle\xi\rangle^\alpha,$$

we have

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}^3}g(\tau)\big(\partial_tG^2_\delta(\tau)\big)\langle D\rangle^{-8}g(\tau)dvd\tau\right|\\&\le 2\int_0^t\|\Lambda^\alpha G_\delta(\tau)\langle D\rangle^{-4}g(\tau)\|_{L^2}\|G_\delta(\tau)\langle D\rangle^{-4}g(\tau)\|_{L^2}d\tau,\end{aligned}$$

and

$$\|G_\delta(0)\langle D\rangle^{-4}g_0\|^2_{L^2}\le\|\langle D\rangle^{-4}g_0\|^2_{L^2}\le C\|g_0\|^2_{L^1}.$$

Thus, for any $\varepsilon>0$, we have

$$\begin{aligned}&\|G_\delta\langle D\rangle^{-4}g(t)\|^2_{L^2}+\Big(\frac{1}{2C_\mu}-\varepsilon\Big)\int_0^t\|\Lambda^\alpha G_\delta(\tau)\langle D\rangle^{-4}g(\tau)\|^2_{L^2}d\tau\\&\le C_0\|g_0\|^2_{L^1}+C_\varepsilon\int_0^t\|G_\delta(\tau)\langle D\rangle^{-4}g(\tau)\|^2_{L^2}d\tau+C_1\int_0^t\|g(\tau)\|^2_{L^1_2}d\tau.\end{aligned}$$

By choosing $\varepsilon=\frac{1}{4C_\mu}>0$, the above inequality shows that there exists a constant $C_2>0$ independent of $\delta\in]0,1[$, such that for any $t\in]0,T_0]$

$$\begin{aligned}&\|G_\delta(t)\langle D\rangle^{-4}g(t)\|^2_{L^2}\\&\le C_0\|g_0\|^2_{L^1}+C_2\left\{\int_0^t\|G_\delta(\tau)\langle D\rangle^{-4}g(\tau)\|^2_{L^2}d\tau+\int_0^t\|g(\tau)\|^2_{L^1_2}d\tau\right\}.\end{aligned}$$

Then the Gronwall inequality yields

$$\|G_\delta(t)\langle D\rangle^{-4}g(t)\|^2_{L^2}\le C_0e^{C_2t}\|g_0\|^2_{L^1}+C_2e^{C_2t}\int_0^te^{-C_1\tau}\|g(\tau)\|^2_{L^1_2}d\tau,$$

where the positive constants C_0 and C_2 are independent of δ. Hence, by noticing

$$e^{-t\langle\xi\rangle^\alpha}\langle\xi\rangle^8\le C_\alpha t^{-\frac{8}{\alpha}},$$

for any fixed $0<t\le T_0$, we have

$$e^{\frac{1}{2}t\langle D_v\rangle^\alpha}g(t,v)\in L^2.$$

And this completes the proof of Theorem 1.2. □

The rest of this section is devoted to

Proof of Proposition 1.1. The construction of the weak solutions which meet the requirement of Theorem 1.2 is based on the following estimate of the operator L in the weighted space $L^2_\ell=L^2_\ell(\mathbb{R}^3)$.

Proposition 4.1. *Assume* $0<\alpha<1/2$ *and let* $\ell>2\alpha+3/2$. *Then, there exists a positive constant* C *such that for any* $g\in L^2_\ell$, *it holds that*

$$(Lg,g)_{L^2_\ell}\le C\|g\|_{L^2_\ell}(\|g\|_{L^1_1}+\|g\|_{L^2_\ell}).$$

Actually, this is just part of the coercivity estimate of L as stated in Remark 4.1 given below, but is enough for the present purpose. Notice that the restriction $0<\alpha<1/2$ comes from (4.18). The proof of this proposition will be given at the end of this section, and we now proceed to the proof of Proposition 1.1.

Let L_δ be the operator L with a cutoff kernel

$$b_\delta(\cos\theta)=\chi(|\theta|>\delta)b(\cos\theta),$$

where χ is the usual characteristic function. Although this is not a bounded operator on $L^2=L^2_0$, so is the operator $L_{R,\delta}=\mathcal{I}_RL_\delta\mathcal{I}_R$ where $\mathcal{I}_R$ is a smooth cutoff function

$$\mathcal{I}_R\in C^\infty_0(\mathbb{R}^3),\quad 0\le\mathcal{I}_R(x)\le1,\quad \mathcal{I}_R(v)=\left\{\begin{array}{ll}1 & (|v|\le R),\\ 0 & (|v|\ge R+1).\end{array}\right.$$

The proof of this boundedness is straightforward and hence is omitted. Thus, $L_{R,\delta}$ is a generator of C_0 semi-group $e^{tL_{R,\delta}}$ on L^2. For any $g_0\in L^2$, define

$$h_{R,\delta}(t)=e^{tL_{R,\delta}}g_0.$$

Since $L_{R,\delta}$ is a bounded operator, $h_{R,\delta}(t)$ has strong regularity in the t variable, i.e.,

$$h_{R,\delta}\in C^\infty([0,\infty);L^2).$$

Thus, it is a unique strong solution to the Cauchy problem

$$\frac{dh_{R,\delta}}{dt}=L_{R,\delta}h_{R,\delta}\quad\text{in}\quad L^2\quad(t\ge0),\qquad h_{R,\delta}(0)=g_0.\tag{4.9}$$

Moreover, the following holds.

Lemma 4.2. *For any* $\ell\in\mathbb{N}$ *and* $g_0\in L^2_\ell$, $h_{R,\delta}(t)$ *is in* $C^\infty([0,\infty);L^2_\ell)$ *and satisfies* (4.9) *strongly in* L^2_ℓ.

Proof. Put $W_\ell(v)=(1+|v|)^\ell$. Since $W_\ell\,\mathcal{I}_R\in L^\infty(\mathbb{R}^3)$ and since $L_{R,\delta}$ is a bounded operator, the series

$$W_\ell h_{R,\delta}=W_\ell e^{t\mathcal{I}_RL^\delta\mathcal{I}_R}g_0=W_\ell g_0+\sum_{k=1}^\infty\frac{t^k}{k!}W_\ell\mathcal{I}_R(\mathcal{I}_RL^\delta\mathcal{I}_R)^kg_0$$

converges in the norm of L^2. In addition, the series can be differentiated term by term in t. This completes the proof of the lemma. □

On the other hand, it is easy to see from its proof that Proposition 4.1 applies also to $L_{R,\delta}$ with the same constant C which is independent of R,δ. This and Lemma 4.2 then yield for $\ell>2\alpha+3/2$

$$\begin{aligned}\frac{d}{dt}\|h_{R,\delta}\|^2_{L^2_\ell}&=2(\frac{dh_{R,\delta}}{dt},h_{R,\delta})_{L^2_\ell}=2(L_{R,\delta}h_{R,\delta},h_{R,\delta})_{L^2_\ell}\\&\le C\|h_{R,\delta}\|_{L^2_\ell}(\|h_{R,\delta}\|_{L^1_1}+\|h_{R,\delta}\|_{L^2_\ell}).\end{aligned}\tag{4.10}$$

From now on, assume $\ell>5/2$ so that $\ell>2\alpha+3/2$ and $L^2_\ell\subset L^1_1$. Then (4.10) yields

$$\|h_{R,\delta}(t)\|_{L^2_\ell}\le e^{Ct}\|g_0\|_{L^2_\ell},\tag{4.11}$$

for all $t \geq 0$.

To simplify the notations, put

$$X_\ell = L^\infty([0,T];\ L^2_\ell), \quad Y_\ell = L^2([0,T];\ L^2_\ell).$$

It follows from (4.11) that

$$h_{R,\delta} \in X_\ell \cap Y_\ell, \quad \|h_{R,\delta}\|_{X_\ell} \leq e^{CT}\|g_0\|_{L^2_\ell}, \quad \|h_{R,\delta}\|_{Y_\ell} \leq \sqrt{T}e^{CT}\|g_0\|_{L^2_\ell},$$

for any $T > 0$.

Now, fix $\delta > 0$ and let $R \to \infty$. It is clear from the above estimates that there exist a function h_δ and a subsequence $\{h_{R,\delta}\}$ (with abuse of notation) such that for any $T > 0$,

$$\begin{array}{ll} h_\delta \in X_\ell \cap Y_\ell, & \|h_\delta\|_{X_\ell} \leq e^{CT}\|g_0\|_{L^2_\ell}, \quad \|h_\delta\|_{Y_\ell} \leq \sqrt{T}e^{CT}\|g_0\|_{L^2_\ell}, \\ h_{R,\delta} \to h_\delta & \text{weakly* in } \ X_\ell \ \text{ and weakly in } \ Y_\ell. \end{array}$$

Consider the weak formulation of (4.9):

$$-(g_0, \phi(0))_{L^2} - \int_0^T (h_{R,\delta}(\tau), \phi_\tau(\tau))_{L^2} d\tau = \int_0^T (h_{R,\delta}(\tau), \mathcal{I}_R L^*_\delta \mathcal{I}_R \phi(t))_{L^2} d\tau, \tag{4.12}$$

where ϕ is any test function in $C^\infty_0([0,T] \times \mathbb{R}^3)$ satisfying $\phi(T) = 0$ and L^* is the adjoint operator of L defined in the sense given in Definition 3.1. Take the limit of (4.12) as $R \to \infty$. Clearly,

$$W_\ell^{-1}\mathcal{I}_R L^*_\delta \mathcal{I}_R \phi \to W_\ell^{-1} L^*_\delta \phi \quad \text{strongly in } Y_0,$$

so that we have

$$-(g_0, \phi(0))_{L^2} - \int_0^T (h_\delta(\tau), \phi_\tau(\tau))_{L^2} d\tau = \int_0^T (W_\ell h_\delta(\tau), W_\ell^{-1} L^*_\delta \phi(t))_{L^2} d\tau. \tag{4.13}$$

Now, let $\delta \to 0$. Then, there exist a function g and a subsequence $\{h_\delta\}$ (again with abuse of notation) such that

$$\begin{array}{ll} g \in X_\ell \cap Y_\ell, & \|g\|_{X_\ell} \leq e^{CT}\|g_0\|_{L^2_\ell}, \quad \|g\|_{Y_\ell} \leq \sqrt{T}e^{CT}\|g_0\|_{L^2_\ell}, \\ h_\delta \to g & \text{weakly* in } \ X_\ell \ \text{ and weakly in } \ Y_\ell. \end{array}$$

The function g is indeed the desired weak solution. To see this, note that

$$W_\ell^{-1} L^*_\delta \phi \to W_\ell^{-1} L^* \phi \quad \text{strongly in } Y_0 \quad \text{as} \quad \delta \to 0.$$

By taking the limit $\delta \to 0$ in (4.13), we deduce

$$-(g_0, \phi(0))_{L^2} - \int_0^T (g(\tau), \phi_\tau(\tau))_{L^2} d\tau = \int_0^T (W_\ell g(\tau), W_\ell^{-1} L^* \phi(\tau))_{L^2} d\tau. \tag{4.14}$$

Finally, set

$$\phi(t,v) = \int_t^T \eta(s) ds \psi(t,v), \quad \eta \in C^\infty([0,T]), \ \psi \in C^\infty_0([0,T] \times \mathbb{R}^N).$$

Then (4.14) yields

$$\begin{aligned} \int_0^T \eta(t) \Big\{ (g(t), \psi(t))_{L^2} - (g_0, \psi(0))_{L^2} - \int_0^t (g(\tau), \psi_\tau(\tau))_{L^2} d\tau \\ - \int_0^t (W_\ell g(\tau), W_\ell^{-1} L^* \psi(\tau))_{L^2} d\tau \Big\} dt = 0, \end{aligned}$$

which implies

$$(g(t),\psi(t))_{L^2}-(g_0,\psi(0))_{L^2}-\int_0^t(g(\tau),\psi_\tau(\tau))_{L^2}d\tau$$
$$-\int_0^t(W_\ell g(\tau),W_\ell^{-1}L^*\psi(\tau))_{L^2}d\tau=0\quad\text{a.a.}\quad t.$$

This is just the last equation in Definition 4.1, which, then, gives for any test function of the form $\psi(t,v)=\bar\psi(v)\in C_0^\infty(\mathbb{R}^N)$,

$$(g(t),\bar\psi)_{L^2}=(g_0,\bar\psi)_{L^2}+\int_0^t w(\tau)d\tau,$$

where $w(t)=(W_\ell g(t),W_\ell^{-1}L^*\bar\psi)_{L^2}\in L^1(0,T)$. Thus, $g\in C(\mathbb{R}^+,\mathcal{D}')$. In summary, g meets all the requirement stated in Definition 4.1. The proof of Proposition 1.1 is now complete, except for □

Proof of Proposition 4.1. Firstly, consider $L_1g=Q(\mu,g)$. Recall $W_\ell(v)=(1+|v|)^\ell$ and use the notation $W'_\ell=W_\ell(v')$, etc, to deduce

$$(L_1g,g)_{L^2_\ell}=(L_1g,W_\ell^2g)_{L^2}=\int_{\mathbb{R}^6\times S^2}b(\mu'_*g'-\mu_*g)W_\ell^2gdvdv_*d\sigma$$
$$=\int_{\mathbb{R}^6\times S^2}b\mu_*g\{(W'_\ell)^2g'-W_\ell^2g\}dvdv_*d\sigma=\int_{\mathbb{R}^6\times S^2}b\mu_*g(W_\ell W'_\ell g'-W_\ell^2g)dvdv_*d\sigma$$
$$+\int_{\mathbb{R}^6\times S^2}b\mu_*g(W'_\ell-W_\ell)W'_\ell g'dvdv_*d\sigma=A_1+A_2.$$

We note that

$$A_1=-\frac{1}{2}\int_{\mathbb{R}^6\times S^2}b\mu_*(W_\ell g-W'_\ell g')^2dvdv_*d\sigma$$
$$-\frac{1}{2}\int_{\mathbb{R}^6\times S^2}b\mu_*\{(W_\ell g)^2-(W'_\ell g')^2\}dvdv_*d\sigma=A_{11}+A_{12}.$$

Clearly,

$$A_{11}\le 0. \tag{4.15}$$

A_{12} can be computed just by the cancellation lemma in [1] with $\gamma=0$, yielding

$$A_{12}=-\frac{1}{2}\int_{\mathbb{R}^3}\mu_*\{S*_v(W_\ell g)^2\}dv_*,$$

where "$*_v$" is the convolution in v and S is the function introduced in [1], which is a constant function in our case, that is,

$$S=2\pi\int_0^{\pi/2}\sin\theta\Big[(\cos\theta)^{-3}-1\Big]b(\cos\theta)d\theta.$$

Hence,

$$|A_{12}|=\frac{S}{2}\int_{\mathbb{R}^3}\mu_*\Big\{\int_{\mathbb{R}^3}(W_\ell g)^2dv\Big\}dv_*=C\|W_\ell g\|^2=C\|g\|_\ell^2. \tag{4.16}$$

In order to evaluate A_2, first, we compute

$$|W'_\ell-W_\ell|\le C|v'-v|(W'_{\ell-1}+W_{\ell-1})\le C\theta(|v|+|v_*|)(W'_{\ell-1}+W_{\ell-1}), \tag{4.17}$$

where we used $|v'-v| = |v-v_*||\sin(\theta/2)|$, which comes from (1.4). Therefore, recalling that μ is a Maxwellian, we have

$$\begin{aligned}\mu_*|W_\ell' - W_\ell| &\le C\theta\mu_*^{1/2}(|v|+1)(W_{\ell-1}' + W_{\ell-1}) \le C\theta\mu_*^{1/2}(W_\ell' + W_\ell)\\ &\le C\theta\mu_*^{1/4}W_\ell(\mu_*^{1/4}W_\ell^{-1}W_\ell' + 1) \le C\theta\mu_*^{1/4}W_\ell,\end{aligned}$$

because

$$\frac{\mu^r(v_*)W_\ell(v')}{W_\ell(v)} \le C\frac{W_\ell(v')}{W_\ell(v_*)W_\ell(v)} \le C, \quad (r>0),$$

by virtue of the conservation law $|v|^2+|v_*|^2 = |v'|^2+|v_*'|^2$. We then get

$$\begin{aligned}|A_2|^2 &\le C\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}\theta b(\cos\theta)\mu_*^{1/4}|W_\ell g||(W_\ell g)'|dvdv_*d\sigma\Big|^2\\ &\le C\Big\{\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}\theta b(\cos\theta)\mu_*^{1/4}|W_\ell g|^2dvdv_*d\sigma\Big\}\\ &\qquad\times\Big\{\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}\theta b(\cos\theta)\mu_*^{1/4}|(W_\ell g)'|^2dvdv_*d\sigma\Big\} = CA_{21}A_{22}.\end{aligned}$$

It is at this stage that we need to assume $0<\alpha<1/2$ for the collision cross section B satisfying (1.5) so that

$$\int_{S^2}\theta b(k\cdot\sigma)d\sigma \sim \int_0^{\pi/2}\theta^{-2\alpha}d\theta < +\infty \qquad (k\in S^2), \tag{4.18}$$

can hold. Then,

$$A_{21} \le C\int_{\mathbb{R}^3}\mu_*^{1/4}dv_*\int_{\mathbb{R}^3}|W_\ell g|^2dv \le C\|g\|^2_{L^2_\ell}.$$

The estimation of A_{22} is done by the help of change of variables

$$v\mapsto v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \tag{4.19}$$

which was introduced in [1] where the Jacobian is found to be

$$\Big|\frac{\partial v}{\partial v'}\Big| = \frac{8}{\Big|I+k\otimes\sigma\Big|} = \frac{8}{|1+k\cdot\sigma|} = \frac{4}{\cos^2(\theta/2)} \le 8, \qquad \theta\in[0,\frac{\pi}{2}]. \tag{4.20}$$

We shall be careful, as in [1], that after this change of variables, $k=(v-v_*)/|v-v_*|$ is a function of v_*, v', σ so that θ plays no longer the role of polar angle because the "pole" k moves with σ and hence the measure $d\sigma$ is no longer given by $\sin\theta d\theta d\phi$. Therefore we need a new pole which is independent of σ to carry out the integration in σ. A possible (and indeed the best) choice is $k' = (v'-v_*)/|v'-v_*|$, for which the polar angle ψ defined by $\cos\psi = k'\cdot\sigma$ satisfies (cf. [1, Fig. 1]),

$$\psi = \frac{\theta}{2}, \qquad d\sigma = \sin\psi d\psi d\phi, \qquad \psi\in[0,\frac{\pi}{4}].$$

This implies that θ works almost as polar angle and we can write

$$A_{22} = \int_{\mathbb{R}^3}\mu_*^{1/4}\Big\{\int_{\mathbb{R}^3}D_0(v_*,v')|(W_\ell g)'|^2dv'\Big\}dv_*$$

with

$$\begin{aligned}D_0(v_*,v') &= \int_{S^2}\theta(v_*,v,\sigma)b(\cos\theta(v_*,v',\sigma))d\sigma\\ &\le C\int_0^{\pi/4}\psi b(\cos 2\psi)\sin\psi d\psi \le C\int_0^{\pi/4}\psi^{-2\alpha}d\psi < +\infty,\end{aligned}$$

which deduces

$$A_{22}\le C\int_{\mathbb{R}^3}\mu_*^{1/4}\Big\{\int_{\mathbb{R}^3}|(W_\ell g)'|^2dv'\Big\}dv_*\le C\|g\|^2_{L^2_\ell}$$

and finally

$$|A_2|\le C(A_{21}A_{22})^{1/2}\le C\|g\|^2_{L^2_\ell}. \tag{4.21}$$

We shall now estimate $L_2g=Q(g,\mu)$. Write

$$\begin{aligned}&(L_2g,g)_{L^2_\ell}=(L_2g,W_\ell^2g)_{L^2}=\int_{\mathbb{R}^6\times S^2}b(g'_*\mu'-g_*\mu)W_\ell^2gdvdv_*d\sigma\\ &=\int_{\mathbb{R}^6\times S^2}bg_*\mu\{(W'_\ell)^2g'-W_\ell^2g\}dvdv_*d\sigma\\ &=\int_{\mathbb{R}^6\times S^2}bg_*\mu W_\ell(W'_\ell g'-W_\ell g)dvdv_*d\sigma+\int_{\mathbb{R}^6\times S^2}bg_*\mu(W'_\ell-W_\ell)W'_\ell g'dvdv_*d\sigma\\ &=A_3+A_4.\end{aligned}$$

In order to evaluate A_3, we again apply Bobylev's identity [4] with $\hat g(\xi)=\mathcal{F}(g)$, $\Phi(\xi)=\mathcal{F}(W_\ell g),\ \Psi(\xi)=\mathcal{F}(W_\ell\mu)$, to deduce

$$\begin{aligned}A_3&=\int_{\mathbb{R}^6\times S^2}b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)\{g'_*(W'_\ell\mu')-g_*(W_\ell\mu)\}W_\ell gdvdv_*d\sigma\\ &=\int_{\mathbb{R}^3\times S^2}b(\frac{\xi}{|\xi|}\cdot\sigma)\{\hat g(\xi^-)\Psi(\xi^+)-\hat g(0)\Psi(\xi)\}\overline{\Phi(\xi)}d\xi d\sigma,\end{aligned}$$

where

$$\xi^+=\frac12(\xi+|\xi|\sigma),\qquad \xi^-=\frac12(\xi-|\xi|\sigma).$$

Split A_3 as follows.

$$\begin{aligned}A_3&=\int_{\mathbb{R}^3\times S^2}b(\frac{\xi}{|\xi|}\cdot\sigma)\hat g(\xi^-)\{\Psi(\xi^+)-\Psi(\xi)\}\overline{\Phi(\xi)}d\xi d\sigma\\ &+\int_{\mathbb{R}^3\times S^2}b(\frac{\xi}{|\xi|}\cdot\sigma)\{\hat g(\xi^-)-\hat g(0)\}\Psi(\xi)\overline{\Phi(\xi)}d\xi d\sigma=A_{31}+A_{32}.\end{aligned}$$

Without loss of generality, we may take $W_\ell(v)=(1+|v|^2)^\ell$. Then, since $\hat\mu(\xi)=(2\pi)^{3/2}\mu(\xi)$ for the absolute Maxwellian (1.9), we have

$$\Psi(\xi)=\mathcal{F}(W_\ell\mu)=(I-\Delta_\xi)^\ell\hat\mu(\xi)=P(\xi)\mu(\xi),$$

where $P(\xi)$ is a polynomial in ξ of order 2ℓ. By noticing that

$$|\xi^+-\xi|=|\xi||\sin\frac{\theta}{2}|\qquad \frac{|\xi|}{\sqrt2}\le|\xi^+|\le|\xi|,$$

with $(\xi\cdot\sigma)/|\xi| = \cos\theta$, $\theta \in [0,\pi/2]$, we get

$$\begin{aligned}|\Psi(\xi^+) - \Psi(\xi)| &\le |P(\xi^+)||\mu(\xi^+) - \mu(\xi)| + |P(\xi^+) - P(\xi)|\mu(\xi)\\ &\le |P(\xi^+)||\mu^{1/2}(\xi^+) - \mu^{1/2}(\xi)||\mu^{1/2}(\xi^+) + \mu^{1/2}(\xi)| + |P(\xi^+) - P(\xi)|\mu(\xi)\\ &\le C(1+|\xi|^{2\ell})\theta|\xi|\mu^{1/4}(\xi) \le C\theta\mu^{1/8}(\xi),\end{aligned}$$

which yields, together with (4.18),

$$\begin{aligned}|A_{31}| &\le C\int_{\mathbb{R}^3} |\hat g(\xi^-)|\mu^{1/8}(\xi)|\Phi(\xi)|d\xi \\ &\le C\|\hat g\|_{L^\infty_\xi}\|\mu^{1/8}(\xi)\|_{L^2_\xi}\|\Phi\|_{L^2_\xi} \le C\|g\|_{L^1}\|W_\ell g\|_{L^2}.\end{aligned} \tag{4.22}$$

On the other hand, by recalling the estimate

$$|\hat g(\xi^-) - \hat g(0)| \le |(\nabla_\xi \hat g)(\tilde\xi)||\xi^-| \le C|\xi||\theta|\|\nabla_\xi \hat g\|_{L^\infty_\xi} \le C|\xi||\theta|\||v|g\|_{L^1},$$

we get

$$\begin{aligned}|A_{32}| &\le C\||v|g\|_{L^1}\int_{\mathbb{R}^3}|\Psi(\xi)||\Phi(\xi)|d\xi \\ &\le C\||v|g\|_{L^1}\|W_\ell\mu\|_{L^2}\|W_\ell g\|_{L^2} \le C\|g\|_{L^1_1}\|g\|_{L^2_\ell}.\end{aligned} \tag{4.23}$$

It remains to evaluate A_4. Since

$$\begin{aligned}|W'_\ell - W_\ell| &\le C|v'-v|\{W'_{\ell-1} + W_{\ell-1}\} \le C|v'-v|\{|v'-v|^{\ell-1} + W_{\ell-1}\}\\ &\le C\{\theta^\ell|v-v_*|^\ell + \theta|v-v_*|W_{\ell-1}\},\end{aligned}$$

we get

$$\mu(v)|W'_\ell - W_\ell| \le C\mu^{1/2}(v)\{\theta^\ell W_\ell(v_*) + \theta W_1(v_*)\},$$

and hence

$$\begin{aligned}|A_4| \le C\Big\{&\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}\theta^\ell b(\cos\theta)\mu^{1/2}(v)|(W_\ell g)_*(W_\ell g)'|dvdv_*d\sigma\\ &+\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}\theta b(\cos\theta)\mu^{1/2}(v)|(W_1 g)_*||(W_\ell g)'|dvdv_*d\sigma\Big\} = C(A_{41}+A_{42})\end{aligned}$$

By the Schwarz inequality, we have

$$\begin{aligned}A_{41}^2 \le & C\Big\{\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}\theta^{\ell-\frac32}b(\cos\theta)\mu^{1/2}(v)|(W_\ell g)_*|^2dvdv_*d\sigma\Big\}\\ &\times\Big\{\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}\theta^{\ell+\frac32}b(\cos\theta)\mu^{1/2}(v)|(W_\ell g)'|^2dvdv_*d\sigma\Big\} = CA_{411}A_{412}.\end{aligned}$$

From now on we choose $\ell > 2\alpha + \frac32$. A straightforward calculation gives

$$A_{411} \le \int_0^{\pi/2}\theta^{-1-2\alpha+\ell-\frac32}d\theta\int_{\mathbb{R}^3}\mu^{1/2}(v)dv\int_{\mathbb{R}^3}|(W_\ell g)_*|^2dv_* = C\|g\|^2_{L^2_\ell}.$$

To estimate A_{412} we need the "singular" change of variables

$$v_* \mapsto v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \tag{4.24}$$

whose Jacobian is computed as

$$\Big|\frac{\partial v_*}{\partial v'}\Big| = \frac{8}{\Big|I - k\otimes\sigma\Big|} = \frac{8}{|1-k\cdot\sigma|} = \frac{4}{\sin^2(\theta/2)} \le 16\theta^{-2},\quad \theta\in[0,\pi/2].$$

This gives rise to an additional singularity. Actually, the situation is much worse. Recall that θ is no longer legitimate polar angle. In this case the best choice of pole is $k'' = (v'-v)/|v'-v|$ for which polar angle ψ defined by $\cos\psi = k''\cdot\sigma$ satisfies (cf. [1, Fig. 1])

$$\psi = \frac{\pi-\theta}{2}, \qquad d\sigma = \sin\psi d\psi d\phi, \qquad \psi \in [\frac{\pi}{4}, \frac{\pi}{2}].$$

Unlike for (4.19), this measure does not cancel the singularity of $b(\cos\theta)$ coming from $\sin\theta$ in (1.5). Nevertheless we have by (4.24)

$$A_{412} \le C\int_{\mathbb{R}^3\times\mathbb{R}^3} D_1(v,v')\mu^{1/2}(v)|(W_\ell g)'|^2 dv dv'$$

with

$$D_1(v,v') = \int_{S^2} \theta^{\ell+\frac{3}{2}-2} b(\cos\theta) d\sigma \le C\int_{\pi/4}^{\pi/2} (\frac{\pi}{2}-\psi)^{-1-2\alpha-1+\ell+\frac{3}{2}-2} d\psi \le C.$$

The last inequality comes since $\ell > 2\alpha + 3/2$. Consequently, we have

$$A_{41} \le C\|g\|^2_{L^2_\ell}.$$

Finally, A_{42} can be estimated exactly in the same way as A_{22} by using the change of variables (4.19):

$$A_{42} \le C\|g\|_{L^1_1}\|g\|_{L^2_\ell}.$$

Thus we obtained

$$A_4 \le C(\|g\|_{L^1_1} + \|g\|_{L^2_\ell})\|g\|_{L^2_\ell}. \tag{4.25}$$

Combining (4.15), (4.16), (4.21), (4.22), (4.23) and (4.25) together completes the proof of Proposition 4.1. □

Remark 4.1. A_{11} has the coercivity estimate,

$$-A_{11} \ge C_1\||D_v|^\alpha (W_\ell g)\|_{L^2} - C_2\|g\|^2_{L^2_\ell},$$

for some positive constants C_1 and C_2, which comes from [1, §6]. This estimate gives a generalized version of the sub-elliptic estimate (4.2) in the weighted space L^2_ℓ.

5. **Appendix. The collision cross-section of the Debye-Yukawa potential.**

Following the computation given in [6, 20], we will give an asymptotic description of the Boltzmann collision kernel $B(z,\sigma)$ for the potential $U(\rho)$ defined in (1.6). Here, ρ is the distance between two interacting particles, $z = v - v_*$ is relative velocity, $\sigma \in \mathbb{S}^2$ and $< \frac{z}{|z|}, \sigma >= \cos(\pi - 2\vartheta)$, $\theta = \pi - 2\vartheta$ is the deviation angle. Let $p \ge 0$ be the impact parameter which is a function of ϑ and z. Then Boltzmann collision cross-section is defined by

$$B(|z|,\vartheta) = \frac{|z|s(|z|,\vartheta)}{4\cos\vartheta} = |z|\frac{p}{2\sin 2\vartheta}\frac{\partial p}{\partial\vartheta}, \tag{5.1}$$

where $s(|z|,\vartheta)$ is called the differential scattering cross-section.

If ρ and φ are the radial and angular coordinates in the plane of motion, then the impact parameter $p(V,\vartheta)$ is determined by the conservation of energy and angular momentum respectively:

$$\begin{cases} \frac{1}{2}\left(\dot\rho^2 + \rho^2\dot\varphi^2\right) + U(\rho) = \frac{1}{2}V^2 + U(\sigma), \qquad (\rho \le \sigma), \\ \rho^2\dot\varphi = pV^2. \end{cases}$$

Here the relative speed is now denoted by $V = |v - v_*|$. As usual, it is impossible to give an explicit expression of solutions to this nonlinear system of ordinary differential equations. Hence, in the following, we will study the singular behavior of the solutions around the grazing collisions, that it, the situation when $\theta \sim 0$.

By using φ as the independent variable to eliminate the time derivative, after integration, we have

$$\vartheta = \frac{1}{\sqrt{2}} Vp \int_{\rho_0}^{\sigma} \rho^{-2} \big[\frac{V^2}{2}\big(1 - \frac{p^2}{\rho^2}\big) - U(\rho) + U(\sigma)\big]^{-1/2} d\rho + \sin^{-1}\Big(\frac{p}{\sigma}\Big),$$

where ρ_0 is the smallest distance between two particles which satisfies

$$\frac{1}{2} V^2 \big(1 - \frac{p^2}{\rho_0^2}\big) = U(\rho_0) - U(\sigma) > 0.$$

Note that $p < \rho_0 < \rho \leq \sigma$. By the transformation $u = \frac{p}{\rho}$, we have

$$\vartheta = \int_{p/\sigma}^{u_0} \big[1 - u^2 - \frac{2}{V^2}\big(U\big(\frac{p}{u}\big) - U(\sigma)\big)\big]^{-1/2} du + \sin^{-1}\big(\frac{p}{\sigma}\big),$$

where $u_0 = p/\rho_0$ satisfies

$$1 - u_0^2 - \frac{2}{V^2}\big(U\big(\frac{p}{u_0}\big) - U(\sigma)\big) = 0.$$

Therefore,

$$\begin{aligned}
\frac{\theta}{2} &= \frac{\pi}{2} - \vartheta = \frac{\pi}{2} - \int_{p/\sigma}^{u_0} \big[1 - u^2 - \frac{2}{V^2}\big(U\big(\frac{p}{u}\big) - U(\sigma)\big)\big]^{-1/2} du - \sin^{-1}\big(\frac{p}{\sigma}\big) \\
&= \int_0^1 \frac{dt}{\sqrt{1-t^2}} - \int_0^{p/\sigma} \frac{dt}{\sqrt{1-t^2}} - \int_{p/\sigma}^{u_0} \big[1 - u^2 - \frac{2}{V^2}\big(U\big(\frac{p}{u}\big) - U(\sigma)\big)\big]^{-1/2} du.
\end{aligned}$$

By setting $u = u_0 t$, we get

$$\begin{aligned}
\frac{\theta}{2} &= \int_{p/\sigma}^1 \frac{dt}{\sqrt{1-t^2}} - \int_{\frac{p}{u_0\sigma}}^1 \big[1 - u_0^2 t^2 - \frac{2}{V^2}\big(U\big(\frac{p}{u_0 t}\big) - U(\sigma)\big)\big]^{-1/2} u_0 dt \\
&= \int_{p/\sigma}^1 \frac{dt}{\sqrt{1-t^2}} - \int_{\frac{p}{u_0\sigma}}^1 \big[1 - t^2 + \frac{2}{V^2 u_0^2}\big(U\big(\frac{p}{u_0}\big) - U\big(\frac{p}{u_0 t}\big)\big)\big]^{-1/2} dt \\
&= -\int_{\frac{p}{u_0\sigma}}^{p/\sigma} \frac{dt}{\sqrt{1-t^2}} + \int_{\frac{p}{u_0\sigma}}^1 \frac{1}{\sqrt{1-t^2}} \big[1 - \big(1 + \frac{2U(\frac{p}{u_0}) - 2U(\frac{p}{u_0 t})}{(1-t^2)V^2 u_0^2}\big)^{-1/2}\big] dt,
\end{aligned}$$

where we have used the fact that

$$\frac{1 - u_0^2}{u_0^2} = \frac{2}{V^2 u_0^2}\big(U\big(\frac{p}{u_0}\big) - U(\sigma)\big).$$

It is clear that there is no explicit formula for $\theta = \theta(p, V)$. To study its asymptotic behavior when $\theta \sim 0$, we let $\sigma \to \infty$ which is equivalent to let $p \to \infty$. In this limit, we have $u_0 \approx 1$ and

$$\big(1 + \frac{2U(\frac{p}{u_0}) - 2U(\frac{p}{u_0 t})}{(1-t^2)V^2 u_0^2}\big)^{-1/2} \approx 1 - \frac{U(\frac{p}{u_0}) - U(\frac{p}{u_0 t})}{(1-t^2)V^2 u_0^2}.$$

Thus,

$$\frac{\theta}{2} \approx \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{U(p) - U(\frac{p}{t})}{(1-t^2)V^2} dt.$$

By plugging $U(\rho) = \rho^{-1} e^{-\rho^s}$ into the above integral, we have

$$\frac{\theta}{2} \approx \frac{1}{V^2 p} e^{-p^s} \int_0^1 (1-t^2)^{-3/2} \left(1 - t e^{-p^s(t^{-s}-1)}\right) dt.$$

Since

$$\begin{aligned} 0 &\le \frac{\partial}{\partial p} \left(\int_0^1 (1-t^2)^{-3/2} \left(1 - t e^{-p^s(t^{-s}-1)}\right) dt \right) \\ &= \int_0^1 (1-t^2)^{-3/2} t (t^{-s}-1) s p^{s-1} e^{-p^s(t^{-s}-1)} dt \le C_s s p^{s-1}, \end{aligned}$$

it holds that

$$0 < c_0 \le \int_0^1 (1-t^2)^{-3/2} \left(1 - t e^{-p^s(t^{-s}-1)}\right) dt \le C_s p^s + c_0.$$

where $c_0 = \int_0^1 (1-t^2)^{-3/2}(1-t)dt$. Finally, for $p \to \infty$ (equivalently $\theta \to 0$), we have

$$\log \theta \approx -K' p^s.$$

In summary, we have the Boltzmann collision cross-section for the Debye-Yukawa type potentials as

$$B(V,\theta) = -\frac{V}{2\sin\theta} \frac{\partial p^2}{\partial \theta} \approx K V \theta^{-2} \left(\log \theta^{-1}\right)^{\frac{2}{s}-1}, \tag{5.2}$$

for some constant $K > 0$ when $\theta \sim 0$. Note that the cross-section $B(V,\theta)$ satisfies for any $s > 0$,

$$\int_0^{\pi/2} B(V,\theta) \sin\theta d\theta = +\infty, \quad \text{and} \quad \int_0^{\pi/2} B(V,\theta) \sin^2\theta d\theta < +\infty.$$

Added in the proof. After the paper had been submitted, Prof. R. Alexandre communicated the paper [8] to us which proves the uniform propagation in all time of the Gevrey regularity for Maxwellian molecules, including the case when the Gevrey index s is not less than $1/2$. Here we stress that Theorem 1.2 is focused on the Gevrey smoothing effect, not the Gevrey propagation, and moreover the method presented here yields the Gevrey smoothing effect of the index $1/2\alpha$ if one replace $\langle \xi \rangle^\alpha$ in the weight $G_\delta(t,\xi)$ by $\varepsilon \langle \xi \rangle^{2\alpha}$ because, for example, the right hand side of (4.4) is only changed to $\varepsilon C_\mu \|\Lambda^\alpha G_\delta \langle D \rangle^{-4} g\|^2_{L^2}$.

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