THE GEVREY HYPOELLIPTICITY FOR LINEAR AND NON-LINEAR FOKKER-PLANCK EQUATIONS*

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Abstract. In this paper, we study the Gevrey regularity of weak solution for a class of linear and semilinear Fokker-Planck equations.

1. Introduction

Recently, a lot of progress has been made on the study for the spatially homogeneous Boltzmann equation without angular cutoff, cf. [2, 3, 7, 21] and references therein, which shows that the singularity of collision cross-section yields some gain of regularity in the Sobolev space frame on weak solutions for Cauchy problem. That means, this gives the C^∞ regularity of weak solution for the spatially homogeneous Boltzmann operator without angular cutoff. The local solutions having the Gevrey regularity have been constructed in [20] for initial data having the same Gevrey regularity, and a genearal Gevrey regularity results have given in [16] for spatially homogeneous and linear Boltzmann equation of Cauchy problem for any initial data. In the other word, there is the smoothness effet similary to heat equation.

However, there is no general theory for the spatially inhomogeneous problems. It is now a kinetic equation and diffusion part is nonlinear operator of velocity variable. In [1], by using the uncertainty principle and microlocal analysis, they obtain a C^∞ regularity results for linear spatially inhomogeneous Boltzmann equation without angular cutoff.

Consider the following linear kinetic operator

$$\mathcal{P} = \partial_t + v \cdot \partial_x + a(t,x,v)(-\triangle_v)^\sigma, \quad t \in \mathbb{R}, (x,v) \in \mathbb{R}^{2n}, \tag{1.1}$$

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where $0 < \sigma \leq 1$, $v \cdot \partial_x = \Sigma_{j=1}^n v_j \partial_{x_j}$, $a(x, v, t) \in C_b^\infty(\mathbb{R}^{2n+1})$, $a(t, x, v) \geq c_0 > 0$ and $(-\triangle_v)^\sigma$ is the Fourier multiplier defined by

$$(-\triangle_v)^\sigma u(t, x, v) = (2\pi)^{-(2n+1)} \int_{\mathbb{R}^{2n+1}} e^{i(t\cdot\tau+x\cdot\xi+v\cdot\eta)} |\eta|^{2\sigma} \hat{u}(\tau, \xi, \eta) d\tau d\xi d\eta.$$

$\mathcal{P}$ is not a pseudo-differential operator in $\mathbb{R}^{2n+1}$ since the coefficient of kinetic part is not bounded in $\mathbb{R}^{2n+1}$ and the symbole of $(-\triangle_v)^\sigma$ is not smooth. Here and all throughout the paper, we denote by $\hat{u}(\tau, \xi, \eta)$ the Fourier transform of u in the (t, x, v) variables. In this paper we only consider the partial case of $\sigma = 1$, and the case $\sigma \in (0,\ 1)$ will be pursued in the future. we have a linear partial differential operator in $\mathbb{R}^{2n+1}$

$$\mathcal{L} = \partial_t + v \cdot \partial_x - a(t, x, v)\triangle_v, \tag{1.2}$$

where $\triangle_v$ is Laplace operator of velocity variables v.

The motivation of study for this class of operator is , as in [14], attempt to study inhomogenous Boltzmann equations without angular cutoff and non linear Vlasov-Fokker-Planck equation (see [10, 11]).

Before stating the result, we recall the definition of Gevrey class function. Let U be an open subset of $\mathbb{R}^N$ and f be a real function defined in U. We say $f \in G^s(U)(s \geq 1)$ if $f \in C^\infty(U)$ and for any compact subset K of U, there exists a constant $C = C_K$, depending only on K, such that for all multi-indices $\alpha \in \mathbb{N}^N$ and for all $x \in K$

$$|\partial^\alpha f(x)| \leq C_K^{|\alpha|+1} (\alpha!)^s. \tag{1.3}$$

Denote by $\bar{U}$ the closure of U in $\mathbb{R}^N$. we say $f \in G^s(\bar{U})$ if $f \in G^s(W)$ for some open neighborhood W of $\bar{U}$. The estimate (1.3) for $x \in K$ is valid if and only if the following one is valid (cf.Chen hua-Rodino[5] or Rodino[17]):

$$\|\partial^\alpha f\|_{L^2(K)} \leq C_K^{|\alpha|+1} (|\alpha|)^{s|\alpha|}.$$

In this paper, we use the above estimate in L^2.

We say an operator P is G^s hypoelliptic in U if $u \in \mathscr{D}', Pu \in G^s(U)$ implies $u \in G^s(U)$. Likewise, we say an operator P is C^∞ hypoelliptic in U if $u \in \mathscr{D}', Pu \in C^\infty(U)$ implies $u \in C^\infty(U)$.

The operator $\mathcal{L}$ satisfies the Hörmander' condition. By virtue of the results of Hörmander [5], we know that $\mathcal{L}$ is C^∞ hypoelliptic, and Morimoto-Xu [14] have proved that $\mathcal{P}$ is also C^∞ hypoelliptic if $1/3 < \sigma \leq 1$. In the context of Gevrey class, Derridj-Zuily [6] proved that $\mathcal{L}$ is G^s-hypoelliptic for $s > 6$ in a general form of Hörmander's operators.

In this paper, we improve firstly the results of [6] for Fokker-Planck operators as the following theorems.

Theorem 1.1. *For any $s \geq 3$, the operator $\mathcal{L}$ given in (1.2) is G^s hypoelliptic in $\mathbb{R}^{2n+1}$, provided the coefficient a is in $G^s(\mathbb{R}^{2n+1})$.*

Of course, Theorem 1.1 is also true for the following general operators,

$$\tilde{\mathcal{L}} = \partial_t + A(v) \cdot \partial_x - \sum_{j,k=1}^n a_{jk}(t, x, v) \partial^2_{v_j v_k}$$

in an open domain U of $\mathbb{R}^{2n+1}$, where A is a non singular $n\times n$ constant matrix, $\big(a_{jk}(t,x,v)\big)$ is positive defined on U and belongs to $G^s(U)$.

Remark Our results is a local and interior regularity results, that means if there exists a weak solution in $\mathscr{D}'$, then this solution is in Gevrey class in interior of domain. So that if the weak solution is a solution of the Cauchy problem, we don't need the regularity of initial data, and there exists not the problem of weight as in [14] where they consider the global problem for the velocity variable v.

We consider now the semi-linear equation

$$\partial_t u + v\cdot\nabla_x u - a\triangle_v u = F(t,x,v,u) \tag{1.4}$$

where F is nonlinear function of real variable (t,x,v,s).

Theorem 1.2. *Let $u\in L^\infty_{loc}(\mathbb{R}^{2n+1})$ be a weak solution of equation (1.4), then*

$$u\in G^s(\mathbb{R}^{2n+1})$$

for any $s>3$, provided the coefficients a is in $G^s(\mathbb{R}^{2n+1})$ and nonlinear function $F(t,x,v,s)$ is in $G^3(\mathbb{R}^{2n+2})$.

The plan of this paper is as follows : In section 2, we obtain a sharp subelliptic estimate for the Fokker-Planck operator $\mathcal{L}$ via direct computation, and then prove the Gevrey hypoellipticity of $\mathcal{L}$. In section 3, we prove the Gevrey regularity for the solutions of the semi-linear Fokker-Planck equation.

2. Subelliptic estimate

We recall firstly some notations, $\|\cdot\|_\kappa, \kappa\in\mathbb{R}$, is the classical Sobolev norm in $H^\kappa(\mathbb{R}^{2n+1})$, and $(h,\ k)$ is the inner product of $h,k\in L^2(\mathbb{R}^{2n+1})$. Moreover if $f,g\in C_0^\infty(\mathbb{R}^{2n+1})$, from Hölder inequality and Young inequality, for any $\varepsilon>0$,

$$|(f,\ g)|\le \|h\|_\kappa\|g\|_{-\kappa}\le \frac{\varepsilon\|h\|_\kappa^2}{2}+\frac{\|g\|_{-\kappa}^2}{2\varepsilon}. \tag{2.1}$$

We have also the interpolation inequality for Sobolev space, for any $\varepsilon>0$ and any $0<r_1<r_2$,

$$\|h\|_{r_1}\le \varepsilon\|h\|_{r_2}+C_\varepsilon\|h\|_0. \tag{2.2}$$

Let Ω be an open subset of $\mathbb{R}^{2n+1}$. We denote by $S^m=S^m(\Omega), m\in\mathbb{R}$, the symbol space of classical pseudo-differential operator and $P=P(t,x,v,D_t,D_x,D_v)\in \mathrm{Op}(S^m)$ a pseudo-differential operator of symbol $p(t,x,v;\tau,\xi,\eta)\in S^m$. If $P\in\mathrm{Op}(S^m)$, then P is a continuous operator from $H^\kappa_c(\Omega)$ to $H^{\kappa-m}_{loc}(\Omega)$. Here $H^\kappa_c(\Omega)$ is the subspace of $H^\kappa(R^{2n+1})$ consisting of the distributions having their compact support in Ω, and $H^{\kappa-m}_{loc}(\Omega)$ consists of the distributions h such that $\phi h\in H^{\kappa-m}(\mathbb{R}^{2n+1})$ for any $\phi\in C_0^\infty(\Omega)$. The more properties can be found in the Treves' book [19]. Remark that if $P_1\in\mathrm{Op}(S^{m_1})$, $P_2\in\mathrm{Op}(S^{m_2})$, then $[P_1,\ P_2]\in\mathrm{Op}(S^{m_1+m_2-1})$.

Now we show a sharp subelliptic estimate for the operator $\mathcal{L}$, our proof bases on the work of Bouchut [4] and Morimoto-Xu [14].

Proposition 2.1. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $r \geq 0$, there exists a constant $C_{K,r}$, depending only on K and r, such that for any $f \in C_0^\infty(K)$,*

$$\|f\|_r \leq C_{K,r}\{\ \|\mathcal{L}f\|_{r-2/3} + \|f\|_0\ \}. \tag{2.3}$$

To simplify the notation, in this section we will denote by C_K the different suitable constants depending only on K. We have firstly the following three lemmas, which establish the gain of regularity in the velocity variable v, in the space variable x and in the time variable t, respectively.

Lemma 2.1. *For any $r \geq 0$, there exists a constant $C_{K,r}$ such that for any $f \in C_0^\infty(K)$,*

$$\|\nabla_v f\|_r \leq C_{K,r}(\|\mathcal{L}f\|_r + \|f\|_r).$$

We get a gain of regularity of order 1 for v variable. This is obtained directly by the positivity of coefficient a and compact support of f. For the space variable x, we have also the following subelliptic estimate.

Lemma 2.2. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\|D_x^{2/3} f\|_0 \leq C_K(\|\mathcal{L}f\|_0 + \|f\|_0),$$

where $D_x^{2/3} = (-\triangle_x)^{1/3}$.

This is a resluts of [4], and it is deduced by follwing two estimates

$$\|D_x^{2/3} f\|_0 \leq C_K\ \|\triangle_v f\|_0^{1/3} \|\partial_t f + v \cdot \partial_x f\|_0^{2/3},$$

and

$$\|\triangle_v f\|_0 \leq C_K(\ \|\mathcal{L}f\|_0 + \|f\|_0\ ).$$

For the time variable t, we have alos a gain of regularity of order 2/3.

Lemma 2.3. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\|\partial_t f\|_{-1/3} \leq C_K(\|\mathcal{L}f\|_0 + \|f\|_0).$$

In fact, we have

$$\|\partial_t f\|_{-1/3} = \|\Lambda^{-1/3}\partial_t f\|_0 \leq \|\Lambda^{-1/3}(\partial_t + v \cdot \partial_x) f\|_0 + \|\Lambda^{-1/3} v \cdot \partial_x f\|_0,$$

where $\Lambda = (1 + |D_t|^2 + |D_x|^2 + |D_v|^2)^{1/2}$. From Lemma 2.2, we have

$$\|\Lambda^{-1/3} v \cdot \partial_x f\|_0 \leq C_K \|D_x^{2/3} f\|_0 \leq C_K(\|\mathcal{L}f\|_0 + \|f\|_0).$$

The estimation for the term $\|\Lambda^{-1/3}(\partial_t + v \cdot \partial_x) f\|_0$ can be obtained by direct calculus as in [14].

Proof of proposition 2.1. The Lemma 2.1, Lemma 2.2 and Lemma 2.3 deduce immediately

$$\|f\|_{2/3} \quad \leq \quad C_K\{\ \|\mathcal{L}f\|_0 + \|f\|_0\ \}. \tag{2.4}$$

Moreover, choose a function $\psi \in C_0^\infty(\mathbb{R}^{2n+1})$ such that $\psi|_K \equiv 1$, Supp ψ is a neigbourhood of K. Then for any $f \in C_0^\infty(K)$ and any $r \geq 0$,

$$\|f\|_r \quad = \quad \|\psi f\|_r \leq C_K\{\ \|\psi \Lambda^{r-2/3} f\|_{2/3} + \|[\Lambda^{r-2/3},\ \psi] f\|_{2/3}\ \}.$$

By virtue of (2.4) and the interpolation inequality (2.2), we have

$$\begin{aligned}\|f\|_r &\leq C_K\{\ \|\mathcal{L}\psi\Lambda^{r-2/3}f\|_0+\|f\|_{r-2/3}\ \}\\ &\leq C_{\varepsilon,K}\{\ \|\mathcal{L}\psi\Lambda^{r-2/3}f\|_0+\|f\|_0\ \}+\varepsilon\|f\|_r.\end{aligned}$$

Taking ε small enough, we get

$$\|f\|_r\leq C_K\{\ \|\mathcal{L}f\|_{r-2/3}+\|f\|_0+\|[\mathcal{L},\ \psi\Lambda^{r-2/3}]f\|_0\ \}.$$

Direct verification gives

$$\begin{aligned}[\mathcal{L},\ \psi\Lambda^{r-2/3}] &= [\partial_t+v\cdot\partial_x,\ \psi\Lambda^{r-2/3}]-\sum_{j=1}^n\{\ \|[a,\ \psi\Lambda^{r-2/3}]\partial^2_{v_j}\\ &\qquad +a[\partial_{v_j},\ [\partial_{v_j},\psi\Lambda^{r-2/3}]\ ]+2a[\partial_{v_j},\ \psi\Lambda^{r-2/3}]\partial_{v_j}\ \},\end{aligned}$$

This along with Lemma 2.1 yields

$$\begin{aligned}\|[\mathcal{L},\ \psi\Lambda^{r-2/3}]f\|_0 &\leq C_K\{\ \|f\|_{r-2/3}+\sum_{j=1}^n\|\partial_{v_j}f\|_{r-2/3}\ \}\\ &\leq C_K\{\ \|\mathcal{L}f\|_{r-2/3}+\|f\|_{r-2/3}\ \}.\end{aligned}$$

These three estimates gives immediately

$$\|f\|_r\leq C_K\{\ \|\mathcal{L}f\|_{r-2/3}+\|f\|_0+\|f\|_{r-2/3}\ \}.$$

Applying interpolation inequality (2.2) again and taking ε small enough, we prove Proposition 2.1.

We consider now the commutators of the operators $\mathcal{L}$ with derivation and cut-off function.

Proposition 2.2. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $r\geq 0$, there exist constants $C_{K,r}, C_{K,r,\varphi}$ such that for any $f\in C_0^\infty(K)$,*

$$\|[\mathcal{L},\ D]f\|_r\ \leq\ C_{K,r}\{\ \|\mathcal{L}f\|_{r+1-2/3}+\|f\|_0\ \},$$

and

$$\|[\mathcal{L},\ \varphi]f\|_r\leq C_{K,r,\varphi}\{\ \|\mathcal{L}f\|_{r-1/3}+\|f\|_0\ \},$$

where $\varphi\in C_b^\infty(\mathbb{R}^{2n+1})$ and we denote by D the differential operator ∂_t,∂_x or ∂_v.

Proof. By using the positivity of coefficient a, we have

$$\|\triangle_v f\|_r\ \leq\ C_K\{\ \|\mathcal{L}f\|_r+\|f\|_{r+1}\ \}.$$

And $[\mathcal{L},\ D]=[\partial_t+v\cdot\partial_x,\ D]-[a,\ D]\triangle_v$ deduce

$$\|[\mathcal{L},\ D]f\|_r\leq C_K\{\ \|f\|_{r+1}+\|\triangle_v f\|_r\ \}.$$

The above two inequalities along with the subelliptic estimate (2.3) yiled the first desired inequality in Proposition 2.2.

To treat $\|[\mathcal{L},\ \varphi]f\|_r$, the subelliptic estimate (2.3) give

$$\|\nabla_v f\|_r\leq C_K(\|\mathcal{L}f\|_{r-1/3}+\|f\|_0).$$

Now simple verification gives

$$\begin{aligned}\|[\mathcal{L},\ \varphi]f\|_r &\leq C_K\{\ \|f\|_r+\sum_{j=1}^{n}\ \|\partial_{v_j}f\|_r\ \}\\ &\leq C_{K,r}\{\ \|\mathcal{L}f\|_{r-1/3}+\|f\|_0\ \}.\end{aligned}$$

This completes the proof of Proposition 2.2.

We prove now the Gevrey hypoellipticity of $\mathcal{L}$. Our starting point is the following result due to M.Durand [8]:

Proposition 2.3. *Let P be a linear differential operator with smooth coefficients in $\mathbb{R}_y^N$ and ϱ, ς two fixed positive numbers. If for any $r \geq 0$, any compact $K \subseteq \mathbb{R}^N$ and any $\varphi \in C^\infty(\mathbb{R}^N)$, there exist constants $C_{K,r}$ and $C_{K,r}(\varphi)$ such that for all $f \in C_0^\infty(K)$,the following conditions are fulfilled:*

$$\|f\|_r \leq C_{K,r}(\|Pf\|_{r-\varrho}+\|f\|_0), \tag{H_1}$$

$$\|[P,\ D_j]f\|_r \leq C_{K,r}(\|Pf\|_{r+1-\varsigma}+\|f\|_0), \tag{H_2}$$

$$\|[P,\ \varphi]f\|_r \leq C_{K,r}(\varphi)(\|Pf\|_{r-\varsigma}+\|f\|_0), \tag{H_3}$$

where

$$D_j=\frac{1}{i}\frac{\partial}{\partial y_j}, j=1,2,\cdots,N.$$

Then for $s \geq \max(1/\varsigma, 2/\varrho), P$ is $G^s(\mathbb{R}^N)$ hypoelliptic, provided the coefficients of P are in the class of $G^s(\mathbb{R}^N)$.

Proposition 2.1 shows that the operator $\mathcal{L}$ satisfies the conditions (H_1) with $\varrho = 2/3$, Proposition 2.2 assures the conditions (H_2) and (H_3) with $\varsigma = 1/3$. Then $\mathcal{L}$ is $G^s(\mathbb{R}^{2n+1})$ hypoelliptic, $s \geq 3$, and we have proved Theorem 1.1.

3. Gevrey regularity of nonlinear equations

The existence and smoothness of weak solution for non-linear Cauchy problems was proved In [14]. Now let $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ be a weak solution of (1.4). Firstly, we will prove $u \in C^\infty(\mathbb{R}^{2n+1})$, and we need the following two lemmas (see for example [22]).

Lemma 3.1. *Let $r > (2n+1)/2$ and $u_1, u_2 \in H^r(\mathbb{R}^{2n+1})$, Then $u_1u_2 \in H^r(\mathbb{R}^{2n+1})$, moreover*

$$\|u_1u_2\|_r \leq \tilde{C}\|u_1\|_r\|u_2\|_r, \tag{3.1}$$

where $\tilde{C}$ is a constant depending only on n, r.

Lemma 3.2. *Let $F(t,x,v,u) \in C^\infty(\mathbb{R}^{2n+1}\times\mathbb{R})$ and $r \geq 0$. If $u \in L^\infty_{loc}(\mathbb{R}^{2n+1}) \cap H^r_{loc}(\mathbb{R}^{2n+1})$, then $F(\cdot, u(\cdot)) \in H^r_{loc}(\mathbb{R}^{2n+1})$.*

Now we are ready to prove

Proposition 3.1. *Let $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ be a weak solution of (1.4). Then u is $C^\infty(\mathbb{R}^{2n+1})$.*

In fact, from the subelliptic estimate (2.3) and the fact $\mathcal{L}u(\cdot) = F(\cdot, u(\cdot))$, it then follows that

$$\|\psi_1 u\|_{r+2/3} \le \bar{C}\{\ \|\psi_2 F(\cdot, u(\cdot))\|_r + \|\psi_2 u\|_0\ \}, \tag{3.2}$$

where $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\psi_2 = 1$ on the support of ψ_1.

Combining Lemma 3.2 and subelliptic estimate (3.2), we have $u \in H^\infty_{loc}(\mathbb{R}^{2n+1})$ by induction. This completes the proof of Proposition 3.1.

Since the Gevrey hypoellipticity is a local property, it suffices to show the Gevrey regularity in the open unit ball

$$\Omega = \{(t, x, v) \in \mathbb{R}^{2n+1} : t^2 + |x|^2 + |v|^2 < 1\}.$$

Set

$$\Omega_\varepsilon = \{(t, x, v) \in \Omega : t^2 + |x|^2 + |v|^2 < 1 - \varepsilon\}.$$

and suppose $\Omega_\varepsilon = \emptyset$ if $\varepsilon \ge 1$. For any $\varepsilon, \varepsilon_1 > 0$ with $\varepsilon + \varepsilon_1 < 1$, take the characteristic function $\chi_{\varepsilon,\varepsilon_1}$ on the set $\Omega_{\varepsilon+\varepsilon_1}$, i.e., $\chi_{\varepsilon,\varepsilon_1}(t, x, v)$ equals to 1 if $(t, x, v) \in \Omega_{\varepsilon+\varepsilon_1}$ and equals to 0 otherwise. Choose a function $\rho \in C_0^\infty(\Omega)$ such that $\int_{\mathbb{R}^{2n+1}} \rho(t, x, v)dtdxdv = 1$. Set $\rho_\varepsilon(t, x, v) = \varepsilon^{-2n-1}\rho(t/\varepsilon, x/\varepsilon, v/\varepsilon)$. Now we define a family of functions $\varphi_{\varepsilon,\varepsilon_1} \in C_0^\infty(\Omega_{\varepsilon_1})$ by setting

$$\varphi_{\varepsilon,\varepsilon_1}(t, x, v) = \rho_{\varepsilon/2} * \chi_{\varepsilon/2,\varepsilon_1}(t, x, v) = \int_{\mathbb{R}^{2n+1}} \rho_{\varepsilon/2}(t - s, x - y, v - w)\chi_{\varepsilon/2,\varepsilon_1}(s, y, w)dsdydw.$$

Then clearly, $\varphi_{\varepsilon,\varepsilon_1}$ satisfies the following property:

$$\left\{ \begin{array}{l} \varphi_{\varepsilon,\varepsilon_1}(t, x, v) = 1, \quad \text{for all } (t, x, v) \in \Omega_{\varepsilon+\varepsilon_1}, \\ \sup\limits_{\Omega_{\varepsilon_1}} |D^\alpha \varphi_{\varepsilon,\varepsilon_1}| \le C_\alpha \varepsilon^{-|\alpha|}. \end{array} \right. \tag{3.3}$$

Let U be an open subset of $\mathbb{R}^{2n+1}$. Denote by $H^r(U)$ the space consisting of the functions which are defined in U and can be extended to $H^r(\mathbb{R}^{2n+1})$. Define

$$\|u\|_{H^r(U)} = \inf\big\{\|\tilde{u}\|_{H^s(\mathbb{R}^{n+1})} : \tilde{u} \in H^s(\mathbb{R}^{2n+1}), \tilde{u}|_U = u\big\}.$$

We denote $\|u\|_{s,U} = \|u\|_{H^r(U)}$.

We denote

$$\|D^j u\|_r = \sum_{|\beta|=j} \|D^\beta u\|_r.$$

In order to treat the nonlinear terms on the right hand of (1.4), we need the following lemma which is an analogue of Lemma 1 in [9]. In the sequel $C_j > 1$ will be used to denote suitable constants depending only on n or the function F.

Lemma 3.3. *Let M_j be a sequence of positive numbers and for some $B_0 > 0$, the M_j satisfy the monotonicity conditions*

$$\frac{j!}{i!(j-i)!} M_i M_{j-i} \le B_0 M_j,\ \ (i = 1, 2, \cdots, j;\ j = 1, 2, \cdots). \tag{3.4}$$

Let $\psi_0 \in C^\infty(\mathbb{R}^{2n+1})$ such that $\psi = 1$ in a neighborhood of Ω and N be any fixed positive integer. Define a family of functions ψ_j by setting

$$\psi_j = \begin{cases} \varphi_{\varepsilon,(N-1)\varepsilon}, & \text{if } 0 < j \le N-1, \\ \varphi_{\varepsilon,N\varepsilon}, & \text{if } j = N,\ N+1. \end{cases} \tag{3.5}$$

Suppose $F(t,x,v,u)$ satisfy

$$\left\|\psi_0 \left(D^k_{t,x,v} D^l_u F\right)(\cdot, u(\cdot))\right\|_{r+n+1} \le C_1^{k+l} M_{k-2} M_{l-2}, \quad \forall\ k+l \le N+1, \tag{3.6}$$

Then there exist two constants C_2, C_3 such that for any H_0, H_1 satisfying $H_0, H_1 \ge 1$ and $H_1 \ge C_2 H_0$, if $u(t,x,v)$ satisfy the following conditions

$$\|\psi_j D^j u\|_{r+n+1} \le H_0, \qquad 0 \le j \le 1, \tag{3.7}$$

$$\|\psi_j D^j u\|_{r+n+1} \le H_0 H_1^{j-2} M_{j-2}, \qquad 2 \le j \le N+1, \tag{3.8}$$

where $r \ge 0$, Then for all α with $|\alpha| = N+1$,

$$\left\|\varphi_{\varepsilon,N\varepsilon} D^\alpha \left[F(\cdot, u(\cdot))\right]\right\|_{r+n+1} \le C_3 H_0 H_1^{N-1} M_{N-1}. \tag{3.9}$$

Proof. From Faa di Bruno, $D^\alpha[F(\cdot,u(\cdot))]$ is the linear combination of terms of the form

$$\varphi_{\varepsilon,N\varepsilon}(D^\beta_{t,x,v}\partial^l_u F) \cdot \prod_{j=1}^{l} D^{\gamma_j} u = (\varphi_0 D^\beta_{t,x,v}\partial^l_u F) \cdot \prod_{j=1}^{l} \psi_{|\gamma_j|} D^{\gamma_j} u, \tag{3.10}$$

where $|\beta| + l \le |\alpha|$ and $\gamma_1 + \gamma_2 + \cdots + \gamma_l = \alpha - \beta$, and if $\lambda_i = 0$, we just mean $D^{\gamma_i} u$ doesn't appear in (3.10). Note that $n+1+r > (2n+1)/2$, and hence applying Lemma 3.1, we have

$$\left\|\varphi_{\varepsilon,N\varepsilon}(D^\beta_{t,x,v}\partial^l_u F) \textstyle\prod_{j=1}^{l} D^{\gamma_j} u\right\|_{r+n+1} \le \tilde{C} \|\psi_0(D^\beta_{t,x,v}\partial^l_u F)\|_{r+n+1} \textstyle\prod_{j=1}^{l} \|\psi_{|\gamma_j|} D^{\gamma_j} u\|_{r+n+1}. \tag{3.11}$$

In virtue of (3.6)-(3.8) and (3.11), the situation is entirely similar to [9], and the Hölder norm $|u|_k$ in [9] is replaced by $\|\psi_k D^k u\|_{r+n+1}$. Then the same argument as the proof of Lemma 1 in [9] yields (3.9). This completes the proof of Lemma 3.3.

Now starting from C^∞ solution, we prove the Gevrey regularity as following proposition.

Proposition 3.2. *Let $s > 3$. Choose s_0 such that $s > s_0 > 3$. Suppose $u(t,x,v) \in C^\infty(\bar{\Omega})$ be a solution of (1.4). Then there exits a constant A such that for any $r \in [0,1]$ and any $k \in \mathbb{N}$,*

$$\|D^\alpha u\|_{r+n+1,\Omega_{k\varepsilon}} + \|D_v D^\alpha u\|_{r-2/3+1/3+n+1,\Omega_{k\varepsilon}} \le A^{|\alpha|-2}\big((|\alpha|-3)!\big)^{s-s_0} \varepsilon^{-s_0|\alpha|+3s_0} \varepsilon^{-s_0 r} \tag{$(I)'_{r,k}$}$$

holds for all α with $|\alpha| \le k$ and for all $\varepsilon > 0$ with $k\varepsilon < 1$.
In particular, for any k, letting $r = 0$, we have for all $|\alpha| \le k$ and for all ε with $k\varepsilon < 1$,

$$\|D^\alpha u\|_{L^2(\Omega_{k\varepsilon})} \le A^{|\alpha|-(n+1)-2}\big((|\alpha|-n-1-2)!\big)^{s-s_0} \varepsilon^{-s_0(|\alpha|-n-1)+3s_0} \le A^{|\alpha|+1} \varepsilon^{-s|\alpha|}. \tag{3.12}$$

Remark. From (3.12), it follows immediately that $u \in G^s(\Omega)$.

Proof. We use the induction on k. Assuming $(I)'_{r,k}$ holds for any r with $0 \le r \le 1$, and we will show $(I)'_{r,k+1}$ still holds for any r with $0 \le r \le 1$. For any $\alpha, |\alpha| \le k+1$, the truth of $(I)'_{r,k+1}$ is reduced to the following

Lemma 3.4. *For any nonnegative integer m satisfying $m \le 4$, we have*

$$\|\varphi_{\varepsilon,k\varepsilon} D^\alpha u\|_{r+n+1} + \|\varphi_{\varepsilon,k\varepsilon} T^\sigma D^\alpha u\|_{r-\delta+\theta+n+1} \le A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-rs_0}. \tag{3.13}$$

holds for all r with $0 \le r \le m/3$.

Proof of Lemma 3.4. We use the induction on m. Firstly we observe that (3.13) holds for $m = 0$. In fact, write $|\alpha| = |\beta| + 1$. Then $|\beta| = k$ and

$$\begin{aligned}\|\varphi_{\varepsilon,k\varepsilon} D^\alpha u\|_{n+1} &\le \|\varphi_{\varepsilon,k\varepsilon} D^\beta u\|_{1+n+1} + \|(D\varphi_{\varepsilon,k\varepsilon})D^\beta u\|_{n+1}\\ &\le C_4\{\ \|D^\beta u\|_{1+n+1,\Omega_{k\varepsilon}} + \varepsilon^{-1}\|D^\beta u\|_{n+1,\Omega_{k\varepsilon}}\ \}.\end{aligned}$$

Since $(I)_{r,k}$ holds for any r with $0 \le r \le 1$, we have immediately

$$\|D^\beta u\|_{1+n+1,\Omega_{k\varepsilon}} + \varepsilon^{-1}\|D^\beta u\|_{n+1,\Omega_{k\varepsilon}} \le 2A^{|\beta|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-rs_0},$$

and taking A large enough such that $A > 4C_4$, we have

$$\|\varphi_{\varepsilon,k\varepsilon} D^\alpha u\|_{n+1} \le \frac{1}{2}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-rs_0}. \tag{3.14}$$

The same arguments as above shows that

$$\|\varphi_{\varepsilon,k\varepsilon} D_v D^\alpha u\|_{-2/3+1/3+n+1} \le \frac{1}{2}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-3}\varepsilon^{-3|\alpha|+9}\varepsilon^{-3r}.$$

Now assuming (3.13) is true for m, we will prove its truth for $m+1$. For any fixed $r, 0 \le r \le (m+1)/3 \le 4/3$, write $r = \tilde r + 1/3$ then $\tilde r \le m/3$. We assume $\tilde r \ge 0$, for the case $\tilde r \le 0$ is trivial. And we will proceed to prove the conclusion by the following four steps.

Step 1. Claim

$$\begin{aligned}&\|[\mathcal{L},\ \varphi_{\varepsilon,k\varepsilon} D^\alpha]u\|_{r-2/3+n+1}\\ \le\ &\big(C_6\varepsilon^{2s_0/3-1} + C_9\varepsilon^{s_0/3-1} + C_{11}\varepsilon^{s_0/3}\big)A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}.\end{aligned} \tag{3.15}$$

In fact, write $\mathcal{L} = X_0 - a\triangle_v$ with $X_0 = \partial_t + v\cdot\partial_v$. Then direct verification deduces

$$\begin{aligned}\|[\mathcal{L},\ \varphi_{\varepsilon,k\varepsilon} D^\alpha]u\|_{r-2/3+n+1} &\le \|[X_0,\ \varphi_{\varepsilon,k\varepsilon} D^\alpha]u\|_{r-2/3+n+1} + \|a[\triangle_v,\ \varphi_{\varepsilon,k\varepsilon} D^\alpha]u\|_{r-2/3+n+1}\\ &\quad + \|\varphi_{\varepsilon,k\varepsilon}[a,\ D^\alpha]\triangle_v u\|_{\tilde r+1/3-2/3+n+1}\\ &=: (I) + (II) + (III).\end{aligned}$$

Denote $[X_0,\ D^\alpha]$ by D^{α_0}. Then $|\alpha_0| \le k+1$ and

$$(I) \ \le\ \|[X_0,\ \varphi_{\varepsilon,k\varepsilon}]D^\alpha u\|_{r-2/3+n+1} + \|\varphi_{\varepsilon,k\varepsilon} D^{\alpha_0} u\|_{r-2/3+n+1}.$$

Note that $r-2/3\le m/3$ and hence

$$\begin{aligned}(I) &\le C_5\varepsilon^{-1}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0(r-2/3)}\\ &\quad +A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0(r-2/3)}\\ &\le C_6\varepsilon^{2s_0/3-1}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}.\end{aligned}\tag{3.16}$$

Now we will estimate (II). It is easy to see that

$$\begin{aligned}\|[\triangle_v,\ \varphi_{\varepsilon,k\varepsilon}]D^\alpha u\|_{r-2/3+n+1} &\le 2\|[D_v,\ \varphi_{\varepsilon,k\varepsilon}]D_vD^\alpha u\|_{\tilde r+1/3-2/3+n+1}\\ &\quad+\|[D_v,\ [D_v,\ \varphi_{\varepsilon,k\varepsilon}]\ ]D^\alpha u\|_{r-2/3+n+1}.\end{aligned}$$

We firstly treat the first term on the right hand, and

$$\|[D_v,\ \varphi_{\varepsilon,k\varepsilon}]D_vD^\alpha u\|_{\tilde r+1/3-2/3+n+1}\le C_7\varepsilon^{-1}\|\varphi_{\varepsilon,k\varepsilon}D_vD^\alpha u\|_{\tilde r+1/3-2/3+n+1}.$$

Note that $\tilde r\le m\theta$ and hence from the assumption that (3.13) is true for m, it follows immediately

$$\|\varphi_{\varepsilon,k\varepsilon}D_vD^\alpha u\|_{\tilde r+1/3-2/3+n+1}\le A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0\tilde r}.$$

Combining the above two inequalities, we have

$$\|[D_v,\ \varphi_{\varepsilon,k\varepsilon}]D_vD^\alpha u\|_{\tilde r+1/3-2/3+n+1}\le C_7\varepsilon^{s_0/3-1}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}.$$

Similarly, we can estimate the second term

$$\begin{aligned}\|[D_v,\ [D_v,\ \varphi_{\varepsilon,k\varepsilon}]\ ]D^\alpha u\|_{r-2/3+n+1} &\le C_8\varepsilon^{-2}\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{r-2/3}\\ &\le C_8\varepsilon^{2s_0/3-2}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}\\ &\le C_8\varepsilon^{s_0/3-1}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}.\end{aligned}$$

Hence we have

$$(II)\le C_9\varepsilon^{s_0/3-1}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}.\tag{3.17}$$

Using Leibniz formula,

$$(III)\le\sum_{0\le|\gamma|<|\alpha|}\Big(\begin{array}{c}\alpha\\ \gamma\end{array}\Big)\big\|\varphi_{\varepsilon,k\varepsilon}(D^{\alpha-\gamma}a)\triangle_vD^\gamma u\big\|_{\tilde r+1/3-2/3}.$$

Since $a\in G^3(\mathbb{R}^{2n+1})\subset G^{s_0}(\mathbb{R}^{2n+1})$, so $D^2u\in G^{s_0}(\mathbb{R}^{2n+1})$. Then

$$\sup_\Omega|D^{\alpha-\gamma}a|=\sup_\Omega|D^{\alpha-\gamma-2}D^2a|\le C_{10}^{|\alpha-\gamma|-2}\big((|\alpha|-|\gamma|-2)!\big)^{s_0}.$$

The above two inequalities yield

$$(III)\le C_{10}^{|\alpha|}\sum_{0\le|\gamma|<|\alpha|}\Big(\begin{array}{c}\alpha\\ \gamma\end{array}\Big)C_{10}^{-|\gamma|}\big((|\alpha|-|\gamma|-2)!\big)^{s_0}\big\|\varphi_{\varepsilon,k\varepsilon}\triangle_vD^\gamma u\big\|_{\tilde r+1/3-2/3+n+1}.$$

Moreover, for each $\gamma, 0 \le |\gamma| \le |\alpha| - 1 = k$,

$$\begin{aligned}
\|\varphi_{\varepsilon,k\varepsilon} \triangle_v D^\gamma u\|_{\tilde{r}+1/3-2/3+n+1} &\le \|\varphi_{\varepsilon,k\varepsilon} D_v D^{\gamma+1} u\|_{\tilde{r}+1/3-2/3+n+1} \\
&\le A^{|\gamma+1|-2}\big((|\gamma+1|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\gamma+1|+3s_0}\varepsilon^{-s_0\tilde{r}} \\
&\le \varepsilon^{s_0/3}A^{|\gamma|-1}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\gamma|+2s_0}\varepsilon^{-s_0 r}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(III) \\
\le\ & C_{10}^{|\alpha|-2}\sum_{0\le|\gamma|<|\alpha|}\left(\begin{array}{c}\alpha\\ \gamma\end{array}\right)C_{10}^{|\alpha|-2}\big((|\alpha|-|\gamma|-2)!\big)^{s_0}\varepsilon^{s_0/3}A^{|\gamma|-1}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\gamma|+2s_0}\varepsilon^{-s_0 r} \\
\le\ & C_{10}^{|\alpha|-2}A^{-1}\varepsilon^{s_0/3}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0 r}\sum_{0\le|\gamma|<|\alpha|}\frac{\alpha!\big((|\alpha|-|\gamma|-2)!\big)^{s_0-1}}{\gamma!}(A/C_{10})^{|\gamma|}\varepsilon^{-s_0|\gamma|+2s_0} \\
\le\ & C_{10}^{|\alpha|-2}A^{-1}\varepsilon^{s_0/3}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0 r} \\
&\times\sum_{0\le|\gamma|<|\alpha|}|\alpha|^{|\alpha|-|\gamma|}\big(|\alpha|-|\gamma|-2\big)^{(s_0-1)(|\alpha|-|\gamma|-2)}(A/C_{10})^{|\gamma|}\varepsilon^{-s_0|\gamma|+2s_0} \\
\le\ & C_{10}^{|\alpha|-2}A^{-1}\varepsilon^{s_0/3}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0 r} \\
&\times\sum_{0\le|\gamma|<|\alpha|}|\alpha|^{|\alpha|-|\gamma|}\varepsilon^{-s_0|\alpha|+s_0|\gamma|+2s_0}|\alpha|^{-(|\alpha|-|\gamma|-2)}(A/C_{10})^{|\gamma|}\varepsilon^{-s_0|\gamma|+2s_0} \\
\le\ & C_{10}^{|\alpha|-2}A^{-1}\varepsilon^{s_0/3+s_0}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}\sum_{0\le|\gamma|<|\alpha|}(A/C_{10})^{|\gamma|}|\alpha|^2 \\
\le\ & C_{11}C_{10}^{|\alpha|-2}A^{-1}\varepsilon^{s_0/3+s_0}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}(A/C_{10})^{|\alpha|-1}|\alpha|^3 \\
\le\ & C_{11}\varepsilon^{s_0/3+s_0}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}\varepsilon^{-3} \\
\le\ & C_{11}\varepsilon^{s_0/3+s_0-3}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r} \\
\le\ & C_{11}\varepsilon^{s_0/3}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}.
\end{aligned}$$

This along with (3.16) and (3.17) yields the conclusion (3.15).

Step 2. Claim

$$\|\varphi_{\varepsilon,k\varepsilon}D^\alpha[F(\cdot,u(\cdot))]\|_{r-2/3+n+1} \le C_{14}\varepsilon^{2s_0/3}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0 r}. \tag{3.18}$$

Firstly, we will prove F and u satisfy the condition (3.6)-(3.8) for some M_j. Since $r-\delta \le m\theta$, and then from the assumption, we have

$$\|\psi_j D^j u\|_{r-2/3+n+1} \le A^{j-2}\big((j-2)!\big)^{s-s_0}\varepsilon^{-s_0 j+3s_0}\varepsilon^{-s_0(r-2/3)}, \quad j=0,1,\cdots,|\alpha|, \tag{3.19}$$

where ψ_j are the functions defined in (3.5). Moreover, for any $|\beta|+l \le |\alpha| < \varepsilon^{-1}$, it is easy to verify

$$\big((|\beta|+l-6)!\big)^{s_0} \le (|\beta|+l-6)^{s_0(|\beta|+l-6)} \le \varepsilon^{-s_0(|\beta|+l-6)}.$$

Since $F \in G^s(\mathbb{R}^{2n+1}\times\mathbb{R})$, then $D^{n+8}_{t,x,v;u}F \in G^s(\mathbb{R}^{2n+1}\times\mathbb{R})$. This along with the above inequality shows that

$$\|\psi_0(D^{\beta}_{t,x,v}\partial^l_u F)\|_{r-2/3+n+1} \le C_{12}^{|\beta|+l+1}\big((|\beta|+l-6)!\big)^{s-s_0}\varepsilon^{-s_0(|\beta|+l-6)}. \tag{3.20}$$

Define M_j, H_0, H_1 by setting $M_j=(j!)^{s-s_0}\varepsilon^{-s_0(j-1)}\varepsilon^{-s_0(r-2/3)}, H_0=\|u\|_{H^{n+3}(\Omega)}+1$ and $H_1=A$, respectively. We can choose A large enough such that $H_1=A\ge C_2H_0$. Then (3.19) and (3.20) can be rewritten

$$\|\psi_j D^j u\|_{r-2/3+n+1}\le H_0, \qquad 0\le j\le 1 \tag{3.21}$$

$$\|\psi_j D^j u\|_{r-2/3+n+1}\le H_0H_1^{j-2}M_{j-2}, \qquad 2\le j\le|\alpha|\le k+1, \tag{3.22}$$

$$\|\psi_0(D^{j}_{t,x,v}\partial^l_u F)\|_{r-2/3+n+1}\le C_{13}^{j+l+1}M_{j-2}M_{l-2}. \tag{3.23}$$

In proving (3.23) we used the inequality

$$\big(p!\big)^s\big(q!\big)^s\le\big((p+q)!\big)^s\le 2^{s(p+q)}\big(p!\big)^s\big(q!\big)^s, \qquad \forall\ p,\ q\in\mathbb{N}.$$

For each j, we compute

$$\begin{aligned}\frac{j!}{i!(j-i)!}M_iM_{j-i} &= j!\big(i!\big)^{s-s_0}\big((j-i)!\big)^{s-s_0}\varepsilon^{-s_0(i-1)}\varepsilon^{-s_0(r-\delta)}\varepsilon^{-s_0(j-i-1)}\varepsilon^{-s_0(r-\delta)}\\ &\le (j!)^{s-s_0}\varepsilon^{-s_0(j-1)}\varepsilon^{-s_0(r-\delta)}\varepsilon^{s_0-s_0(r-\delta)}\le M_j.\end{aligned}$$

In the last inequality we used the fact $r-\delta\le 1$. Thus M_j satisfy the monotonicity condition (3.4). In virtue of (3.21)-3.23), using Lemma 3.3, we have

$$\begin{aligned}\|\varphi_{\varepsilon,k\varepsilon}D^\alpha[F(\cdot,u(\cdot))]\|_{r-\delta+n+1} &\le C_3H_0H_1^{|\alpha|-2}M_{|\alpha|-2}\\ &\le \big(C_3H_0\varepsilon^{2s_0/3}\big)A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}\\ &= \big(C_{14}\varepsilon^{2s_0/3}\big)A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}.\end{aligned}$$

This completes the proof of conclusion (3.18).

Step 3. Claim $\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{r+n+1}\le\frac14 A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}$.

In fact, applying the subelliptic estimate (2.3), we obtain

$$\begin{aligned}&\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{r+n+1}\le C_{15}\{\ \|\mathcal{L}\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{r-2/3+n+1}+\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{n+1}\ \}\\ \le\ & C_{15}\{\ \|[\mathcal{L},\ \varphi_{\varepsilon,k\varepsilon}D^\alpha]u\|_{r-2/3+n+1}+\|\varphi_{\varepsilon,k\varepsilon}D^\alpha\mathcal{L}u\|_{r-2/3+n+1}+\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{n+1}\ \}\\ =\ & C_{15}\{\|[\mathcal{L},\ \varphi_{\varepsilon,k\varepsilon}D^\alpha]u\|_{r-2/3+n+1}+\|\varphi_{\varepsilon,k\varepsilon}D^\alpha[F\big(\cdot,u(\cdot)\big)]\|_{r-2/3+n+1}\}+C_{15}\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{n+1}.\end{aligned}$$

For the third term on the right hand, we have proved at the beginning of the proof of Lemma 3.4 that

$$C_{15}\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{n+1}\le 2C_4C_{15}A^{|\beta|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-rs_0}.$$

Letting A large enough such that $A>16C_4C_{15}$, then we have

$$C_{15}\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{n+1}\le\frac18 A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-rs_0}.$$

Combing (3.15) and (3.18), we have

$$\begin{aligned}
&C_{15}\{\|[\mathcal{L},\ \varphi_{\varepsilon,k\varepsilon}D^\alpha]u\|_{r-2/3+n+1}+\|\varphi_{\varepsilon,k\varepsilon}D^\alpha[F\big(\cdot,u(\cdot)\big)]\|_{r-2/3+n+1}\}\\
\le\ & C_{16}\big(\varepsilon^{2s_0/3-1}+\varepsilon^{s_0/3-1}+\varepsilon^{s_0/3}+\varepsilon^{2s_0/3}\big)A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}\\
\le\ & 2C_{16}\big(\varepsilon^{s_0/3-1}+\varepsilon^{s_0/3}\big)A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}.
\end{aligned}$$

Since $s_0/3-1>0$, we can find an integer N such that

$$32C_{16}\le N^{s_0/3-1}\le N^{s_0/3}. \tag{3.24}$$

We can choose A large enough such that $A\ge\|u\|_{H^{2n+3+N}(\Omega)}+1$. Then the conclusion obviously holds for any $|\alpha|$ with $|\alpha|\le n+1+N$. So we only need consider the case $|\alpha|>n+1+N$. Since $N<|\alpha|<\varepsilon^{-1}$, then combining (3.24), we have

$$2C_{16}\big(\varepsilon^{s_0/3-1}+\varepsilon^{s_0/3}\big)\le\frac{1}{16}(N\varepsilon)^{s_0/3-1}+\frac{1}{16}(N\varepsilon)^{s_0/3}\le\frac{1}{8}.$$

Consequently,

$$\begin{aligned}
&C_{15}\{\|[\mathcal{L},\ \varphi_{\varepsilon,k\varepsilon}D^\alpha]u\|_{r-2/3+n+1}+\|\varphi_{\varepsilon,k\varepsilon}D^\alpha[F\big(\cdot,u(\cdot)\big)]\|_{r-2/3+n+1}\}\\
\le\ & \frac{1}{8}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}.
\end{aligned}$$

This completes the proof of the conclusion.

Step 4. Claim $\|\varphi_{\varepsilon,k\varepsilon}D_vD^\alpha u\|_{r-2/3+1/3+n+1}\le\frac{3}{4}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}$.
It is clear

$$\|\varphi_{\varepsilon,k\varepsilon}D_vD^\alpha u\|_{r-2/3+1/3+n+1}\le\|D_v\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{r-1/3+n+1}+\|[D_v,\ \varphi_{\varepsilon,k\varepsilon}]D^\alpha u\|_{r-1/3+n+1}.$$

Firstly, we treat the first term on the right. By direct calculation, it follows that

$$\begin{aligned}
&\|D_v\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|^2_{r-1/3+n+1}\\
=\ & \mathrm{Re}(\mathcal{L}\varphi_{\varepsilon,k\varepsilon}D^\alpha u,\ a^{-1}\Lambda^{2r-2/3+2n+2}\varphi_{\varepsilon,k\varepsilon}D^\alpha u)-\mathrm{Re}(X_0\varphi_{\varepsilon,k\varepsilon}D^\alpha u,\ a^{-1}\Lambda^{2r-2/3+2n+2}\varphi_{\varepsilon,k\varepsilon}D^\alpha u)\\
=\ & \mathrm{Re}(\mathcal{L}\varphi_{\varepsilon,k\varepsilon}D^\alpha u,\ a^{-1}\Lambda^{2r-2/3+2n+2}\varphi_{\varepsilon,k\varepsilon}D^\alpha u)-\frac{1}{2}(\varphi_{\varepsilon,k\varepsilon}D^\alpha u,\ [a^{-1}\Lambda^{2r-2/3+2n+2},\ X_0]\varphi_{\varepsilon,k\varepsilon}D^\alpha u)\\
&-\frac{1}{2}(\varphi_{\varepsilon,k\varepsilon}D^\alpha u,\ [\Lambda^{2r-2/3+2n+2},\ a^{-1}]X_0\varphi_{\varepsilon,k\varepsilon}D^\alpha u)\\
\le\ & C_{17}\{\ \|\mathcal{L}\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|^2_{r-2/3+n+1}+\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|^2_{r+n+1}\ \}\\
\le\ & C_{18}\{\ \|\mathcal{L}\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|^2_{r-2/3+n+1}+\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|^2_{n+1}\ \}.
\end{aligned}$$

Similar to the case in Step 3, we have

$$\|D_v\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{r-1/3+n+1}\le\frac{1}{4}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}. \tag{3.25}$$

Since $r-1/3\leq m/3$, and then from the assumption, we have

$$\begin{aligned} &\|[D_v,\ \varphi_{\varepsilon,k\varepsilon}]D^\alpha u\|_{r-1/3+n+1}\\ \leq\ & C_{19}\varepsilon^{-1}\|\varphi_{\varepsilon,k\varepsilon}D^\alpha u\|_{r-1/3+n+1} \qquad (3.26)\\ \leq\ & C_{19}\varepsilon^{s_0/3-1}A^{|\alpha|-2}\big((|\alpha|-2)!\big)^{s-s_0}\varepsilon^{-s_0|\alpha|+3s_0}\varepsilon^{-s_0r}. \end{aligned}$$

Note that $s_0/3-1>0$ and hence similar to the case in Step 3, the last term on the right can be bounded by $\frac{1}{2}A^{|\alpha|+1}\varepsilon^{-s|\alpha|}\varepsilon^{-rs}$. Combination of (3.25) and (3.26) gives the conclusion.

From Step 3 and Step 4, we know that (3.13) is truth for $m+1$, and hence for any m with $m\theta<1+\theta$. This completes the proof of Lemma 3.4.

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