

## PROPAGATION OF GEVREY REGULARITY FOR SOLUTIONS OF LANDAU EQUATIONS

HUA CHEN

School of Mathematics and Statistics, Wuhan University  
Wuhan 430072, China

WEI-XI LI

School of Mathematics and Statistics, Wuhan University  
Wuhan 430072, China

CHAO-JIANG XU

School of Mathematics and Statistics, Wuhan University  
Wuhan 430072, China

and

Université de Rouen, UMR 6085-CNRS, Mathématiques  
Avenue de l'Université, BR.12,76801 Saint Etienne du Rouvray, France

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**ABSTRACT.** By using the energy-type inequality, we obtain, in this paper, the result on propagation of Gevrey regularity for the solution of the spatially homogeneous Landau equation in the cases of Maxwellian molecules and hard potential.

**1. Introduction.** There are many papers concerning the propagation of regularity for the solution of the Boltzmann equation (cf. [5, 6, 8, 9, 13] and references therein). In these works, it has been shown that the Sobolev or Lebesgue regularity satisfied by the initial datum is propagated along the time variable. The solutions having the Gevrey regularity for a finite time have been constructed in [15] in which the initial data has the same Gevrey regularity. Recently, the uniform propagation in all time of the Gevrey regularity has been proved in [4] in the case of Maxwellian molecules, which was based on the Wild expansion and the characterization of the Gevrey regularity by the Fourier transform.

In this paper, we study the propagation of Gevrey regularity for the solution of Landau equation, which is the limit of the Boltzmann equation when the collisions become grazing, see [3] for more details. Also we know that the Landau equation can be regarded as a non-linear and non-local analog of the hypo-elliptic Fokker-Planck equation, and if we choose a suitable orthogonal basis, the Landau equation in the Maxwellian molecules case will become a non-linear Fokker-Planck equation (cf. [17]). Recently, a lot of progress on the Sobolev regularity has been made for the spatially homogeneous and inhomogeneous Landau equations, cf. [2, 6, 7, 16] and references therein. On the other hand, in the Gevrey class frame, the local Gevrey

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regularity for all variables  $t, x, v$  is obtained in [1] for some semi-linear Fokker-Planck equations.

Let us consider the following Cauchy problem for the spatially homogeneous Landau equation,

$$\begin{cases} \partial_t f = \nabla_v \cdot \left\{ \int_{\mathbb{R}^n} a(v - v_*) [f(v_*) \nabla_v f(v) - f(v) \nabla_v f(v_*)] dv_* \right\}, \\ f(0, v) = f_0(v), \end{cases} \tag{1}$$

where  $f(t, v) \geq 0$  stands for the density of particles with velocity  $v \in \mathbb{R}^n$  at time  $t \geq 0$ , and  $(a_{ij})$  is a nonnegative symmetric matrix given by

$$a_{ij}(v) = \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right) |v|^{\gamma+2}, \quad \gamma \in [0, 1]. \tag{2}$$

Here and throughout the paper, we consider only the hard potential case (i.e.  $\gamma \in (0, 1]$ ) and the Maxwellian molecules case (i.e.  $\gamma = 0$ ).

Set

$$b_i = \sum_{j=1}^n \partial_{v_j} a_{ij} = -2 |v|^\gamma v_i, \quad i = 1, 2, \dots, n; \quad c = \sum_{i,j=1}^n \partial_{v_i v_j} a_{ij} = -2(\gamma + 3) |v|^\gamma,$$

$$\bar{a}_{ij}(t, v) = (a_{ij} * f)(t, v) = \int_{\mathbb{R}^n} a_{ij}(v - v_*) f(t, v_*) dv_*, \quad \bar{b}_i = b_i * f, \quad \bar{c} = c * f.$$

Then the Cauchy problems (1) can be rewritten in the following form:

$$\begin{cases} \partial_t f = \sum_{i,j=1}^n \bar{a}_{ij} \partial_{v_i v_j} f - \bar{c} f, \\ f(0, v) = f_0(v), \end{cases} \tag{3}$$

which is a non-linear diffusion equation with the coefficients  $\bar{a}_{ij}$  and  $\bar{c}$  depending on the solution  $f$ .

The motivation for studying the Cauchy problem (3) (cf. [12]) comes from the study of the inhomogenous Boltzmann equations without angular cutoff and non linear Vlasov-Fokker-Planck equation (see [10, 11]).

Throughout the paper, for a multi-index  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  and an integer  $k$  with  $0 \leq k \leq |\alpha|$ , the notation  $D^{|\alpha|-k}$  is always used to denote  $\partial_v^\gamma$  with the multi-index  $\gamma$  satisfying  $\gamma \leq \alpha$  and  $|\gamma| = |\alpha| - k$ . We denote also by  $M(f(t)), E(f(t))$  and  $H(f(t))$  as the mass, energy and entropy respectively for the function  $f(t, \cdot)$ . That is,

$$M(f(t)) = \int_{\mathbb{R}^n} f(t, v) dv, \quad E(f(t)) = \frac{1}{2} \int_{\mathbb{R}^n} f(t, v) |v|^2 dv,$$

$$H(f(t)) = \int_{\mathbb{R}^n} f(t, v) \log f(t, v) dv.$$

Denote here  $M_0 = M(f(0)), E_0 = E(f(0))$  and  $H_0 = H(f(0))$ . It's known that the solutions of the Landau equation satisfy the formal conservation laws:

$$M(f(t)) = M_0, \quad E(f(t)) = E_0, \quad H(f(t)) \leq H_0, \quad \text{for } \forall t \geq 0.$$

Also in this paper we use the following notations

$$\|f(t, \cdot)\|_{L^1_s} = \int_{\mathbb{R}^n} f(t, v) (1 + |v|^2)^{s/2} dv,$$

$$\|\partial_v^\alpha f(t, \cdot)\|_{L^2_s}^2 = \int_{\mathbb{R}^n} |\partial_v^\alpha f(t, v)|^2 (1 + |v|^2)^{s/2} dv,$$

$$\|f(t, \cdot)\|_{H_s^m}^2 = \sum_{0 \leq |\alpha| \leq m} \|\partial_v^\alpha f(t)\|_{L_s^2}^2.$$

When there is no risk causing confusion, we write  $\|g(t)\|_{L_s^1}$  for  $\|g(t, \cdot)\|_{L_s^1}$ .

Next, let us recall the definition of the Gevrey class function space  $G^\sigma(\mathbb{R}^N)$ , where  $\sigma \geq 1$  is the Gevrey index (cf. [14]). Let  $u \in C^\infty(\mathbb{R}^N)$ . We say  $u \in G^\sigma(\mathbb{R}^N)$  if there exists a constant  $C$ , called the Gevrey constant, such that for all multi-indices  $\alpha \in \mathbb{N}^N$ ,

$$\|\partial^\alpha u\|_{L^2(\mathbb{R}^N)} \leq C^{|\alpha|+1} (|\alpha|!)^\sigma.$$

We denote by  $G_0^\sigma(\mathbb{R}^N)$  the space of Gevrey function with compact support. Note that  $G^1(\mathbb{R}^N)$  is space of all real analytic functions.

In the hard potential case, the existence, uniqueness and Sobolev regularity of the weak solution had been studied in [6], in which they proved that, under suitable assumptions on the initial datum (e.g.  $f_0 \in L_{2+\delta}^1$  with  $\delta > 0$ ), there exists a unique weak solution of the Cauchy problem (3), which moreover is in the space  $C^\infty(\mathbb{R}_t^+; \mathcal{S}(\mathbb{R}^3))$ . Here  $\mathbb{R}_t^+ = (0, +\infty)$  and  $\mathcal{S}$  denotes the space of smooth functions which decay rapidly at infinity.

Assuming the existence of the smooth solution, we state now the main result of the paper as follows:

**Theorem 1.1.** *Let  $f_0$  be an initial datum with finite mass, energy and entropy. Suppose  $f_0 \in G^\sigma(\mathbb{R}^n)$  with  $\sigma > 1$ , and  $f$  is a solution of the Cauchy problem (3) which satisfies*

$$f(t, v) \in L_{loc}^\infty([0, +\infty[; H^m(\mathbb{R}^n)) \cap L_{loc}^2([0, +\infty[; H_\gamma^{m+1}(\mathbb{R}^n)), \quad \text{for all } m \geq 0. \tag{4}$$

*Then  $f(t, \cdot) \in G^\sigma(\mathbb{R}^n)$  for all  $t > 0$  uniformly, namely, the Gevrey constant of  $f(t, \cdot)$  is independent of  $t$ . More precisely, for any fixed  $T > 0$ , there exists a constant  $C > 0$  which is independent of  $t$ , such that for any multi-index  $\alpha$ , one has*

$$\sup_{t \in [0, T]} \|\partial_v^\alpha f(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C^{|\alpha|+1} (|\alpha|!)^\sigma.$$

**Remark 1.** If  $f_0 \in L_{2+\delta}^1$  additionally, then by the result of [6], the Cauchy problem (3) admits a solution which satisfies (4).

**Remark 2.** For simplicity, we shall prove Theorem 1.1 in the case of space dimension  $n = 3$ . The conclusion for general cases can be deduced similarly.

The plan of the paper is as follows: In Section 2 we prove some lemmas. Section 3 is devoted to the proof of the main result.

**2. Some Lemmas.** In this section we give some lemmas, which will be used in the proof of the main result.

**Lemma 2.1.** *For any  $\sigma > 1$ , there exists a constant  $C_\sigma$ , depending only on  $\sigma$ , such that for all multi-indices  $\mu \in \mathbb{N}^3, |\mu| \geq 1$ ,*

$$\sum_{1 \leq |\beta| \leq |\mu|} \frac{1}{|\beta|^3} \leq C_\sigma |\mu|^{\sigma-1}, \tag{5}$$

and

$$\sum_{1 \leq |\beta| \leq |\mu|-1} \frac{1}{|\beta|^2 (|\mu| - |\beta|)} \leq C_\sigma |\mu|^{\sigma-1}. \tag{6}$$

Here and throughout the paper, the notation  $\sum_{1 \leq |\beta| \leq |\mu|}$  denotes the summation over all the multi-indices satisfying  $\beta \leq \mu$  and  $1 \leq |\beta| \leq |\mu|$ . Also the notation  $\sum_{1 \leq |\beta| \leq |\mu|-1}$  denotes the summation over all the multi-indices satisfying  $\beta \leq \mu$  and  $1 \leq |\beta| \leq |\mu| - 1$ .

*Proof.* For each positive integer  $l$ , we denote by  $N\{|\beta| = l\}$  the number of the multi-indices  $\beta$  with  $|\beta| = l$ . In the case when the space dimension is 3, one has

$$N\{|\beta| = l\} = \frac{(l+2)!}{2!l!} = \frac{1}{2}(l+1)(l+2).$$

It is easy to deduce that

$$\sum_{1 \leq |\beta| \leq |\mu|} \frac{1}{|\beta|^3} \leq \sum_{l=1}^{|\mu|} \sum_{|\beta|=l} \frac{1}{l^3} = \sum_{l=1}^{|\mu|} \frac{N\{|\beta| = l\}}{l^3}.$$

Combining the estimates above, it holds that

$$\sum_{1 \leq |\beta| \leq |\mu|} \frac{1}{|\beta|^3} \leq \frac{1}{2} \sum_{l=1}^{|\mu|} \frac{(l+1)(l+2)}{l^3} \leq 3 \sum_{l=1}^{|\mu|} \frac{1}{l}.$$

Observing that  $3 \sum_{l=1}^{|\mu|} l^{-1} \leq C_\sigma |\mu|^{\sigma-1}$  for some constant  $C_\sigma$ , we obtain the desired estimate (5). Similarly we can deduce the estimate (6).  $\square$

The following lemma is crucial to the proof of Theorem 1.1.

**Lemma 2.2.** *Let  $\sigma > 1$ . There exist constant  $C_1, C_2 > 0$ , depending only on  $M_0, E_0, H_0$  and  $\gamma$ , such that for all multi-indices  $\mu \in \mathbb{N}^3$  with  $|\mu| \geq 2$  and all  $t > 0$ , we have*

$$\begin{aligned} & \partial_t \|\partial_v^\mu f(t)\|_{L^2}^2 + C_1 \|\nabla_v \partial_v^\mu f(t)\|_{L^2_\gamma}^2 \leq C_2 |\mu|^2 \|\nabla_v D^{|\mu|-1} f(t)\|_{L^2_\gamma}^2 \\ & + C_2 \sum_{2 \leq |\beta| \leq |\mu|} C_\mu^\beta \|\nabla_v D^{|\mu|-|\beta|+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v D^{|\mu|-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{|\beta|-2} \\ & + C_2 \sum_{0 \leq |\beta| \leq |\mu|} C_\mu^\beta \|\partial_v^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v D^{|\mu|-1} f(t)\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{|\mu|-|\beta|}, \end{aligned}$$

where  $C_\mu^\beta = \frac{\mu!}{(\mu-\beta)! \beta!}$  is the binomial coefficient, and  $[G_\sigma(f(t))]_{|\nu|} = \|\partial_v^\nu f(t)\|_{L^2} + B^{|\nu|} (|\nu|!)^\sigma$  with  $B$  being the constant as given in Lemma 2.4 below.

By the assumption in Theorem 1.1, the solution  $f(t, v)$  of the Cauchy problem (3) is smooth in  $v$ , and so are the coefficients  $\bar{a}_{ij} = a_{ij} * f$ ,  $\bar{b}_i = b_i * f$  and  $\bar{c} = c * f$ . Here and in what follows, we write  $C$  for a constant, depending only on the Gevrey index  $\sigma$ , and  $M_0, E_0$  and  $H_0$  (the initial mass, energy and entropy), which may be different in different contexts.

The proof of Lemma 2.2 can be deduced by the following lemmas:

**Lemma 2.3.** *(uniformly ellipticity) There exists a constant  $K$ , depending only on  $\gamma$  and  $M_0, E_0, H_0$ , such that*

$$\sum_{i,j=1}^3 \bar{a}_{ij}(t, v) \xi_i \xi_j \geq K(1 + |v|^2)^{\gamma/2} |\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \text{ and } \gamma \in [0, 1]. \quad (7)$$

*Proof.* See Proposition 4 of [6]  $\square$

**Remark 3.** Although the ellipticity of  $(\bar{a}_{ij})$  was proved in [6] in the hard potential case  $\gamma \in (0, 1]$ , it still holds for the Maxwellian case  $\gamma = 0$ . This can be seen in the proof of Proposition 4 of [6].

**Lemma 2.4.** *There exists a constant  $B$ , depending only on the Gevrey index  $\sigma > 1$ , such that for all multi-indices  $\beta$  with  $|\beta| \geq 2$  and all  $g, h \in L^2_\gamma(\mathbb{R}^3)$ , one has*

$$\sum_{i,j=1}^3 \int_{\mathbb{R}^3} (\partial_v^\beta \bar{a}_{ij}(t, v))g(v)h(v)dv \leq C\|g\|_{L^2_\gamma}\|h\|_{L^2_\gamma}[G_\sigma(f(t))]_{|\beta|-2}, \quad \text{for all } t > 0,$$

where  $[G_\sigma(f(t))]_{|\beta|-2} = \{\|D^{|\beta|-2}f(t)\|_{L^2} + B^{|\beta|-2}[(|\beta| - 2)!]^\sigma\}$ .

*Proof.* For  $\sigma > 1$ , there exists a function  $\psi \in G_0^\sigma(\mathbb{R}^3)$  (cf. [14]) with compact support in  $\{v \in \mathbb{R}^3 \mid |v| \leq 2\}$ , satisfying that  $\psi(v) = 1$  on the ball  $\{v \in \mathbb{R}^3 \mid |v| \leq 1\}$ , and that for some constant  $\tilde{B} > 4$  depending only on  $\sigma$ ,

$$\sup |\partial_v^\lambda \psi| \leq \tilde{B}^{|\lambda|}(|\lambda| - 1)^\sigma, \quad \text{for all } \lambda \in \mathbb{Z}_+^3. \tag{8}$$

Write  $a_{ij} = \psi a_{ij} + (1 - \psi)a_{ij}$ . Then  $\bar{a}_{ij} = (\psi a_{ij}) * f + [(1 - \psi)a_{ij}] * f$ . It is easy to see that

$$\partial_v^\beta \bar{a}_{ij} = [\partial_v^\nu (\psi a_{ij})] * (\partial_v^{\beta-\nu} f) + \{\partial_v^\beta [(1 - \psi)a_{ij}]\} * f, \quad \text{for } |\nu| = 2.$$

We first treat the term  $[\partial_v^\nu (\psi a_{ij})] * (\partial_v^{\beta-\nu} f)$ . A direct computation shows that

$$\begin{aligned} |[\partial_v^\nu (\psi a_{ij})] * (\partial_v^{\beta-\nu} f)(v)| &= \left| \int_{\mathbb{R}^3} [\partial_v^\nu (\psi a_{ij})](v - v_*) \cdot (\partial_v^{\beta-\nu} f)(v_*) dv_* \right| \\ &\leq C \int_{\{|v_* - v| \leq 2\}} |(\partial_v^{\beta-\nu} f)(v_*)| dv_* \\ &\leq C \|\partial_v^{\beta-\nu} f(t)\|_{L^2}. \end{aligned}$$

Next, for the term  $\{\partial_v^\beta [(1 - \psi)a_{ij}]\} * f$ , one has, by using the Leibniz's formula,

$$\begin{aligned} &|(\partial_v^\beta [(1 - \psi)a_{ij}]) * f(v)| \\ &= \left| \sum_{0 \leq |\lambda| \leq |\beta|} C_\beta^\lambda \int_{\mathbb{R}^3} [\partial_v^{\beta-\lambda} (1 - \psi)](v - v_*) \cdot (\partial_v^\lambda a_{ij})(v - v_*) \cdot f(v_*) dv_* \right| \\ &\leq \left| \sum_{1 \leq |\lambda| \leq |\beta|} C_\beta^\lambda \int_{\{1 \leq |v_* - v| \leq 2\}} (\partial_v^{\beta-\lambda} \psi)(v - v_*) \cdot (\partial_v^\lambda a_{ij})(v - v_*) \cdot f(v_*) dv_* \right| \\ &\quad + \left| \int_{\{|v_* - v| \geq 1\}} [(1 - \psi)](v - v_*) \cdot (\partial_v^\beta a_{ij})(v - v_*) \cdot f(v_*) dv_* \right| \\ &= J_1 + J_2. \end{aligned}$$

In view of (2), we can find a constant  $\tilde{C}$ , depending only on  $\gamma$ , such that

$$|(\partial_v^\lambda a_{ij})(v - v_*)| \leq \tilde{C}^{|\lambda|}(|\lambda|)! \quad \text{for } 1 \leq |v_* - v| \leq 2.$$

And for  $|\beta| \geq 2$ ,

$$|(\partial_v^\beta a_{ij})(v - v_*)| \leq \tilde{C}^{|\beta|}(|\beta|)!(1 + |v_*|^\gamma + |v|^\gamma) \quad \text{for } 1 \leq |v_* - v|.$$

From the estimate (8) we know that  $J_1 + J_2$  can be estimated by

$$\tilde{B}^{|\beta|}(|\beta|)^\sigma \cdot \|f(t)\|_{L^1} \sum_{1 \leq |\lambda| \leq |\beta|} \left(\frac{\tilde{C}}{\tilde{B}}\right)^{|\lambda|} + \tilde{C}^{|\beta|}(|\beta|)^\sigma \cdot \|f(t)\|_{L^1_\gamma} (1 + |v|^\gamma).$$

We can take  $\tilde{B}$  large enough such that  $\tilde{B} \geq 2\tilde{C}$ . Then we get

$$\begin{aligned} |(\partial_v^\beta [(1-\psi)a_{ij}]) * f(v)| &\leq J_1 + J_2 \\ &\leq C \|f(t)\|_{L_\gamma^1} \tilde{B}^{|\beta|} (|\beta|!)^\sigma (1+|v|^2)^{\gamma/2} \\ &\leq C \tilde{B}^{|\beta|} (|\beta|!)^\sigma (1+|v|^2)^{\gamma/2}. \end{aligned}$$

In the last inequality we used the fact  $\|f(t)\|_{L_\gamma^1} \leq M_0 + 2E_0$ . Now we choose a constant  $B$  such that  $\tilde{B}^{|\beta|} (|\beta|!)^\sigma \leq B^{|\beta|-2} [(|\beta|-2)!]^\sigma$ . It follows immediately that

$$|(\partial_v^\beta [(1-\psi)a_{ij}]) * f(v)| \leq CB^{|\beta|-2} [(|\beta|-2)!]^\sigma (1+|v|^2)^{\gamma/2}.$$

Combining with the estimate on  $[\partial_v^\nu (\psi a_{ij})] * (\partial_v^{\beta-\nu} f)$ , we get finally

$$\begin{aligned} |\partial_v^\beta \bar{a}_{ij}(v)| &\leq C \left\{ \|\partial_v^{\beta-\nu} f(t)\|_{L^2} + B^{|\beta|-2} [(|\beta|-2)!]^\sigma \cdot (1+|v|^2)^{\gamma/2} \right\} \\ &\leq C [G_\sigma(f(t))]_{|\beta|-2} \cdot (1+|v|^2)^{\gamma/2}. \end{aligned}$$

Thus, combining with Cauchy's inequality, the estimate above gives the proof of Lemma 2.4.  $\square$

Similar to Lemma 2.4, we can prove that

**Lemma 2.5.** *For all multi-indices  $\beta$  with  $|\beta| \geq 0$  and all  $g, h \in L_\gamma^2(\mathbb{R}^3)$ , one has*

$$\int_{\mathbb{R}^3} (\partial_v^\beta \bar{c}(t, v)) g(v) h(v) dv \leq C \|g\|_{L_\gamma^2} \|h\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\beta|}, \quad \forall t \geq 0.$$

Let us now give the proof of Lemma 2.2. Since

$$\sum_{i=1}^3 \partial_{v_i} \bar{a}_{ij} = \bar{b}_j, \quad \sum_{j=1}^3 \partial_{v_j} b_j = \bar{c},$$

and  $f$  satisfies  $\partial_t f = \sum_{i,j=1}^3 \bar{a}_{ij} \partial_{v_i v_j} f - \bar{c} f$ , then it holds that

$$\begin{aligned} \partial_t \|\partial_v^\mu f(t)\|_{L^2}^2 &= 2 \int_{\mathbb{R}^3} [\partial_t \partial_v^\mu f(t, v)] \cdot [\partial_v^\mu f(t, v)] dv \\ &= 2 \int_{\mathbb{R}^3} \left[ \sum_{i,j=1}^3 \partial_v^\mu (\bar{a}_{ij} \partial_{v_i v_j} f - \bar{c} f) \right] \cdot [\partial_v^\mu f(t, v)] dv. \end{aligned}$$

Moreover, by using Leibniz's formula, we have

$$\begin{aligned} \partial_t \|\partial_v^\mu f(t)\|_{L^2}^2 &= 2 \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \bar{a}_{ij} (\partial_{v_i v_j} \partial_v^\mu f) \cdot (\partial_v^\mu f) dv \\ &\quad + 2 \sum_{i,j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^\beta \bar{a}_{ij}) (\partial_{v_i v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_v^\mu f) dv \\ &\quad + 2 \sum_{i,j=1}^3 \sum_{2 \leq |\beta| \leq |\mu|} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^\beta \bar{a}_{ij}) (\partial_{v_i v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_v^\mu f) dv \\ &\quad - 2 \sum_{0 \leq |\beta| \leq |\mu|} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^{\mu-\beta} \bar{c}) (\partial_v^\beta f) \cdot (\partial_v^\mu f) dv \\ &= (I) + (II) + (III) + (IV). \end{aligned}$$

Thus the proof of Lemma 2.2 depends on the following estimates.

**Step 1. Estimate on the term (I).**

Integrating by parts, one has

$$\begin{aligned} (I) &= -2 \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \bar{a}_{ij} (\partial_{v_j} \partial_v^\mu f) \cdot (\partial_{v_i} \partial_v^\mu f) dv \\ &\quad - 2 \sum_{j=1}^3 \int_{\mathbb{R}^3} \bar{b}_j (\partial_{v_j} \partial_v^\mu f) \cdot (\partial_v^\mu f) dv \\ &= (I)_1 + (I)_2. \end{aligned}$$

For the term  $(I)_1$ , one has, by applying the ellipticity property (7),

$$(I)_1 \leq -2K \int_{\mathbb{R}^3} |\nabla_v \partial_v^\mu f|^2 (1 + |v|^2)^{\gamma/2} dv = -2K \|\nabla_v \partial_v^\mu f\|_{L^2_\gamma}^2.$$

For the term  $(I)_2$ , integrating by parts again, we have

$$(I)_2 = -(I)_2 + 2 \int_{\mathbb{R}^3} \bar{c} (\partial_v^\mu f) \cdot (\partial_v^\mu f) dv.$$

Observing that  $|\bar{c}(v)| \leq C \|f(t)\|_{L^1_\gamma} (1 + |v|^2)^{\gamma/2} \leq C(1 + |v|^2)^{\gamma/2}$ , one has

$$(I)_2 \leq C \|\partial_v^\mu f\|_{L^2_\gamma}^2 \leq C \|\nabla_v D^{|\mu|-1} f\|_{L^2_\gamma}^2.$$

This implies

$$(I) \leq -2K \|\nabla_v \partial_v^\mu f\|_{L^2_\gamma}^2 + C \|\nabla_v D^{|\mu|-1} f\|_{L^2_\gamma}^2. \tag{9}$$

**Step 2. Upper bound for the term (II).**

Recall that  $(II) = 2 \sum_{i,j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^\beta \bar{a}_{ij}) (\partial_{v_i v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_v^\mu f) dv$ . Integrating by parts, we have

$$\begin{aligned} (II) &= -2 \sum_{j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^\beta \bar{b}_j) (\partial_{v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_v^\mu f) dv \\ &\quad - 2 \sum_{i,j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^\beta \bar{a}_{ij}) (\partial_{v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_{v_i} \partial_v^\mu f) dv \\ &= (II)_1 + (II)_2. \end{aligned}$$

Observing that  $|\partial_v^\beta \bar{b}_j(t, v)| \leq C(1 + |v|^2)^{\gamma/2}$  for  $|\beta| = 1$ , one has

$$(II)_1 \leq C |\mu| \cdot \|\nabla_v \partial_v^{\mu-\beta} f(t)\|_{L^2_\gamma} \|\partial_v^\mu f(t)\|_{L^2_\gamma} \leq C |\mu| \cdot \|\nabla_v D^{|\mu|-1} f(t)\|_{L^2_\gamma}^2.$$

For the term  $(II)_2$ , if we write  $\mu = \beta + (\mu - \beta)$ , then it holds that

$$(II)_2 = -2 \sum_{i,j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^\beta \bar{a}_{ij}) (\partial_{v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_v^\beta \partial_{v_i} \partial_v^{\mu-\beta} f) dv.$$

Since  $|\beta| = 1$ , we can integrate by parts to get

$$\begin{aligned}
(II)_2 &= 2 \sum_{i,j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^\beta \bar{a}_{ij}) (\partial_{v_j} \partial_v^\mu f) \cdot (\partial_{v_i} \partial_v^{\mu-\beta} f) dv \\
&\quad + 2 \sum_{i,j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^{\beta+\beta} \bar{a}_{ij}) (\partial_{v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_{v_i} \partial_v^{\mu-\beta} f) dv \\
&= - (II)_2 + 2 \sum_{i,j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^{\beta+\beta} \bar{a}_{ij}) (\partial_{v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_{v_i} \partial_v^{\mu-\beta} f) dv.
\end{aligned}$$

Hence

$$(II)_2 = \sum_{i,j=1}^3 \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^{\beta+\beta} \bar{a}_{ij}) (\partial_{v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_{v_i} \partial_v^{\mu-\beta} f) dv.$$

Observing that  $|\partial_v^{\beta+\beta} \bar{a}_{ij}(v)| \leq C(1+|v|^2)^{\gamma/2}$  for  $|\beta| = 1$ , one has

$$(II)_2 \leq C \sum_{|\beta|=1} C_\mu^\beta \cdot \|\nabla_v \partial_v^{\mu-\beta} f\|_{L_\gamma^2}^2 \leq C |\mu| \cdot \|\nabla_v D^{|\mu|-1} f\|_{L_\gamma^2}^2,$$

which means

$$(II) \leq C |\mu| \cdot \|\nabla_v D^{|\mu|-1} f\|_{L_\gamma^2}^2. \quad (10)$$

**Step 3. Upper bound for the term (III) and (IV).**

Recall that

$$(III) = 2 \sum_{i,j=1}^3 \sum_{2 \leq |\beta| \leq |\mu|} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^\beta \bar{a}_{ij}) (\partial_{v_i v_j} \partial_v^{\mu-\beta} f) \cdot (\partial_v^\mu f) dv,$$

and that

$$(IV) = -2 \sum_{0 \leq |\beta| \leq |\mu|} C_\mu^\beta \int_{\mathbb{R}^3} (\partial_v^{\mu-\beta} \bar{c}) (\partial_v^\beta f) \cdot (\partial_v^\mu f) dv.$$

By Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}
(III) &\leq C \sum_{i,j=1}^3 \sum_{2 \leq |\beta| \leq |\mu|} C_\mu^\beta \|\partial_{v_i v_j} \partial_v^{\mu-\beta} f(t)\|_{L_\gamma^2} \cdot \|\partial_v^\mu f(t)\|_{L_\gamma^2} [G_\sigma(f(t))]_{|\beta|-2} \\
&\leq C \sum_{2 \leq |\beta| \leq |\mu|} C_\mu^\beta \|\nabla_v D^{|\mu|-|\beta|+1} f(t)\|_{L_\gamma^2} \cdot \|\nabla_v D^{|\mu|-1} f(t)\|_{L_\gamma^2} [G_\sigma(f(t))]_{|\beta|-2},
\end{aligned} \quad (11)$$

and

$$(IV) \leq C \sum_{0 \leq |\beta| \leq |\mu|} C_\mu^\beta \|\partial_v^\beta f(t)\|_{L_\gamma^2} \cdot \|\nabla_v D^{|\mu|-1} f(t)\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\mu|-|\beta|}. \quad (12)$$

Combining with the estimates (9)-(12), one has

$$\begin{aligned}
&\partial_t \|\partial_v^\mu f(t)\|_{L_\gamma^2}^2 + C_1 \|\nabla_v \partial_v^\mu f(t)\|_{L_\gamma^2}^2 \leq C_2 |\mu|^2 \|\nabla_v D^{|\mu|-1} f(t)\|_{L_\gamma^2}^2 \\
&\quad + C_2 \sum_{2 \leq |\beta| \leq |\mu|} C_\mu^\beta \|\nabla_v D^{|\mu|-|\beta|+1} f(t)\|_{L_\gamma^2} \cdot \|\nabla_v D^{|\mu|-1} f(t)\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\beta|-2} \\
&\quad + C_2 \sum_{0 \leq |\beta| \leq |\mu|} C_\mu^\beta \|\partial_v^\beta f(t)\|_{L_\gamma^2} \cdot \|\nabla_v D^{|\mu|-1} f(t)\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\mu|-|\beta|}.
\end{aligned}$$

This completes the proof of Lemma 2.2.

3. **Proof of Theorem 1.1.** Theorem 1.1 will be deduced by the following result:

**Proposition 1.** *Let  $\sigma > 1$  and  $f_0 \in G^\sigma(\mathbb{R}^3)$  be the initial datum with finite mass, energy and entropy, and let  $f$  be a smooth solution of the Cauchy problem (3) satisfying (4). Then for any fixed  $T, 0 < T < +\infty$ , there exists a constant  $A$ , which depends only on  $M_0, E_0, H_0, \gamma, T, \sigma$  and the Gevrey constant of  $f_0$ , such that for any  $k \in \mathbb{N}, k \geq 1$ , one has*

$$(Q)_k \quad \sup_{t \in [0, T]} \|\partial_v^\alpha f(t)\|_{L^2} + \left\{ \int_0^T \|\nabla_v \partial_v^{\tilde{\alpha}} f(t)\|_{L_\gamma^2}^2 dt \right\}^{1/2} \leq A^k [(k-1)!]^\sigma$$

for all multi-indices  $\alpha$  and  $\tilde{\alpha}$  with  $|\alpha| = |\tilde{\alpha}| = k$ .

**Remark 4.** From the estimate  $(Q)_k$  in Proposition 1 we can deduce directly the result of Theorem 1.1.

*Proof of Proposition 1.*

We use induction on  $k$  to prove the estimate  $(Q)_k$ . Observe that  $(Q)_1$  holds if we take  $A$  large enough such that

$$A \geq \sup_{t \in [0, T]} \|f(t)\|_{H_\gamma^1} + \|f\|_{L^2([0, T]; H_\gamma^2)} + 2. \tag{13}$$

Assume that the estimate  $(Q)_l$  holds for  $1 \leq l \leq k - 1$  and  $k \geq 2$ . Then we need to prove that the estimate  $(Q)_k$  is true. Firstly we prove that

$$\sup_{t \in [0, T]} \|\partial_v^\alpha f(t)\|_{L^2} \leq \frac{1}{2} A^{|\alpha|} [(|\alpha| - 1)!]^\sigma, \quad \text{for all } |\alpha| = k. \tag{14}$$

Applying Lemma 2.2 with  $\mu = \alpha$ , we obtain

$$\begin{aligned} & \partial_t \|\partial_v^\alpha f(t)\|_{L^2}^2 + C_1 \|\nabla_v \partial_v^\alpha f\|_{L_\gamma^2}^2 \leq C_2 |\alpha|^2 \|\nabla_v D^{|\alpha|-1} f\|_{L_\gamma^2}^2 \\ & + C_2 \sum_{2 \leq |\beta| \leq |\alpha|} C_\alpha^\beta \|\nabla_v D^{|\alpha|-|\beta|+1} f\|_{L_\gamma^2} \cdot \|\nabla_v D^{|\alpha|-1} f\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\beta|-2} \\ & + C_2 \sum_{0 \leq |\beta| \leq |\alpha|} C_\alpha^\beta \|\partial_v^\beta f\|_{L_\gamma^2} \cdot \|\nabla_v D^{|\alpha|-1} f\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\alpha|-|\beta|}. \end{aligned} \tag{15}$$

Next for the last term on the right hand side of the above inequality, one has

$$\begin{aligned} & C_2 \sum_{0 \leq |\beta| \leq |\alpha|} C_\alpha^\beta \|\partial_v^\beta f(t)\|_{L_\gamma^2} \cdot \|\nabla_v \partial_v^{|\alpha|-1} f(t)\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\alpha|-|\beta|} \\ & = C_2 \|f(t)\|_{L_\gamma^2} \cdot \|\nabla_v \partial_v^{|\alpha|-1} f(t)\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\alpha|} \\ & + C_2 \sum_{|\beta|=1} C_\alpha^\beta \|\partial_v^\beta f(t)\|_{L_\gamma^2} \cdot \|\nabla_v \partial_v^{|\alpha|-1} f(t)\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\alpha|-1} \\ & + C_2 \sum_{2 \leq |\beta| \leq |\alpha|} C_\alpha^\beta \|\partial_v^\beta f(t)\|_{L_\gamma^2} \cdot \|\nabla_v \partial_v^{|\alpha|-1} f(t)\|_{L_\gamma^2} \cdot [G_\sigma(f(t))]_{|\alpha|-|\beta|}. \end{aligned}$$

We denote  $[G_\sigma(f)]_{|\beta|} = \sup_{t \in [0, T]} [G_\sigma(f(t))]_{|\beta|}$ . Integrating in both sides of the estimate (15) over the interval  $[0, T]$ , and using the Cauchy inequality, we get

$$\begin{aligned}
& \|\partial_v^\alpha f(t)\|_{L^2}^2 - \|\partial_v^\alpha f(0)\|_{L^2}^2 \leq C_2 |\alpha|^2 \int_0^T \|\nabla_v D^{|\alpha|-1} f(s)\|_{L_\gamma^2}^2 ds \\
& + C_2 \sum_{2 \leq |\beta| \leq |\alpha|} C_\alpha^\beta [G_\sigma(f)]_{|\beta|-2} \left\{ \int_0^T \|\nabla_v D^{|\alpha|-|\beta|+1} f(s)\|_{L_\gamma^2}^2 ds \right\}^{\frac{1}{2}} \\
& \quad \times \left\{ \int_0^T \|\nabla_v D^{|\alpha|-1} f(s)\|_{L_\gamma^2}^2 ds \right\}^{\frac{1}{2}} \\
& + C_2 \max_{0 \leq t \leq T} \|f(t)\|_{L_\gamma^2} \cdot \left\{ \int_0^T [G(f(s))]_{|\alpha|} ds \right\}^{\frac{1}{2}} \left\{ \int_0^T \|\nabla_v D^{|\alpha|-1} f(s)\|_{L_\gamma^2}^2 ds \right\}^{\frac{1}{2}} \\
& + C_2 \sum_{|\beta|=1} C_\alpha^\beta \max_{0 \leq t \leq T} \|f(t)\|_{H_\gamma^1} \cdot [G_\sigma(f)]_{|\alpha|-1} \cdot \int_0^T \|\nabla_v D^{|\alpha|-1} f(s)\|_{L_\gamma^2} ds \\
& + C_2 \sum_{2 \leq |\beta| \leq |\alpha|} C_\alpha^\beta [G_\sigma(f)]_{|\alpha|-|\beta|} \left\{ \int_0^T \|\partial_v^\beta f(s)\|_{L_\gamma^2}^2 ds \right\}^{\frac{1}{2}} \\
& \quad \times \left\{ \int_0^T \|\nabla_v D^{|\alpha|-1} f(s)\|_{L_\gamma^2}^2 ds \right\}^{\frac{1}{2}} \\
& \stackrel{\text{def}}{=} (S_1) + (S_2) + (S_3) + (S_4) + (S_5).
\end{aligned}$$

From the induction assumption and the fact  $\|\partial_v^\beta f\|_{L_\gamma^2} \leq \|\nabla_v \partial_v^{|\beta|-1} f\|_{L_\gamma^2}$  for  $|\beta| \geq 1$ , we have, respectively, the following estimates:

$$\left\{ \int_0^T \|\nabla_v \partial_v^{|\alpha|-1} f(s)\|_{L_\gamma^2}^2 ds \right\}^{\frac{1}{2}} \leq A^{|\alpha|-1} [(|\alpha|-2)!]^\sigma; \quad (16)$$

$$\left\{ \int_0^T \|\nabla \partial_v^{|\alpha|-|\beta|+1} f(s)\|_{L_\gamma^2}^2 ds \right\}^{\frac{1}{2}} \leq A^{|\alpha|-|\beta|+1} [(|\alpha|-|\beta|)!]^\sigma, \quad 2 \leq |\beta| \leq |\alpha|; \quad (17)$$

$$\left\{ \int_0^T \|\partial_v^\beta f(s)\|_{L_\gamma^2}^2 ds \right\}^{\frac{1}{2}} \leq A^{|\beta|-1} [(|\beta|-2)!]^\sigma, \quad 2 \leq |\beta| \leq |\alpha|. \quad (18)$$

We next treat the term  $[G_\sigma(f)]_m$  given by

$$[G_\sigma(f)]_m = \sup_{t \in [0, T]} [G_\sigma(f(t))]_m = \sup_{t \in [0, T]} \|\partial_v^m f(t)\|_{L^2} + B^m (m!)^\sigma,$$

Observe that for some  $\tilde{C}_\sigma > 0$ ,

$$B^m (m!)^\sigma \leq (\tilde{C}_\sigma B)^m ((m-1)!)^\sigma, \quad \text{for } 1 \leq |\beta| \leq |\alpha| - 1.$$

Also from the induction assumption, one has

$$\max_{t \in [0, T]} \|\partial_v^m f(t)\|_{L^2} \leq A^m ((m-1)!)^\sigma, \quad 1 \leq m \leq |\alpha| - 1 = k - 1.$$

Thus for  $A$  large enough, we have

$$[G_\sigma(f)]_m \leq 2A^m ((m - 1)!)^\sigma, \quad 1 \leq m \leq |\alpha| - 1 = k - 1. \tag{19}$$

By using the estimate (16), one has

$$\int_0^T [G(f(s))]_{|\alpha|} ds \leq CA^{|\alpha|-1} [ (|\alpha| - 2)! ]^\sigma. \tag{20}$$

The estimate for the term  $(S_1)$  can be given by (16) directly,

$$(S_1) \leq C_2 |\alpha|^2 [A^{|\alpha|-1} ((|\alpha| - 1)!)^\sigma]^2 \leq \tilde{C}_2 [A^{|\alpha|-1} (|\alpha|!)^\sigma]^2. \tag{21}$$

We write now the term  $(S_2)$  into two parts,

$$\begin{aligned} (S_2) &= C_2 \sum_{|\beta|=2} C_\alpha^\beta [G_\sigma(f)]_0 \cdot \int_0^T \|\nabla D^{|\alpha|-1} f(s)\|_{L_\gamma^2}^2 ds \\ &\quad + C_2 \sum_{3 \leq |\beta| \leq |\alpha|} C_\alpha^\beta [G_\sigma(f)]_{|\beta|-2} \left\{ \int_0^T \|\nabla D^{|\alpha|-|\beta|+1} f(s)\|_{L_\gamma^2}^2 ds \right\}^{1/2} \\ &\quad \times \left\{ \int_0^T \|\nabla D^{|\alpha|-1} f(s)\|_{L_\gamma^2}^2 ds \right\}^{1/2} \\ &= (S_2)' + (S_2)''. \end{aligned}$$

Thus (13) and (16) give that

$$(S_2)' \leq C_2 A |\alpha|^2 \left\{ A^{|\alpha|-1} [ (|\alpha| - 2)! ]^\sigma \right\}^2 \leq \tilde{C}_2 A \left\{ A^{|\alpha|-1} [ (|\alpha| - 1)! ]^\sigma \right\}^2,$$

and (16), (17) and (19) give that

$$\begin{aligned} (S_2)'' &\leq C_2 \sum_{3 \leq |\beta| \leq |\alpha|} \frac{|\alpha|!}{|\beta|!(|\alpha| - |\beta|)!} A^{|\beta|-2} [ (|\beta| - 3)! ]^\sigma A^{|\alpha|-|\beta|+1} [ (|\alpha| - |\beta|)! ]^\sigma \\ &\quad \times A^{|\alpha|-1} [ (|\alpha| - 2)! ]^\sigma. \end{aligned}$$

Observing that

$$\frac{|\alpha|!}{|\beta|!(|\alpha| - |\beta|)!} [ (|\beta| - 3)! ]^\sigma [ (|\alpha| - |\beta|)! ]^\sigma \leq \frac{6|\alpha|}{|\beta|^3} \{ (|\alpha| - 1)! \}^\sigma,$$

we have by the estimate (5)

$$\begin{aligned} (S_2)'' &\leq C_2 (A^{|\alpha|-1})^2 [ (|\alpha| - 2)! ]^\sigma [ (|\alpha| - 1)! ]^\sigma \sum_{3 \leq |\beta| \leq |\alpha|} \frac{6|\alpha|}{|\beta|^3} \\ &\leq 6C_2 (A^{|\alpha|-1})^2 [ (|\alpha| - 2)! ]^\sigma [ (|\alpha| - 1)! ]^\sigma |\alpha|^\sigma \\ &\leq 6C_2 \left\{ A^{|\alpha|-1} [ (|\alpha| - 1)! ]^\sigma \right\}^2. \end{aligned}$$

Therefore

$$(S_2) = (S_2)' + (S_2)'' \leq 6C_2 A \left\{ A^{|\alpha|-1} [ (|\alpha| - 1)! ]^\sigma \right\}^2. \tag{22}$$

Similarly, from the estimates (13), (16) and (20), one has

$$(S_3) \leq \tilde{C}_2 A \left\{ A^{|\alpha|-1} [ (|\alpha| - 2)! ]^\sigma \right\}^2. \tag{23}$$

From (13), (16) and (19), one can deduce that

$$(S_4) \leq \tilde{C}_2 A \left\{ A^{|\alpha|-1} [(|\alpha|-1)!]^\sigma \right\}^2. \quad (24)$$

It remains to estimate the term  $(S_5)$ . From (16), (18) and (19) we have

$$\begin{aligned} (S_5) &\leq \sum_{2 \leq |\beta| \leq |\alpha|-1} \frac{C_2 |\alpha|!}{|\beta|!(|\alpha|-|\beta|)!} A^{|\alpha|-|\beta|} [(|\alpha|-|\beta|-1)!]^\sigma A^{|\beta|-1} [(|\beta|-2)!]^\sigma \\ &\quad \times A^{|\alpha|-1} [(|\alpha|-2)!]^\sigma \\ &\quad + C_2 A \left\{ A^{|\alpha|-1} [(|\alpha|-2)!]^\sigma \right\}^2. \end{aligned}$$

Since for  $2 \leq |\beta| \leq |\alpha|-1$ ,

$$\frac{|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!} [(|\alpha|-|\beta|-1)!]^\sigma [(|\beta|-2)!]^\sigma \leq \frac{2|\alpha|}{|\beta|^2 (|\alpha|-|\beta|)} [(|\alpha|-1)!]^\sigma,$$

then by using the estimate (6), we have

$$\begin{aligned} (S_5) &\leq C_2 \left( A^{|\alpha|-1} \right)^2 [(|\alpha|-1)!]^\sigma [(|\alpha|-2)!]^\sigma \sum_{2 \leq |\beta| \leq |\alpha|-1} \frac{2|\alpha|}{|\beta|^2 (|\alpha|-|\beta|)} \\ &\quad + C_2 A \left\{ A^{|\alpha|-1} [(|\alpha|-2)!]^\sigma \right\}^2 \\ &\leq 3\tilde{C}_2 A \left( A^{|\alpha|-1} \right)^2 [(|\alpha|-1)!]^\sigma [(|\alpha|-2)!]^\sigma |\alpha|^\sigma \\ &\leq 3\tilde{C}_2 A \left\{ A^{|\alpha|-1} [(|\alpha|-1)!]^\sigma \right\}^2. \end{aligned}$$

This, combined with (21)-(24), implies that

$$\|\partial_v^\alpha f(t)\|_{L^2}^2 - \|\partial_v^\alpha f(0)\|_{L^2}^2 \leq C_3 A \left\{ A^{|\alpha|-1} [(|\alpha|)!]^\sigma \right\}^2, \quad \forall t \in [0, T].$$

Since  $f(0) = f_0 \in G^\sigma(\mathbb{R}^3)$ , then there exists a constant  $L$  such that

$$\|\partial_v^\alpha f(0)\|_{L^2}^2 \leq \left\{ L^{|\alpha|} [(|\alpha|-1)!]^\sigma \right\}^2.$$

Thus taking  $A$  large enough, we can deduce that

$$\|\partial_v^\alpha f(t)\|_{L^2}^2 \leq \left\{ \frac{1}{2} A^{|\alpha|} [(|\alpha|-1)!]^\sigma \right\}^2, \quad \text{for all } t \in [0, T], \text{ and } |\alpha| = k.$$

This gives the proof of the inequality (14).

Finally, we need to prove that

$$\int_0^T \|\nabla_v \partial_v^{\tilde{\alpha}} f(t)\|_{L_\gamma^2}^2 dt \leq \left\{ \frac{1}{2} A^{|\tilde{\alpha}|} [(|\tilde{\alpha}|-1)!]^\sigma \right\}^2, \quad \forall |\tilde{\alpha}| = k. \quad (25)$$

The proof of the estimate (25) is similar to that of (14). Let us apply Lemma 2.2 again with  $\mu = \tilde{\alpha}$ . Then we have

$$\begin{aligned} & \partial_t \|\partial_v^{\tilde{\alpha}} f(t)\|_{L^2}^2 + C_1 \|\nabla_v \partial_v^{\tilde{\alpha}} f\|_{L^2_\gamma}^2 \leq C_2 |\tilde{\alpha}|^2 \|\nabla_v \partial_v^{|\tilde{\alpha}|-1} f\|_{L^2_\gamma}^2 \\ & + C_2 \sum_{2 \leq |\beta| \leq |\tilde{\alpha}|} C_\alpha^\beta \|\nabla_v \partial_v^{|\tilde{\alpha}|-|\beta|+1} f\|_{L^2_\gamma} \cdot \|\nabla_v \partial_v^{|\tilde{\alpha}|-1} f\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{|\beta|-2} \\ & + C_2 \sum_{0 \leq |\beta| \leq |\tilde{\alpha}|} C_\alpha^\beta \|\partial_v^\beta f\|_{L^2_\gamma} \cdot \|\nabla_v \partial_v^{|\tilde{\alpha}|-1} f\|_{L^2_\gamma} \cdot [G_\sigma(f(t))]_{|\tilde{\alpha}|-|\beta|} \\ & \stackrel{\text{def}}{=} \mathcal{N}(t). \end{aligned}$$

Integrating the above inequality over the interval  $[0, T]$ , we then have

$$C_1 \int_0^T \|\nabla_v \partial_v^{\tilde{\alpha}} f(s)\|_{L^2_\gamma}^2 ds \leq \|\partial_v^{\tilde{\alpha}} f(0)\|_{L^2}^2 + \int_0^T \mathcal{N}(s) ds.$$

By a similar argument as in the proof of (14), one has

$$\|\partial_v^{\tilde{\alpha}} f(0)\|_{L^2}^2 + \int_0^T \mathcal{N}(s) ds \leq C_1 \left\{ \frac{1}{2} A^{|\tilde{\alpha}|} [ (|\tilde{\alpha}| - 1)! ]^\sigma \right\}^2,$$

which gives the estimate (25). The proof of Proposition 1 can be derived directly by the estimates in (14) and (25).  $\square$

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Received May 2008; revised June 2008.

*E-mail address:* `chenhua@whu.edu.cn`

*E-mail address:* `weixi.li@yahoo.com`

*E-mail address:* `Chao-Jiang.Xu@univ-rouen.fr`