

TRACE THEOREM ON THE HEISENBERG GROUP ON HOMOGENEOUS HYPERSURFACES

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Abstract : We prove in this work the trace and trace lifting theorem for Sobolev spaces on the Heisenberg groups for homogenous hypersurfaces.

Résumé : Dans ce travail, nous démontrons des théorèmes de trace et de relèvement pour les espaces de Sobolev sur le groupe de Heisenberg pour des hypersurfaces homogènes.

Key words Trace and trace lifting, Heisenberg group, Hörmander condition, Hardy's inequality.

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1. INTRODUCTION

In this work, we continue the study of the problem of restriction of functions that belongs to Sobolev spaces associated to left invariant vector fields for the Heisenberg group \mathbb{H}^d initiated in [4]. As observed in [4], the case when $d = 1$ is not very different from the case when $d \geq 2$, but the statement in this particular case are less pleasant. Thus, for the sake of simplicity, we shall assume from now on that $d \geq 2$. Let us recall that the Heisenberg group is the space \mathbb{R}^{2d+1} of the (non commutative) law of product

$$w \cdot w' = (x, y, s) \cdot (s', x', y') = (x + x', y + y', s + s' + (y|x') - (y'|x)).$$

The left invariant vector fields are

$$X_j = \partial_{x_j} + y_j \partial_s, \quad Y_j = \partial_{y_j} - x_j \partial_s, \quad j = 1, \dots, d \quad \text{and} \quad S = \partial_s = \frac{1}{2}[Y_j, X_j].$$

In all that follows, we shall denote by \mathcal{Z} this family and state $Z_j = X_j$ and $Z_{j+d} = Y_j$ for j in $\{1, \dots, d\}$. Moreover, for any C^1 function f , we shall state

$$\nabla_{\mathbb{H}} f \stackrel{\text{def}}{=} (Z_1 \cdot f, \dots, Z_{2d} \cdot f).$$

The key point is that \mathcal{Z} satisfies Hörmander's condition at order 2, which means that the family $(Z_1, \dots, Z_{2d}, [Z_1, Z_{d+1}])$ spans the whole tangent space $T\mathbb{R}^{2d+1}$.

For $k \in \mathbb{N}$ and V an open subset of \mathbb{H}^d , we define the associated Sobolev space as following

$$H^k(\mathbb{H}^d, V) = \left\{ f \in L^2(\mathbb{R}^{2d+1}) / \text{Supp } f \subset V \quad \text{and} \quad \forall \alpha / |\alpha| \leq k, Z^\alpha f \in L^2(\mathbb{R}^{2d+1}) \right\},$$

where if $\alpha \in \{1, \dots, 2d\}^{k'}$, $|\alpha| \stackrel{\text{def}}{=} k'$ and $Z^\alpha \stackrel{\text{def}}{=} Z_{\alpha_1} \dots Z_{\alpha_{k'}}$. As in the classical case, when s is any real number, we can define the function space $H^s(\mathbb{H}^d)$ through duality and complex interpolation, Littlewood-Paley theory on the Heisenberg group (see [6]), or Weyl-Hörmander calculus (see [10], [12] and [13]).

It turns out that these spaces have properties which look very much like the ones of usual Sobolev spaces, see [4] and their references.

The purpose of this paper is the study of the problems of trace and trace lifting on a smooth hypersurface of \mathbb{H}^d in the frame of Sobolev spaces. Let us point out that the problem of existence of trace appears only when s is less than or equal to 1. Indeed, under the subellipticity of system \mathcal{Z} , the space $H^s(\mathbb{H}^d)$ is included locally in $H^{\frac{s}{2}}(\mathbb{R}^{2d+1})$. So if s is strictly larger than 1, this implies that the trace on any smooth hypersurface exists and belongs locally to the usual Sobolev space $H^{\frac{s}{2}-\frac{1}{2}}$ of the hypersurface. Thus the case when $s = 1$ appears as the critical one. It is the case we study here.

1.1. Statement of the results. Two very different cases then appear: the one when the hypersurface is non characteristic, which means that any point w_0 of the hypersurface Σ is such that $\mathcal{Z}|_{w_0} \not\subset T_{w_0}\Sigma$, and the one when some point w_0 of the hypersurface Σ is characteristic, which means that $\mathcal{Z}|_{w_0} \subset T_{w_0}\Sigma$.

The non characteristic case is now well understood. In [4], we give a full account of trace and trace lifting results on smooth non characteristic hypersurfaces for $s > 1/2$. This result generalize various previous results (see among others [9], [14] and [23]).

Let us recall this theorem in the case of H^1 (see [4] for the details). If w_0 is any non characteristic point of Σ , then there exists at last one of the vector fields Z_1, \dots, Z_{2d} which is transverse to Σ at w_0 . We denote by \mathcal{X}_Σ the subspace of $T\Sigma$ define, for w in Σ , by $\mathcal{X}_{\Sigma|w} = T_w\Sigma \cap \mathcal{X}|_w$ where \mathcal{X} is the C^∞ -module of vector fields spanned by $\{Z_1, \dots, Z_{2d}\}$. It is easily checked that, if g is a local defining function of Σ , the family

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j$$

generates \mathcal{X}_Σ and that it satisfies the Hörmander condition at order 2 (see for instance Lemma 4.1 of [4]). We define

$$H^k(\Sigma, Z_\Sigma) = \{f \in L^2(\Sigma) / \text{Supp } f \subset V \cap \Sigma \text{ and } \forall(j, k), R_{j,k}u \in L^2(\Sigma)\}.$$

We have proved the following trace and trace lifting theorem in [4].

Theorem 1.1. *Let us suppose that Σ is non characteristic on an open subset V of \mathbb{H}^d , then the trace operator on Σ denoted by γ_Σ is an onto continuous map from $H^1(\mathbb{H}^d, V)$ onto $[H^1(\Sigma, Z_\Sigma), L^2(\Sigma)]_{\frac{1}{2}} \stackrel{\text{def}}{=} H^{\frac{1}{2}}(\Sigma, Z_\Sigma)$.*

Remark As the system Z_Σ satisfies the Hörmander's condition at order 2, Theorem 1.1 implies in particular that γ_Σ maps $H^1(\mathbb{H}^d, V)$ into $H^{1/4}(\Sigma, V \cap \Sigma)$.

We shall now consider the characteristic case. The set of characteristic points of Σ

$$\Sigma_c = \{w \in \Sigma / \mathcal{Z}|_w \subset T_w\Sigma\},$$

may have a complicated structure. For the sake of simplicity, we shall only consider here a particular case. By translation in the Heisenberg group, we work only in a neighborhood of $w_0 = (0, 0, 0)$. Near w_0 , the hypersurface Σ can always be written as

$$\Sigma = \{w = (x, y, s) / g(w) \stackrel{\text{def}}{=} w - f(x, y) = 0\}$$

with $f(0, 0) = 0$ and $Df(0, 0) = 0$. From now on, we assume that f is a homogenous polynomial of degree 2 on \mathbb{R}^{2d} . In this case, the equation is homogenous of order 2 with respect to the dilation of Heisenberg group $d_\lambda(x, y, s) \stackrel{\text{def}}{=} (\lambda x, \lambda y, \lambda^2 s)$. Then the set of characteristic points Σ_c is a submanifold defined by

$$\Sigma_c = \{w = (s, x, y) \in \mathbb{H}^d / g(w) = 0 \text{ and } \langle L_j, (x, y) \rangle = 0\}$$

where L_j is the linear form (on \mathbb{R}^{2d}) defined by $Z_j(g)(x, y)$.

Let us denote by r the rank of the family $(L_j)_{1 \leq j \leq 2d}$ of linear forms on \mathbb{R}^{2d} . Let us observe that r is also the rank of the matrix $(Z_i \cdot Z_j \cdot g)_{1 \leq i, j \leq 2d}$ at w_0 . Let us notice that, because, if $i \in \{1, \dots, d\}$ and $j \neq i + d$,

$$(Z_i \cdot Z_{i+d} \cdot g)(w_0) - (Z_{i+d} \cdot Z_i \cdot g)(w_0) = -2\partial_s g(w_0) = -2 \quad \text{and} \quad (Z_i \cdot Z_j \cdot g) = (Z_j \cdot Z_i \cdot g),$$

the rank of the matrix $(Z_i \cdot Z_j \cdot g)_{1 \leq i, j \leq 2d}$ and thus of $(L_j)_{1 \leq j \leq 2d}$ is at least d .

From now on, we always consider this case for the sake of simplicity.

Let us introduce some rings of functions adapted to our situation.

Definition 1.1. Let W be any open subset of Σ and F a closed subset of W . Let us denote by $C_F^\infty(W)$ the set of smooth functions a on $W \setminus F$ such that for any multi-index α , a constant C_α exists such that

$$\forall \alpha \in \mathbb{N}^d \quad |\partial^\alpha a(z)| \leq C_\alpha d(z, F)^{-|\alpha|},$$

where d denotes the distance on Σ induced by the euclidian distance on \mathbb{R}^{2d+1} .

Now let us define the vector fields on Σ which will describe the regularity on Σ .

Definition 1.2. Let W be a neighbourhood of w_0 . We denote by Z_Σ the $C_{\Sigma_c}^\infty(W)$ modulus spanned by the set vector fields of $\mathcal{Z} \cap T\Sigma|_W$ that vanish on Σ_c .

As we shall see in Proposition 3.1, the modulus Z_Σ is of finite type (of course as a $C_{\Sigma_c}^\infty(W)$ modulus) if w_0 is a regular characteristic point and W is chosen small enough. If g is a local defining function of Σ , a generating system is given by

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j \quad \text{for} \quad 1 \leq j < k \leq 2d. \quad (1.1)$$

Now we are ready to introduce the space of traces.

Definition 1.3. Let W be a small enough neighbourhood of w_0 . We denote by $H^1(\mathcal{Z}_\Sigma, W)$ the space of functions v of $L^2(\Sigma)$ supported in W such that

$$\|v\|_{H^1(\mathcal{Z}_\Sigma)}^2 \stackrel{\text{def}}{=} \|v\|_{L^2(\Sigma)}^2 + \sum_{1 \leq j, k \leq 2d} \|R_{j,k}v\|_{L^2(\Sigma)}^2 < \infty.$$

where the family $(R_{j,k})_{1 \leq j, k \leq 2d}$ is given by (1.1). If $s \in [0, 1]$, we define $H^s(\mathcal{Z}_\Sigma, V)$ by complex interpolation.

Our theorem is the following.

Theorem 1.2. Let V be a small enough neighbourhood of w_0 . Then the restriction map γ_Σ is an onto continuous map from $H^1(\mathbb{H}^d, V)$ onto $H^{\frac{1}{2}}(\mathcal{Z}_\Sigma, V \cap \Sigma)$.

Let us remark that, if $r = 2d$, this theorem is a particular case of Theorem 1.8 of [4].

1.2. Structure of the proof. In our paper [4], which corresponds to the case when $r = 2d$ as thus $\Sigma_c = \{w_0\}$, we use a blow up of the point w_0 . Here we blow up the submanifold Σ_c . In order to do it, let us introduce a function $\varphi \in \mathcal{D}(\mathbb{R}_+ \setminus \{0\})$ such that

$$\forall t \in [-1, 1] \setminus \{0\}, \quad \sum_{p=0}^{\infty} \varphi(2^p t) = 1. \quad (1.2)$$

Let us define the function ρ_c by $\rho_c \stackrel{\text{def}}{=} (g^2 + |\nabla_{\mathbb{H}} g|^4)^{\frac{1}{4}}$. Now writing that for any function u in $L^2(\rho_c \leq 1)$,

$$u = \sum_{p=0}^{\infty} \varphi_p u \quad \text{with} \quad \varphi_p(w) \stackrel{\text{def}}{=} \varphi(2^p \rho_c(w)), \quad (1.3)$$

we apply Theorem 1.1 of trace and trace lifting to each piece $\varphi_p u$ which is supported in a domain where Σ is non characteristic because $\rho_c \sim 2^{-p}$ in this domain. This decomposition leads immediately to the problem of estimating the norm $H^1(\mathbb{H}^d)$ of each piece $\varphi_p u$. Leibnitz formula and the chain rule tell us that

$$\nabla_{\mathbb{H}}(\varphi_p u) = \varphi_p \nabla_{\mathbb{H}} u + 2^p \varphi'(2^p \rho_c) u \nabla_{\mathbb{H}} \rho_c.$$

Let us observe that, as

$$Z_j \rho_c^4 = 2g(Z_j \cdot g) + 4|\nabla_{\mathbb{H}} g|^2 \sum_{k=1}^{2d} (Z_k \cdot g) Z_j \cdot (Z_k \cdot g),$$

we have, for any real number s , $|\nabla_{\mathbb{H}} \rho_c^s| \leq C_s \rho_c^{s-1}$. As the support of $\varphi'(2^p \rho_c)$ included in $\rho_c \sim 2^{-p}$, the supports of $\varphi'(2^p \rho_c)$ and $\varphi'(2^{p'} \rho_c)$ are disjoint if $|p - p'| \geq N_0$ for some N_0 . Thus, we get that

$$\sum_{p=0}^{\infty} 2^{2p} \|\varphi'(2^p \rho_c) u \nabla_{\mathbb{H}} \rho_c\|_{L^2}^2 \leq C \left\| \frac{u}{\rho_c} \right\|_{L^2}^2.$$

This leads to the proof of the following Hardy type inequality.

Theorem 1.3. *A neighbourhood V of w_0 exists such that, for any u in the space $H^1(\mathbb{H}^d, V)$ of $H^1(\mathbb{H}^d)$ functions supported in V ,*

$$\int_{\mathbb{H}^d} \frac{u^2}{\rho_c^2} dw \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}^2. \quad \text{with } \rho_c = (g^2 + |\nabla_{\mathbb{H}} g|^4)^{\frac{1}{4}}.$$

This theorem implies that, for any u in $H^1(\mathbb{H}^d, V)$,

$$\sum_{p=0}^{\infty} \|\nabla_{\mathbb{H}}(\varphi_p u)\|_{L^2}^2 \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}^2. \quad (1.4)$$

The proof of this theorem, which is the core of this work, is the purpose of the second section.

In the third section, after dilation, we apply Theorem 1.1. This gives a rather unpleasant description on the trace space. Then, we use an interpolation result which allows to conclude the proof of Theorem 1.2.

2. A HARDY TYPE INEQUALITY

2.1. The classical Hardy inequality. As a warm up, let us recall briefly the usual proof of the classical Hardy inequality ¹.

$$\int_{\mathbb{H}^d} \frac{u^2}{\rho^2} dw \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}^2 \quad \text{with } \rho(w) = (s^2 + (|x|^2 + |y|^2)^2)^{\frac{1}{4}}. \quad (2.5)$$

As $\mathcal{D}(\mathbb{H}^d \setminus \{0\})$ is dense in $H^1(\mathbb{H}^d)$, we have restrict ourselves to functions u in $\mathcal{D}(\mathbb{H}^d \setminus \{0\})$. Then the proof mainly consists in an integration by parts with respect to the radial vector field $R_{\mathbb{H}}$ adapted to the structure of \mathbb{H}^d , namely

$$R_{\mathbb{H}} \stackrel{\text{def}}{=} 2s\partial_s + \sum_{j=1}^d (x_j \partial_{x_j} + y_j \partial_{y_j}) = s[Y_1, X_1] + \sum_{j=1}^d (x_j X_j + y_j Y_j)$$

¹For a different approach based on Fourier analysis, see [2].

once noticed that $R_{\mathbb{H}} \cdot \rho^{-2} = -2\rho^{-2}$ and $\operatorname{div} R_{\mathbb{H}} = 2d + 2$. More precisely, this gives

$$-d \int \frac{u^2}{\rho^2} dw = \int \sum_{j=1}^d \frac{u}{\rho} \left(\frac{x_j}{\rho} X_j + \frac{y_j}{\rho} Y_j \right) u dw - \int \left(Y_1 \frac{s}{\rho^2} \right) u (X_1 u) dw + \int \left(X_1 \frac{s}{\rho^2} \right) u (Y_1 u) dw.$$

As we have $\left| Z_j \left(\frac{s}{\rho^2} \right) \right| \leq C\rho^{-1}$, Cauchy-Schwarz inequality gives (2.5).

2.2. Construction of substitute of ρ and $R_{\mathbb{H}}$. The classical case studied above corresponds to the case when $r = 2d$. Let us assume from now on that $r < 2d$ and let us consider $(L_{j_\ell})_{1 \leq \ell \leq r}$ a basis of the vector space generated by $(L_j)_{1 \leq j \leq 2d}$. First, we have the following lemma.

Lemma 2.1. *A couple of vector fields (Z_0, \bar{Z}_0) exists in $(\mathcal{Z} \setminus \{Z_{j_1}, \dots, Z_{j_r}\}) \times (\pm\mathcal{Z})$ such that*

$$[Z_0, \bar{Z}_0] = 2\partial_s \quad \text{and} \quad D(\bar{Z}_0 \cdot g)(w_0) \neq 0.$$

Proof of Lemma 2.1 Let us consider $Z_0 \in \mathcal{Z} \setminus \{Z_{j_1}, \dots, Z_{j_r}\}$ and \underline{Z}_0 in $\pm\mathcal{Z}$ such that $[Z_0, \underline{Z}_0] = 2\partial_s$. If $\pm\underline{Z}_0$ belongs to $\{Z_{j_1}, \dots, Z_{j_r}\}$, we infer from the definition of the family $(L_{j_\ell})_{1 \leq \ell \leq r}$ that $D(\underline{Z}_0 \cdot g)(w_0)$ is different from 0 and then $\bar{Z}_0 = \underline{Z}_0$ fits. If $\pm\underline{Z}_0$ is not in $\{Z_{j_1}, \dots, Z_{j_r}\}$, as

$$(Z_0 \cdot (\underline{Z}_0 \cdot g))(w_0) - (\underline{Z}_0 \cdot (Z_0 \cdot g))(w_0) = 2,$$

either $D(Z_0 \cdot g)(w_0)$ or $D(\underline{Z}_0 \cdot g)(w_0)$ is different from 0. Thus if $D(\underline{Z}_0 \cdot g)(w_0) = 0$, we get the lemma interchanging the role of \underline{Z}_0 and Z_0 . \blacksquare

Let us state the following theorem, which immediatly implies the Hardy type inequality stated in Theorem 1.3.

Theorem 2.1. *A neighbourhood V of w_0 exists such that, for any u in $H^1(\mathbb{H}^d, V)$,*

$$\int \frac{u^2}{\rho_0^2} dw \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}^2 \quad \text{with} \quad \rho_0 \stackrel{\text{def}}{=} (g^2 + (\bar{Z}_0 \cdot g)^4)^{\frac{1}{4}}.$$

The above inequality is obviously better than the one of Theorem 1.3 and it is surprisingly the one we are able to prove.

Proof of Theorem 2.1

By definition of the family $(L_j)_{1 \leq j \leq r}$, a family of real numbers $(\alpha_\ell)_{1 \leq \ell \leq r}$ exists such that

$$Z_0 \cdot g = \sum_{\ell=1}^r \alpha_\ell (Z_{j_\ell} \cdot g). \quad (2.6)$$

Let us define an analogous to $R_{\mathbb{H}}$ in our situation by

$$R_1 = 2g\partial_s + \frac{1}{2}(\bar{Z}_0 \cdot g)\tilde{Z}_0 \quad \text{with} \quad \tilde{Z}_0 \stackrel{\text{def}}{=} Z_0 - \sum_{\ell=1}^r \alpha_\ell Z_{j_\ell}. \quad (2.7)$$

In order to check that R_1 is analogous to the radial field in the case of the classical Hardy inequality, let us prove that

$$R_1 \cdot \rho_0^4 = 4\rho_0^4 \quad \text{and} \quad \operatorname{div} R_1 = 3. \quad (2.8)$$

By definition of the function ρ_0 , we have

$$R_1 \cdot \rho_0^4 = 2g(R_1 \cdot g) + 4(\bar{Z}_0 \cdot g)^3 (R_1 \cdot (\bar{Z}_0 \cdot g)).$$

Inequality (2.6) implies that \tilde{Z}_0 is tangent to Σ . Using that $\partial_s g \equiv 1$, we get that $R_1 \cdot g = 2g$. Let us compute $R_1 \cdot (\bar{Z}_0 \cdot g)$. As $\partial_s(\bar{Z}_0 \cdot g) = 0$, we have

$$R_1 \cdot (\bar{Z}_0 \cdot g) = \frac{1}{2}(\bar{Z}_0 \cdot g) \left(\tilde{Z}_0 \cdot (\bar{Z}_0 \cdot g) \right).$$

Then we have $R_1 \cdot \rho_0^4 = 4\rho_0^4$. Let us notice that Z_0 does not belong to the family $(Z_{j_\ell})_{1 \leq \ell \leq r}$. Thus \bar{Z}_0 commutes with the vector fields Z_{j_ℓ} . By definition of \tilde{Z}_0 , we infer

$$[\tilde{Z}_0, \bar{Z}_0] = [Z_0, \bar{Z}_0] + \sum_{\ell=1}^r \alpha_\ell [Z_\ell, \bar{Z}_0] = 2\partial_s. \quad (2.9)$$

By definition of \tilde{Z}_0 , we have $\tilde{Z}_0 \cdot g = 0$. Thus we get

$$\tilde{Z}_0 \cdot (\bar{Z}_0 \cdot g) = \bar{Z}_0 \cdot (\tilde{Z}_0 \cdot g) + 2\partial_s g = 2. \quad (2.10)$$

It turns out that $R_1 \cdot \rho_1^4 = 4g^2 + 4(\bar{Z}_0 \cdot g)^4 = 4\rho_1^4$. Now, let us compute $\operatorname{div} R_1$. We have

$$\operatorname{div} R_1 = 2\partial_s g + \frac{1}{2}\tilde{Z}_0 \cdot (\bar{Z}_0 \cdot g) + (\bar{Z}_0 \cdot g) \operatorname{div} \tilde{Z}_0.$$

Using that the vector fields Z_j are divergence free, the fact that $\partial_s g \equiv 1$ and (2.10), we get that $\operatorname{div} R_1 = 3$. Thus Assertion (2.8) is proved.

In order to continue the proof of Theorem 2.1, let us observe that, near w_0 , the set $\rho_0^{-1}(0)$ is a submanifold of \mathbb{H}^d of codimension 2. The following lemma will allow us to assume all along the proof that u belongs to $\mathcal{D}(V \setminus \rho_0^{-1}(0))$.

Lemma 2.2. *Let V be a bounded domain of \mathbb{H}^d and Γ a submanifold of codimension ≥ 2 . Then $\mathcal{D}(V \setminus \Gamma)$ is dense in the space $H_0^1(\mathbb{H}^d, V)$ of functions of $H_0^1(\mathbb{H}^d)$ supported in V equipped with the norm*

$$\left(\|u\|_{L^2}^2 + \|\nabla_{\mathbb{H}} u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Proof of Lemma 2.2 As $H_0^1(\mathbb{H}^d, V)$ is a Hilbert space, it is enough to prove that the orthogonal of $\mathcal{D}(V \setminus \Gamma)$ is $\{0\}$. Let u be in this space. For any v in $\mathcal{D}(V \setminus \Gamma)$, we have

$$(u|v)_{L^2} + (\nabla_{\mathbb{H}} u | \nabla_{\mathbb{H}} v)_{L^2} = 0.$$

By integration by part, this implies that

$$\forall v \in \mathcal{D}(V \setminus \Gamma), \quad \langle u - \Delta_{\mathbb{H}} u, v \rangle = 0.$$

Thus the support of $u - \Delta_{\mathbb{H}} u$ is included in Γ . As $Z_j u$ belongs to L^2 , then $Z_j^2 u$ belongs to $H^{-1}(\mathbb{R}^{2d+1})$ (the classical Sobolev space). And except 0, no distribution of $H^{-1}(\mathbb{R}^{2d+1})$ can be supported in a submanifold of codimension greater than 1. Thus $u - \Delta_{\mathbb{H}} u = 0$. Taking the L^2 scalar product with u implies that $u \equiv 0$. \blacksquare

Thanks to Inequality (2.7), we have

$$\rho_0^{-2} = -\frac{1}{2}R_1 \cdot \rho_0^{-2}. \quad (2.11)$$

Thus by integration by part, we have, using Inequality (2.7),

$$\int \frac{u^2}{\rho_0^2} dw = \frac{3}{2} \int \frac{u^2}{\rho_0^2} dw + I \quad \text{with} \quad I \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} (R_1 \cdot u) dw.$$

In order to estimate I , which contains terms of the type $g\partial_s u$, we have to compute the vector field R_1 in term of elements of \mathcal{Z} . Using (2.9), we infer that

$$R_1 = 2g[\tilde{Z}_0, \bar{Z}_0] + \frac{1}{2}(\bar{Z}_0 \cdot g)\tilde{Z}_0.$$

We deduce that

$$\begin{aligned} I &= J_1 + J_2 \quad \text{with} \\ J_1 &\stackrel{\text{def}}{=} \frac{1}{2} \int \frac{u}{\rho_0} \frac{(\bar{Z}_0 \cdot g)}{\rho_0} (\tilde{Z}_0 \cdot u) dw \quad \text{and} \\ J_2 &\stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g[\tilde{Z}_0, \bar{Z}_0] \cdot u dw. \end{aligned}$$

By definition of ρ_0 , Cauchy Schwarz inequality yields

$$|J_1| \leq C \left\| \frac{u}{\rho_0} \right\|_{L^2} \|\nabla_{\mathbb{H}} u\|_{L^2}. \quad (2.12)$$

The estimate about J_2 is a little bit more difficult to obtain. Let us write that $J_2 = K_1 - K_2$ with

$$K_1 \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g \tilde{Z}_0 \cdot (\bar{Z}_0 \cdot u) dw \quad \text{and} \quad K_2 \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g Z_0 \cdot (\tilde{Z}_0 \cdot u) dw.$$

By integration by parts, we have $K_1 = -K_{11} - K_{12}$ with

$$\begin{aligned} K_{11} &\stackrel{\text{def}}{=} \int \frac{g}{\rho_0^2} (\tilde{Z}_0 \cdot u) (\bar{Z}_0 \cdot u) dw \quad \text{and} \\ K_{12} &\stackrel{\text{def}}{=} \int f \frac{u}{\rho_0} (\bar{Z}_0 \cdot u) dw \quad \text{with} \quad f \stackrel{\text{def}}{=} \rho_0 \left(\tilde{Z}_0 \cdot \frac{g}{\rho_0^2} \right). \end{aligned}$$

By definition of ρ_0 , it is obvious that

$$|K_{11}| \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}^2. \quad (2.13)$$

Using that $\tilde{Z}_0 \cdot g = 0$, we get

$$\left| \tilde{Z}_0 \cdot \frac{g}{\rho_0^2} \right| = \frac{2g}{\rho_0^6} \left| \tilde{Z}_0 \cdot (\bar{Z}_0 \cdot g) \right| |\bar{Z}_0 \cdot g|^3 \leq C \frac{g}{\rho_0^3} \leq \frac{C}{\rho_0}.$$

This ensures that f is bounded on V and thus by Cauchy-Schwarz inequality,

$$\|K_{12}\| \leq C \left\| \frac{u}{\rho_0} \right\|_{L^2} \|\nabla_{\mathbb{H}} u\|_{L^2}.$$

Together with (2.13), this proves that

$$|K_1| \leq C \left(\left\| \frac{u}{\rho_0} \right\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2} \right) \|\nabla_{\mathbb{H}} u\|_{L^2}. \quad (2.14)$$

In order to estimate K_2 , let us write that, by integration by parts,

$$K_2 = \int \frac{g}{\rho_0^2} (\bar{Z}_0 \cdot u) (\tilde{Z}_0 \cdot u) dw + \int \rho_0 \left(\bar{Z}_0 \cdot \frac{g}{\rho_0^2} \right) \frac{u}{\rho_0} (\tilde{Z}_0 \cdot u) dw.$$

Using that

$$\bar{Z}_0 \cdot \rho_0^4 = 2g(\bar{Z}_0 \cdot g) + 4(\bar{Z}_0 \cdot (\bar{Z}_0 \cdot g)) (\bar{Z}_0 \cdot g)^3,$$

we immediatly get that the function $\rho_0 \left(\bar{Z}_0 \cdot \frac{g}{\rho_0^2} \right)$ is bounded on V and we deduce that

$$|K_2| \leq C \left(\left\| \frac{u}{\rho_0} \right\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2} \right) \|\nabla_{\mathbb{H}} u\|_{L^2}.$$

Together with (2.12) and (2.14), we infer that

$$\left\| \frac{u}{\rho_0} \right\|_{L^2}^2 \leq C \left(\left\| \frac{u}{\rho_0} \right\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2} \right) \|\nabla_{\mathbb{H}} u\|_{L^2}$$

which concludes the proof of Theorem 2.1.

3. THE PROOF OF THE TRACE AND TRACE LIFTING THEOREM

3.1. Some preliminary properties.

Proposition 3.1. *A neighbourhood W of w_0 exists such that the $C_{\Sigma_c}(W)$ modulus \mathcal{Z}_{Σ} spanned by the vector fields of $\mathcal{Z} \cap T\Sigma|_W$ which vanish on the characteristic submanifold Σ_c is of finite type and generated by*

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j.$$

Proof of Proposition 3.1 It is enough to prove that any element L of $\mathcal{Z} \cap T\Sigma$ which vanish on Σ_c is a combinaison (with coefficients in $C_{\Sigma_c}^{\infty}(W)$) of the $R_{j,k}$. By definition

$$L = \sum_{j=1}^{2d} \beta_j Z_j \quad \text{with} \quad \beta_j|_{\Sigma_c} = 0 \quad \text{and} \quad \sum_{j=1}^{2d} \beta_j (Z_j \cdot g) = 0.$$

Let us introduce a partition of unity $(\tilde{\psi}_j)_{1 \leq j \leq 2d}$ of the sphere \mathbb{S}^{2d-1} such that the support of $\tilde{\psi}_j$ is included in the set of ζ of \mathbb{S}^{2d-1} such that $|\zeta_j| \geq (4d)^{-1}$. Let us state

$$\psi_j \stackrel{\text{def}}{=} \tilde{\psi}_j \left(\frac{\nabla_{\mathbb{H}} g}{|\nabla_{\mathbb{H}} g|} \right).$$

It is an exercise left to the reader to check that ψ_j belongs to $C_{\Sigma_c}^{\infty}(W)$. On $\Sigma \setminus \Sigma_c$, we have, for any j in $\{1, \dots, 2d\}$,

$$\psi_j(L \cdot g) = \sum_{k=1}^{2d} \psi_j \beta_k (Z_k \cdot g) = 0.$$

By definition of ψ_j , $(Z_j \cdot g)$ does not vanish on the support of ψ_j . Thus we have

$$\beta_j \psi_j = -\frac{1}{(Z_j \cdot g)} \sum_{k \neq j} \psi_j \beta_k (Z_k \cdot g).$$

From this, we deduce that

$$\begin{aligned} \psi_j L &= \sum_{k \neq j} \psi_j \beta_k \left(Z_k - \frac{(Z_k \cdot g)}{(Z_j \cdot g)} Z_j \right) \\ &= \sum_{k \neq j} \frac{\psi_j \beta_k}{(Z_j \cdot g)} ((Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j). \end{aligned}$$

Now the facts that $\beta_k \in C_{\Sigma_c}^{\infty}$ and that $(Z_j \cdot g)$ does not vanish on the support of ψ_j ensure that

$$\beta_{j,k} \stackrel{\text{def}}{=} \frac{\varphi_j \beta_k}{(Z_j \cdot g)} \in C_{\Sigma_c}^{\infty}.$$

So we have

$$L = \sum_{1 \leq j < k \leq 2d} \beta_{j,k} ((Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j)$$

and the proposition is proved. ■

3.2. The blow up procedure. Using (1.3), Theorem 1.3 and its consequence (1.4), we have

$$\sum_{p=0}^{\infty} \left(\|\nabla_{\mathbb{H}}(\varphi_p u)\|_{L^2}^2 + 2^{2p} \|\varphi_p u\|_{L^2}^2 \right) \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}^2. \quad (3.15)$$

Now let us use the Heisenberg dilation $d_{2^{2p}}$ and state

$$u_p(w) \stackrel{\text{def}}{=} \varphi_0(w) u(2^p x, 2^p y, 2^{2p} s)$$

From (3.15), we immediatly infer that

$$\sum_{p=0}^{\infty} 2^{-2pd} \|u_p\|_{H^1(\mathbb{H}^d)}^2 \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}^2. \quad (3.16)$$

On the support of φ_0 , the hypersurface defined by $g(w) = s - f(x, y) = 0$ (which invariant under the action of Heisenberg dilation) is non characteristic because on the support of φ_0 , we have that ρ_c is between two positive constants. Thus, on $g = 0$, we have that $|\nabla_{\mathbb{H}} g|$ is between two positive constants. Thus Theorem 1.2 implies that a constant C exists (independent of p) such that

$$\|\gamma(u_p)\|_{[L^2(\Sigma), H^1(\mathcal{Z}_{\Sigma}, \Sigma)]_{\frac{1}{2}}} \leq C \|u_p\|_{H^1(\mathbb{H}^d)} \quad (3.17)$$

where $[A, B]_{\theta}$ denotes the complex interpolation between A and B .

Using the Heisenberg dilation d_{λ} and the fact that

$$\forall \lambda > 0, \forall v, \|v \circ d_{\lambda}\|_{L^2(\Sigma)} = \lambda^{-2d} \|v\|_{L^2(\Sigma)} \quad \text{and} \quad \|v \circ d_{\lambda}\|_{H^1(\mathcal{Z}_{\Sigma}, \Sigma)} = \lambda^{-2d} \|v\|_{H^1(\mathcal{Z}_{\Sigma}, \Sigma)}. \quad (3.18)$$

Then, using (3.15)–(3.17), we get that

$$\sum_{p=0}^{\infty} \|\gamma(\varphi_p u)\|_{[L^2(\Sigma), H^1(\mathcal{Z}_{\Sigma}, \Sigma)]_{\frac{1}{2}}}^2 \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}^2.$$

Moreover, an integer N_0 exists such that, if $|p - p'| \geq N_0$, then the support of φ_p and $\varphi_{p'}$ are disjoint. Thus, we have in particular that

$$\|\gamma(u)\|_{L^2(\Sigma)} \leq C \|\nabla_{\mathbb{H}} u\|_{L^2}.$$

Stating $\varphi_p^{\Sigma} = \varphi_p|_{\Sigma}$, let us define the following space $T^s(\Sigma)$.

Definition 3.1. For $s \in [0, 1]$, let us state

$$T^s(\Sigma) \stackrel{\text{def}}{=} \left\{ v \in L^2 / \|v\|_{T^s(\Sigma)}^2 \stackrel{\text{def}}{=} \sum_p \|\varphi_p^{\Sigma} v\|_{[L^2(\Sigma), H^1(\mathcal{Z}_{\Sigma}, \Sigma)]_s} < \infty \right\}.$$

We have (almost) proved the following theorem.

Theorem 3.1. *The restriction map on the hypersurface Σ can be extended in a continuous onto map from $H^1(\mathbb{H}^d \{ \rho_c \leq 1 \})$ onto the space $T^{\frac{1}{2}}(\Sigma)$.*

Proof of Theorem 3.1 The only think we still have to prove is the fact that γ is onto. Let us consider a function v in $T^{\frac{1}{2}}(\Sigma)$. By definition of $T^{\frac{1}{2}}(\Sigma)$, let us write

$$v = \sum_{p=0}^{\infty} \varphi_p^{\Sigma} v$$

Stating $v_p \stackrel{\text{def}}{=} \varphi_0^\Sigma v \circ d_{2^p}$, we infer from (3.18) that

$$\sum_{p=0}^{\infty} 2^{2pd} \|v_p\|_{[L^2(\Sigma), H^1(\mathcal{Z}_\Sigma, \Sigma)]_{\frac{1}{2}}}^2 \leq C \|v\|_{T^{\frac{1}{2}}(\Sigma)}^2.$$

As the support φ_0^Σ , and thus the support of the functions v_p , is included in the set $|\nabla_{\mathbb{H}} g| \sim 1$, Theorem 1.1 tells us that a function u_p exists in $H^1(\mathbb{H}^d)$ such that

$$\gamma(u_p) = v_p \quad \text{and} \quad \|\tilde{u}_p\|_{H^1(\mathbb{H}^d)} \leq C \|v_p\|_{[L^2(\Sigma), H^1(\mathcal{Z}_\Sigma, \Sigma)]_{\frac{1}{2}}}.$$

Let us consider a function $\tilde{\varphi}_0$ such that the support of $\tilde{\varphi}_0$ is included in a set where $\rho_c \sim 1$ and $\tilde{\varphi}_0$ has value 1 near the support of φ_0 . We obviously have

$$\gamma(\tilde{\varphi}_0 u_p) = v_p \quad \text{and} \quad \|\tilde{\varphi}_0 u_p\|_{H^1(\mathbb{H}^d)} \leq C \|\tilde{u}_p\|_{H^1(\mathbb{H}^d)}.$$

Then using (3.18) and stating $\tilde{\varphi}_p \stackrel{\text{def}}{=} \tilde{\varphi}_0 \circ d_{2^p}$, and $\tilde{u}_p \stackrel{\text{def}}{=} u_p \circ d_{2^p}$, we get after dilation that

$$\sum_{p=0}^{\infty} (\|\nabla_{\mathbb{H}}(\tilde{\varphi}_p \tilde{u}_p)\|_{L^2}^2 + 2^{2p} \|\tilde{\varphi}_p \tilde{u}_p\|_{L^2}^2) \leq C \|v\|_{T^{\frac{1}{2}}(\Sigma)}^2.$$

As an integer N_1 exists such that, if $|p - p'| \geq N_1$, then the support of $\tilde{\varphi}_p$ and $\tilde{\varphi}_{p'}$, we have that

$$\sum_{p=0}^{\infty} \tilde{\varphi}_p \tilde{u}_p \in H^1(\mathbb{H}^d) \quad \text{and} \quad \left\| \sum_{p=0}^{\infty} \tilde{\varphi}_p \tilde{u}_p \right\|_{H^1(\mathbb{H}^d)}^2 \leq C \|v\|_{T^{\frac{1}{2}}(\Sigma)}^2.$$

This ends the proof of Theorem 3.1.

Remarks –The trace lifting theorem provides functions in $H^1(\mathbb{H}^d)$ the support of which is included in a set of the form $s^2 \leq C(|x|^2 + |y|^2)^2$. It obviously prevents, using this method, to prove trace lifting theorem for very regular (for instance continuous) functions.

–The description given by Theorem 3.1 is not totally satisfactory. We want to describe this space of trace as an interpolation space to get Theorem 1.2.

3.3. Conclusion of the proof of Theorem 1.2. It is a point of interpolation theory which is to prove that

$$T^s(\Sigma) = [L^2(\Sigma), H^1(\mathcal{Z}_\Sigma, \Sigma)]_s.$$

This is the consequence the following two lemmas, the proof of which is omitted (we refer to [1] for the details).

Lemma 3.1. *The space $T^1(\Sigma)$ is equal to $H^1(\mathcal{Z}_\Sigma, \Sigma)$ and the norm are equivalent.*

Lemma 3.2. *Let us consider $(\mathcal{H}_j, \|\cdot\|_j)_{j \in \{0,1\}}$ two Hilbert spaces such that \mathcal{H}_1 is densely included in \mathcal{H}_0 and a family $(\mathcal{H}_{j,p})_{(j,p) \in \{0,1\} \times \mathbb{N}}$ such that, for any p , $\mathcal{H}_{j,p}$ is a closed subset of \mathcal{H}_j .*

Let us assume that a family of $(\Lambda_p)_{p \in \mathbb{N}}$ of (unbounded) selfadjoints operators on $\mathcal{H}_{0,p}$ exists such that $\mathcal{H}_{1,p}$ equals to the domain of Λ_p and

$$\forall u \in \mathcal{H}_{1,p}, \quad \|u\|_{\mathcal{H}_1} \sim \|\Lambda_p u\|_{\mathcal{H}_0}. \quad (3.19)$$

Let us assume in addition that a family of operators $(A_p)_{p \in \mathbb{N}}$ exists such that, for any (j,p) in $\{0,1\} \times \mathbb{N}$, the operator A_p is continuous from \mathcal{H}_j into $\mathcal{H}_{j,p}$ and

$$\forall v \in \mathcal{H}_j, \quad \lim_{p \rightarrow \infty} \left\| v - \sum_{p=0}^N A_p v \right\|_{\mathcal{H}_j} = 0 \quad \text{and} \quad \|v\|_{\mathcal{H}_j}^2 \sim \sum_p \|A_p v\|_{\mathcal{H}_j}^2. \quad (3.20)$$

Then,

$$[\mathcal{H}_0, \mathcal{H}_1]_s = \left\{ v \in \mathcal{H}_0 / \sum_{p=0}^{\infty} \|A_p v\|_{\mathcal{H}_{s,p}}^2 < \infty \right\} \quad \text{with} \quad \mathcal{H}_{s,p} \stackrel{\text{def}}{=} [\mathcal{H}_{0,p}, \mathcal{H}_{1,p}]_s.$$

4. CONCLUDING REMARKS

In fact, Theorem 1.2 is valid in more genreal situation than homogeneous hypersurface. But it remains necessary to do some assumptions on the nature of the characterisitic set Σ_c . In fact, this characteristic must be a hypersurface the codimension of which fits which the rank of the matrix $(Z_i \cdot Z_j \cdot g(w_0))_{1 \leq i, j \leq 2d}$. More precisely, we introduce in [1] the following notion of regular characteristic point.

Definition 4.1. *A characteristic point w_0 of a hypersurface Σ is a regular point of order r if and only if*

- i) *for any local defining function g of Σ , the rank of the matrix $(Z_i \cdot Z_j \cdot g(w_0))_{1 \leq i, j \leq 2d}$ is equal to r ;*
- ii) *near w_0 , the characteristic set Σ_c is a submanifold of Σ of codimension r in Σ .*

Let us notice that, as w_0 is a characteristic point, for any defining function g of Σ , $Z_{j \cdot g}(w_0) = 0$ for all j . Thus the first condition of the definition does not depend of the choice of the function g . In this more general, the strategy of the proof is the same than in the present simplified case, but the proof are more delicate because the function $Z_{j \cdot g}$ are of course not anymore linear form and the hypersurface Σ is not invariant under the action of the Heisenberg dilation d_λ . We refer to [1] for complete proofs in this case.

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