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Nonlinear Hypoellipticity of Infinite Type

By

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Abstract. We study the regularity of weak solutions for a class of second order semilinear infinitely degenerate elliptic equations. We get the regularity of weak solutions up to the boundary for Dirichlet problem, by noting the logarithmic regularity estimate for a linear principal part. In relation to this linear part, we also show the controllability and strong maximum principle for second order hypoelliptic operators even in the case where they degenerate infinitely. Model equations naturally come from some variational problems, if one replace the Laplace operator in such as Yamabe problems by degenerate elliptic operators. In the infinitely degenerate case, a permissible nonlinear term is not fractional power, compared with elliptic or subelliptic case. To treat this nonlinear term, the nonlinear microlocal analysis is developed in the logarithmic Sobolev space.

Key Words and Phrases. Nonlinear hypoellipticity, Hörmander's operators, Controllability, Maximum principle.

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1. Introduction

In this work, we study the C^∞ regularity of weak solutions for a class of second order semi-linear infinitely degenerate elliptic equation. Consider a system of vector fields $X = (X_1, \dots, X_m)$ defined on an open domain $\tilde{\Omega} \subset \boldsymbol{R}^n$. In the infinite degenerate case, the following is called logarithmic regularity estimate,

$$\|(\log \Lambda)^s u\|_{L^2}^2 \le C\left\{\sum_{j=1}^m \|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2\right\}, \qquad \forall u \in C_0^\infty(\tilde{\Omega}), \tag{1.1}$$

where $\Lambda = (e + |D|^2)^{1/2} = \langle D \rangle$. If the system X satisfies the finite type of Hörmander's condition then (1.1) holds for any real $s > 0$. On the other hand (1.1) admits the infinite degeneracy of the system X, and the estimate (1.1) with $s > 1$ always implies the interior hypoellipticity of the second order operator $\triangle_X = \sum_{j=1}^m X_j^* X_j$, where X_j^* is the formal adjoint of X_j (see [16]). We recall that the linear partial differential operator P is hypoelliptic in $\tilde{\Omega}$ if, for any open $\Omega \subset \tilde{\Omega}$, $u \in \mathscr{D}'(\Omega)$ and $Pu \in C^\infty(\Omega)$ imply $u \in C^\infty(\Omega)$. Some sufficient conditions for the estimate (1.1) can be seen in the Appendix of [19], and further sufficient conditions for variants of (1.1) have been investigated in the research of linear hypoelliptic operators (see [17, 18] and references therein, cf., [5, 11, 12] as for recent related works in the complex analysis). The simplest example for (1.1) is the system in $\boldsymbol{R}^3$ such as $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2}$, $X_3 = \exp(-|x_1|^{-1/s})\partial_{x_3}$ with $s > 0$ (see [14, 15, 16]). The operator $\triangle_X$ for this example degenerates infinitely on $\Gamma_0 = \{x_1 = 0\}$. The example with infinite degeneracy on a union of surfaces $\Gamma = \bigcup_j \Gamma_j$ is the system in $\boldsymbol{R}^2$ such as $X_1 = \partial_{x_1}$, $X_2 = \exp(-(x_1^2 \sin^2(\pi/x_1))^{-1/2s})\partial_{x_2}$, and we see that if $\Gamma_j = \{x_1 = 1/j\}$, $j \in \boldsymbol{Z}\backslash\{0\}$, $\Gamma_0 = \{x_1 = 0\}$, then X_1 is transversal to all Γ_j, $j \in \boldsymbol{Z}$ and X_2 vanishes infinitely on $\Gamma = \bigcup_{j \in \boldsymbol{Z}} \Gamma_j$.

It is known in [16] that the estimate (1.1) with $s = 1$ is not sufficient for hypoellipticity, but the following weak form of estimates is sufficient: For any small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|\log \Lambda v\|_{L^2}^2 \le \varepsilon \sum_{j=1}^{m} \|X_j v\|_{L^2}^2 + C_\varepsilon \|v\|_{L^2}^2, \qquad \forall v \in C_0^\infty(\tilde{\Omega}). \tag{1.2}$$

The estimate (1.1) with $s > 1$ implies immediately the estimate (1.2) by interpolation. We have a very simple example which satisfies the estimate (1.2), but not (1.1) for any $s > 1$. It is the system in $\boldsymbol{R}^3$ such as $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2}$, $X_3 = \exp(-(|x_1|\,|\log|x_1|\,|)^{-1})\partial_{x_3}$, (see [15, 19]).

Associated with the system of vector fields $X = (X_1, \dots, X_m)$, we define function spaces:

$$H_X^1(\tilde{\Omega}) = \{u \in L^2(\tilde{\Omega}); X_j u \in L^2(\tilde{\Omega}), j = 1, \dots, m\},$$

which are Hilbert spaces. We say that $u \in H_{X,\mathrm{loc}}^1(\tilde{\Omega})$, if $\alpha u \in H_X^1(\tilde{\Omega})$ for any $\alpha \in C_0^\infty(\tilde{\Omega})$.

For a smooth surface S of $\tilde{\Omega}$, we say that $x_0 \in S$ is a non characteristic point for the system of vector fields X, if there exists at least one vector field of $X_1, \dots, X_m$ which is transversal to S at x_0. We say that S is non characteristic for X if it is non characteristic for any point $x_0 \in S$. In this case, if $u \in H_X^1(\tilde{\Omega})$, the trace exists and belongs to $L^2(S)$, see [7].

Take $\Omega \subset\subset \tilde{\Omega}$ and suppose that $\partial\Omega$ is C^∞ and non characteristic for X. We define $H_{X,0}^1(\Omega) = \{u \in H_X^1(\Omega); u|_{\partial\Omega} = 0\}$. Then $H_{X,0}^1(\Omega)$ is a Hilbert subspace in $H_X^1(\Omega)$, containing $C_0^\infty(\Omega)$, and moreover the extension of an element of $H_{X,0}^1(\Omega)$ by 0 belongs to $H_X^1(\tilde{\Omega})$ (see Lemma 2.1 of [19]). Since for any $v \in H_{X,0}^1(\Omega)$ there exists a mollifier family $\{\rho_\varepsilon, \varepsilon > 0\}$ such that $\rho_\varepsilon * v \in C_0^\infty$, $\lim_{\varepsilon \to 0} \rho_\varepsilon * v = v$ in L^2 and $\|X(\rho_\varepsilon * v)\|_{L^2} \le C\{\|Xv\|_{L^2} + \|v\|_{L^2}\}$ with C independent of ε, the estimate (1.2) holds for any $v \in H_{X,0}^1(\Omega)$, which implies that $H_{X,0}^1(\Omega)$ is embedded compactly in $L^2(\Omega)$.

We consider the following semi-linear equation:

$$(1.3) \qquad \triangle_X u + X_0 u = F(x, u) \qquad \text{in } \Omega$$

where $F \in C^\infty(\Omega \times \boldsymbol{R})$ and X_0 a vector field on $\tilde{\Omega}$.

We have now the following nonlinear hypoelliptic result:

Theorem 1.1. *Suppose that the system of vector fields X satisfies the logarithmic regularity estimate* (1.2), *and $u \in H^1_{X,\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ is a weak solution of equation* (1.3). *Then we have the hypoellipticity, namely, $u \in C^\infty(\Omega)$.*

Furthermore, for the regularity up to the boundary, assume that $\partial\Omega$ is C^∞ and non characteristic. If $g \in C^\infty(\partial\Omega)$ and $u \in H^1_X(\Omega) \cap L^\infty(\Omega)$ is a weak solution of equation (1.3) *with $u|_{\partial\Omega} = g$, then $u \in C^\infty(\bar{\Omega})$.*

Remark. If the function F in (1.3) is linear for u, the interior regularity is just hypoellipticity of operators $\triangle_X + X_0$ (see [16]), but the regularity up to the boundary for linear Dirichlet problem is new. We remark that the regularity up to the boundary $\partial\Omega$ even in linear case can not be expected in general if $\partial\Omega$ possesses characteristic points for the system X.

We give here an example of equation (1.3) coming from a variational problem. From (1.1), we have the following logarithmic Sobolev inequality (see [19]),

$$(1.4) \qquad \int_\Omega |v|^2 \left| \log\left(e + \frac{|v|^2}{\|v\|^2_{L^2}} \right) \right|^{2s-1} dx \le C_0 \left\{ \sum_{j=1}^m \|X_j v\|^2_{L^2} + \|v\|^2_{L^2} \right\},$$

for all $v \in H^1_{X,0}(\Omega)$. Suppose that $1 \le k < 2(s-1)$, take $A = (a_1, \dots, a_k) \in \boldsymbol{R}^k$, and consider the following variational problems:

$$(1.5) \qquad I_A = \inf_{\|v\|_{L^2}=1,\, v \in H^1_{X,0}(\Omega)} \left\{ \sum_{j=1}^m \|X_j v\|^2_{L^2} - \sum_{j=1}^k a_j \int_\Omega |v|^2 (\log(e + v^2))^j dx \right\}.$$

Theorem 1.2. *Assume that the system of vector fields X verifies the regularity estimate* (1.1) *for $s > 3/2$, $\partial\Omega$ is C^∞ and non characteristic. Then I_A is an attained minimum in $H^1_{X,0}(\Omega)$, and the minimizer belongs to $C^\infty(\bar{\Omega})$.*

In fact, by exactly the same calculus as in [19], the inequality (1.4) and Poincaré inequality (see Lemma 2.1) give the existence of minimizer $u \in H^1_{X,0}(\Omega)$ for the variational problems (1.5), and the minimizer is a bounded non trivial positive weak solution of the following Euler-Lagrange equation;

$$(1.6) \qquad \triangle_X u = F(u),$$

with nonlinear term

$$F(t) = \sum_{j=1}^{k} a_j \left(t(\log(e+t^2))^j + \frac{j}{2} \frac{t^3}{e+t^2} (\log(e+t^2))^{j-1} \right) + b_0 t \in C^\infty(\boldsymbol{R}),$$

where b_0 is a constant depending on the minimizer u.

Since b_0 can not be freely chosen, Theorem 1.2 is just a result of variational problems, which gives certain semi-linear partial differential equations.

As natural semi-linear partial differential equations from which one can start, we can consider the semi-linear Dirichlet problems

$$\begin{cases} \triangle_X u = au \log|u| + bu & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.7}$$

where $a, b \in \boldsymbol{R}$.

These Dirichlet problems (1.7) correspond to those in the elliptic and subelliptic case (see [3, 9, 23]), where the nonlinear terms are the form of $u^{p-1} + \lambda u$, and $p > 2$ is the corresponding Sobolev's index. For our infinitely degenerate operators, we have only the logarithmic Soboloev inequality (1.4), so that our nonlinear terms are logarithmic. Here the nonlinear function $\tilde{F}(t) = at \log|t| + bt$ are only continuous at $t = 0$. We can not use directly Theorem 1.1 to problem (1.7).

Theorem 1.3. *Suppose that $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X. Assume that the system of vector fields X verifies the estimate* (1.1) *for $s > 3/2$. Then, if $a < 0$, the Dirichlet problem* (1.7) *has at least one solution $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ and $u(x) > 0$ for $x \in \Omega$.*

As to this semi-linear problem (1.7), let us recall the results obtained in [19]; by using the variational principle we proved that the semi-linear Dirichlet problem (1.7) possesses at least one non-negative weak solution $u \in H^1_{X,0}(\Omega) \cap L^\infty(\Omega)$ with $\|u\|_{L^2(\Omega)} > 0$. The proof of boundedness of weak solution is principal result of [19], where we need $s > 3/2$ in the estimate (1.1). For the regularity of this weak solutions of (1.7), we see that if Γ is the set of infinitely degenerate points of $\tilde{\Omega}$ for X, then the system of vector fields X satisfies the finite type of Hörmander's condition on $\tilde{\Omega} \backslash \Gamma$, so that the regularity results of [20, 22] and maximal principle of J.-M. Bony [1] imply that $u \in C^\infty(\Omega \backslash \Gamma) \cap C^0(\bar{\Omega} \backslash \Gamma)$ and $u(x) > 0$ for all $x \in \Omega \backslash \Gamma$.

In this work, we study the crucial part for the C^∞ regularity of solution at the infinitely degenerate points Γ of Ω. In the equation (1.7), the nonlinear functions $\tilde{F}(t) = at \log|t| + bt \in C^\infty(\boldsymbol{R} \backslash \{0\})$, but they are only continuous at $t = 0$. To get the higher regularity of solution u, we need that the nonlinear composition $\tilde{F}(u)$ has (almost) same regularity as u (see precisely Theorem 4.1), and hence we have to show $u(x) > 0$ for any $x \in \Omega$. To this end, we prove, firstly in section 2, the controllability and maximum principle for hypoelliptic operators $\triangle_X$.

Theorem 1.4. *Let Ω be a bounded and connected open subdomain of $\tilde{\Omega}$, $\mathscr{L}(X_1, \dots, X_m)$ be the Lie algebra spanned by the system of vector fields X and their commutators. If $\triangle_X + c(x)$ is hypoelliptic in Ω for any $c \in C^\infty(\Omega)$, then any two points of Ω can be linked by continuous curve made of a finite numbers of the integral paths of vector fields belonging to $\mathscr{L}(X_1, \dots, X_m)$,* (*this property is called the controllability*).

Remark. 1) It should be noted that the controllability follows only from the hypoellipticity of $\triangle_X + c(x)$. Conversely, it is known (see [14, 15, 16, 18]) that the controllability does not imply the hypoellipticity of $\triangle_X$ in general. The first example with $0 < s \le 1$ given at the beginning of this introduction, satisfies the controllability but not the hypoellipticity.

2) Logarithmic regularity estimate (1.2) implies the hypoellipticity of $L = \triangle_X + c(x)$ in any open subdomain of $\tilde{\Omega}$ and for any $c \in C^\infty(\Omega)$ (see [16, 18]).

3) The controllability results given in this proposition enable us to define the distance (Carnot-Carathéodory metric) associated with $\triangle_X$ similarly to the finitely degenerate case (cf., [10, 8]). This metric might set light aglow in the analysis for infinitely degenerate hypoelliptic operators (cf., another aspect by [4]).

At the present, from this controllability results we have immediately the strong maximum principle, on account of Bony's result ([1]).

Theorem 1.5. *Suppose that $\Delta_X + c(x)$ is hypoelliptic in Ω for any $c \in C^\infty(\Omega)$. We consider the operators $L = \Delta_X + a(x)$ with $a \in C^\infty(\Omega)$, $a(x) \ge 0$. If $u \in C^2(\Omega)$, $Lu \le 0$, then u can not take its positive maximum on interior points of Ω except that it is constant on the connected component of those points.*

The structure of the paper is as follows: We prove the controllability and maximum principle for hypoelliptic operators in the second section. The third section consists of the proof for the continuity and strict positivity of weak solutions of problems (1.7). The fourth section is devoted to the Littlewood-Paley theory for non-homogeneous Sobolev spaces defined by logarithmic type weights, and the nonlinear calculus. In the fifth section, we prove our Theorem 1.1 and Theorem 1.3. In the last section, we study pseudo-differential calculus for logarithmic symbol.

2. Controllability and maximum principle

We prove now Theorem 1.4, that is, the controllability results for hypoelliptic operators. Let Ω be a bounded and connected open sub-domain of $\tilde{\Omega}$. Denote by $\mathscr{L}(X_1, \dots, X_m)$ the Lie algebra spanned by the system of vector fields X and their commutators. For $x_0 \in \Omega$, we denote by Ω_{x_0} a maximal subset of Ω whose points can be linked to x_0 along a continuous curve made of finite numbers of the integral paths of vector fields belonging to $\mathscr{L}(X_1, \dots, X_m)$. We have that $\bigcup_{x_0 \in \Omega} \Omega_{x_0}$ is a partition of Ω, i.e. for any $x_1, x_2 \in \Omega$, we have that $\Omega_{x_1} = \Omega_{x_2}$ or $\Omega_{x_1} \cap \Omega_{x_2} = \emptyset$. We prove Theorem 1.4 as the following proposition.

Proposition 2.1. *If $L = \Delta_X + c(x)$ is hypoelliptic in Ω for any $c \in C^\infty(\Omega)$, then we have*

$$\Omega_{x_0} = \Omega \qquad \text{for any } x_0 \in \Omega. \tag{2.1}$$

Furthermore, the conclusion is also valid if the operators L is replaced by $\Delta_X + X_0 + c(x)$, and if Ω_{x_0} is defined by Lie algebra $\mathscr{L}(X_0, X_1, \dots, X_m)$.

Proof. We shall prove the second statement of the proposition by two steps.

Claim 1: If $\mathring{\Omega}_{x_0} \neq \emptyset$ then Ω_{x_0} is open.

It follows from the assumption that there exist $y_0 \in \Omega_{x_0}$ and an $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subset \Omega_{x_0}$, where $B(y_0, \varepsilon)$ denotes a ball with a radius $\varepsilon > 0$ and centered at $y_0 \in \Omega$. Let z_0 be an arbitrary point in Ω_{x_0}. Then we can find a suitable continuous curve, made of integral paths of vector fields belonging to $\mathscr{L}(X_0, \dots, X_m)$, which links z_0 and y_0 via x_0. More precisely, there exist an interval $[0, T] = \bigcup_{j=1}^{N} [T_{j-1}, T_j]$ $(T_0 = 0, T_N = T)$ and a piecewise smooth vector field Z belonging smoothly to $\mathscr{L}(X_0, \dots, X_m)$ in each subinterval $[T_{j-1}, T_j]$ such that $x(T, z_0) = y_0$, where $x(t; z)$ denotes the solution to

$$\frac{dx}{dt} = Z(x(t)), \qquad x(0) = z.$$

By means of the continuity of the solution with respect to the initial value, there exists a $\delta > 0$ such that $x(T; z) \subset B(y_0, \varepsilon)$ if $z \in B(z_0, \delta)$. This shows $B(z_0, \delta) \subset \Omega_{x_0}$. Since z_0 is arbitrary we get the conclusion.

Set $B = \bigcup_{x_0 \in \Omega} \mathring{\Omega}_{x_0}$. Since Ω is connected it suffices to show

Claim 2: $(\Omega \backslash B) = \emptyset$.

Suppose this is not valid. Then $\exists x_0 \notin \mathring{\Omega}_{x_0}$ and it follows from step 1 that $\mathring{\Omega}_{x_0} = \emptyset$. For the multi-index $J = (j_1, \dots, j_k)$, $1 \le j_\ell \le m$, let X_J denote the repeated commutator $[X_{j_1}, [X_{j_2}, \dots, [X_{j_{k-1}}, X_{j_k}] \dots]]$. For $y \in \Omega$, we set

$$\Gamma_y := \text{sub-tangent space of } y \in \Omega \text{ spanned by} \tag{2.2}$$

$$\{X_J(y); \forall J = (j_1, \dots, j_k), 1 \le j_\ell \le m\}.$$

We have obviously

$$\dim(\Gamma_y) \leq n - 1 \qquad \text{for all } y \in \Omega_{x_0}. \tag{2.3}$$

In fact if $y_0 \in \Omega_{x_0}$ such that $\dim(\Gamma_{y_0}) = n$, then the system of vector fields X satisfies the finite Hörmander's condition in a neighborhood of y_0 and $y_0 \in \mathring{\Omega}_{y_0}$, which is contradictory to $\mathring{\Omega}_{x_0} = \emptyset$. For the proof of claim 2, we shall show that if there exists $x_0 \in \Omega$ such that $\exists x_0 \notin \mathring{\Omega}_{x_0}$, then there exists a function $c \in C^\infty(\Omega)$ such that the operator $\Delta_X + X_0 + c(x)$ is non-hypoelliptic; more precisely,

$$\begin{cases} \Omega_{x_0} \text{ contains a submanfold } S \text{ such that all } X_j \text{ are tangent to } S \text{ and} \\ \text{there exist a } c(x) \text{ and a singular solution } v \text{ on } S \text{ to} \\ \Delta_X v + X_0 v + c(x) v = 0. \end{cases}$$

a) We want to prove that $\dim(\Gamma_y) \leq n - 2$ for any $y \in \Omega_{x_0}$. If there exists a $y_0 \in \Omega_{x_0}$ such that $\dim(\Gamma_{y_0}) = n - 1$, then one can find linearly independent vectors $X_{J_1}(y_0), \dots, X_{J_{n-1}}(y_0)$ such that

$$\Gamma_{y_0} = \text{sub-tangent space spanned by } \{X_{J_1}(y_0), \dots, X_{J_{n-1}}(y_0)\}.$$

It follows from the continuity with respect to y that $X_{J_1}(y), \dots, X_{J_{n-1}}(y)$ are linearly independent in a sufficiently small neighborhood ω_0 of y_0. By means of (2.3) we have $\dim(\Gamma_y) = n - 1$ for any $y \in \omega_0 \cap \Omega_{x_0}$ and

$$\Gamma_y = \text{sub-tangent space spanned by } \{X_{J_1}(y), \dots, X_{J_{n-1}}(y)\} \tag{2.4}$$

$$\forall y \in \omega_0 \cap \Omega_{x_0}.$$

Consider C^∞ mapping

$$\Phi_t : \boldsymbol{R}^{n-1} \ni a = (a_1, \dots, a_{n-1}) \to \left(\exp t \sum_{j=1}^{n-1} a_j X_{J_j} \right) y_0 \in \boldsymbol{R}^n$$

for $t > 0$. If a is in a sufficiently small neghborhood U of 0, this mapping is well defined for $0 \leq t \leq 1$. We have

$$\operatorname{rank} \left. \frac{\partial \Phi_1}{\partial a} \right|_{a=0} = \operatorname{rank}(X_{J_1}, \dots, X_{J_{n-1}})|_{x=y_0} = n - 1$$

because the solution of differential equation with respect to t is differentiable with respect to the parameter a. Therefore the image of a small neighborhood $U \subset \boldsymbol{R}_a^{n-1}$ by the mapping Φ_1 makes a smooth hypersurface S_{y_0} through y_0, which is generated by $X_{J_1}(y), \dots, X_{J_{n-1}}(y)$ and $S_{y_0} \subset \Omega_{x_0} \cup \omega_0$. By using (2.4), all vector fields $X_0, \dots, X_m$ tangent to smooth surface S_{y_0}.

Remark that our problem is different from Frobenius theorem. We construct only one surface (or submanifold) passing through y_0, and (2.4) implies that $T_y S_{y_0} = \Gamma_y$ for any $y \in S_{y_0} \subset \Omega_{x_0}$.

Decompose ω_0 into two parts $\omega_0 = \omega_0^+ \cup S_{y_0} \cup \omega_0^-$ and denote by $\boldsymbol{n}(y)$ the unit normal vector of S_{y_0} at y. If v is a function such that $v = 1$ in ω_0^+ and $v = 0$ in another side, then it follows from the Gauss theorem that we have for any $\varphi \in C^\infty(\omega_0)$, $j = 0, \dots, m$

$$\langle X_j v, \varphi \rangle = \int_{\omega_0} v X_j^* \varphi \, dx = \int_{\omega_0^+} X_j^* \varphi \, dx = \int_{S_{y_0}} X_j(y, \boldsymbol{n}(y)) \varphi \, dS = 0,$$

since $X_j(y, \boldsymbol{n}(y)) = 0$ for any $y \in S_{y_0}$, $j = 0, \dots, m$. We have proved $X_j v = 0$, $j = 0, \dots, m$ in $\mathscr{D}'(\omega_0)$ and so $(\varDelta_X + X_0)v = 0$ in $\mathscr{D}'(\omega_0)$. This is contradictory to the hypoellipticity of $\varDelta_X + X_0$ in Ω.

b) Suppose that there exists $2 \le k \le n-1$ such that

$$\dim(\Gamma_y) \le n - k \qquad \text{for all } y \in \Omega_{x_0}. \tag{2.5}$$

Then we shall show $\dim(\Gamma_y) \le n - k - 1$ for all $y \in \Omega_{x_0}$. If there exists a $y_0 \in \Omega_{x_0}$ such that $\dim(\Gamma_{y_0}) = n - k$ then, by the same reason as in the step a), there exists a submanifold S_{y_0} of codimension k, generated by a suitable base $X_{J_1}, \dots, X_{J_{n-k}}$, such that $S_{y_0} \subset \Omega_{x_0}$. By taking the change of variables we may assume locally in a small neighborhood ω of $y_0 = (0, y_0'')$ that $S_{y_0} = \{(0, x_{k+1}, \dots, x_n)\}$ and

$$X_{J_i} = \sum_{\ell=1}^{n} \tilde{a}_{\ell, J_i}(x) \partial_{x_\ell}, \qquad \text{with } \tilde{a}_{\ell, J_i}(0, x_{k+1}, \dots, x_n) = 0 \quad \text{for } \ell = 1, \dots, k.$$

On account of (2.5) we have for any $j = 0, \dots, m$

$$X_j = \sum_{\ell=1}^{n} \tilde{a}_{\ell, j}(x) \partial_{x_\ell}, \qquad \text{with } \tilde{a}_{\ell, j}(0, x_{k+1}, \dots, x_n) = 0 \quad \text{for } \ell = 1, \dots, k, \tag{2.6}$$

because of the Cramer formula. In new variables, we have

$$\begin{aligned}
\langle X_j^* X_j \delta(x') \otimes 1, \tilde{\varphi} \rangle &= \langle \delta(x') \otimes 1, X_j^* X_j \tilde{\varphi} \rangle \\
&= -\int_{\boldsymbol{R}_{x''}^{n-k}} \sum_{\ell=1}^{k} \sum_{i=k+1}^{n} (\partial_{x_\ell} \tilde{a}_{\ell, j})(0, x'') \tilde{a}_{i, j}(0, x'') \partial_{x_i} \tilde{\varphi}(0, x'') dx'' \\
&\quad - \int_{\boldsymbol{R}_{x''}^{n-k}} \sum_{\ell, i=k+1}^{n} \partial_{x_\ell} (\tilde{a}_{\ell, j}(0, x'') \tilde{a}_{i, j}(0, x'') \partial_{x_i} \tilde{\varphi}(0, x'')) dx'' \\
&= \int_{\boldsymbol{R}_{x''}^{n-k}} \sum_{\ell=1}^{k} \sum_{i=k+1}^{n} \partial_{x_i} ((\partial_{x_\ell} \tilde{a}_{\ell, j})(0, x'') \tilde{a}_{i, j}(0, x'')) \tilde{\varphi}(0, x'') dx'' \\
&= -\langle c_j(x) \delta(x') \otimes 1, \tilde{\varphi} \rangle \qquad \text{for all } \tilde{\varphi}(x) \in C_0^\infty(\omega),
\end{aligned}$$

where $c_j(x) = -\sum_{\ell=1}^{k}\sum_{i=k+1}^{n}\partial_{x_i}((\partial_{x_\ell}\tilde{a}_{\ell,j})(x)\tilde{a}_{i,j}(x))$. For X_0 we have the term

$$\begin{aligned}\langle X_0\delta(x')\otimes 1,\tilde{\varphi}\rangle &= \langle\delta(x')\otimes 1, X_0^*\tilde{\varphi}\rangle\\ &= -\int_{\boldsymbol{R}^{n-k}_{x''}}\sum_{\ell=1}^{k}(\partial_{x_\ell}\tilde{a}_{\ell,0})(0,x'')\tilde{\varphi}(0,x'')dx''\\ &\quad+\int_{\boldsymbol{R}^{n-k}_{x''}}\sum_{\ell=k+1}^{n}\partial_{x_\ell}(\tilde{a}_{\ell,0}(0,x'')\tilde{\varphi}(0,x''))dx''\\ &= -\langle c_0(x)\delta(x')\otimes 1,\tilde{\varphi}\rangle \qquad \text{for all } \tilde{\varphi}(x)\in C_0^\infty(\omega),\end{aligned}$$

where $c_0(x) = \sum_{\ell=1}^{k}\partial_{x_\ell}\tilde{a}_{\ell,0}(x)$. With $c(x) = \sum_{j=0}^{m}c_j(x)$ we have shown

$$(\varDelta_X + X_0 + c(x))(\delta(x')\otimes 1) = 0 \qquad \text{in } \mathscr{D}'(\omega),$$

which is the contradiction. If we repeat this inductive procedure we attains to the fact that $\dim(\varGamma_y) = 0$ for all $y\in\varOmega_{x_0}$, that is, $\varOmega_{x_0} = \{x_0\}$. In this case we also find δ_{x_0} is a singular solution to $(\varDelta_X + X_0 + a_0)v = 0$ with $a_0(x) = \operatorname{div}(X_0)$. Now the proof is complete.

Remark. If we denote the change of variables by $z = \psi(x)$ with the original variable z, then the singular solution in the step b) is written by the original variables as

$$\left|\frac{\partial\psi^{-1}(z)}{\partial z}\right|(\delta(x')\otimes 1)|_{x=\psi^{-1}(z)}.$$

The construction of the singular solution in the step b) is also applicable in the case S of codimension 1 though the singular solution in the step a) was constructed with $c(x) = 0$.

Exactly the same proof as the last part of the step a) gives also the following results (see [7] for finitely degenerate case)

Proposition 2.2. *If $L = \varDelta_X$ is hypoelliptic in $\varOmega$ and if S a smooth surface of $\varOmega$, then the set of non characteristic points of S for X is an open and dense set in S.*

If $x_0\in S$ and V_{x_0} a neighbourhood of x_0 in $\varOmega$, then $V_{x_0}\cap S$ possesses the non characteristic points. In fact, if it is not true, $V_{x_0}\cap S$ is a smooth surface and all vector fields tangent to it, then $\triangle_X$ is non-hypoelliptic in V_{x_0}.

Proof of Theorem 1.5. From Proposition 2.1, we can joint any two points of $\varOmega$ by a continuous curve made of a finite numbers of the integral paths of vector fields belonging to $\mathscr{L}(X_1,\dots,X_m)$. By using proposition 2.1 of

Bony [1], the continuous curve can be approximated uniformly by a piecewise continuous integral paths of vector fields $X_1, \dots, X_m$, then propagation of maximum of J.-M. Bony (see [1]) deduces Theorem 1.5.

From the maximum principle of Theorem 1.5, we have the following first Poincaré inequality for infinite degenerate hypoelliptic system of vector fields.

Lemma 2.1. *Suppose that the system of vector fields X satisfies the estimate* (1.2). *If $\Omega \subset\subset \tilde{\Omega}$, $\partial\Omega$ is C^∞ and non characteristic for X, then we have the Poincaré inequality*

$$\|v\|_{L^2(\Omega)}^2 \le \frac{1}{\lambda_1} \sum_{j=1}^m \|X_j v\|_{L^2(\Omega)}^2, \qquad \forall v \in H_{X,0}^1(\Omega), \tag{2.7}$$

where λ_1 is the first eigenvalue of Dirichlet problem for $\triangle_X$ in Ω.

Remark. 1) By using this lemma, in the Hilbert space $H_{X,0}^1(\Omega)$, we can use $\|X\varphi\|_{L^2(\Omega)} = (\sum_{j=1}^m \|X_j\varphi\|_{L^2(\Omega)}^2)^{1/2}$ as norm.

2) It will be seen below that (2.7) holds if $\triangle_X + c$ is hypoelliptic in Ω for any $c \in C^\infty(\Omega)$, and if $H_{X,0}^1(\Omega)$ is compactly embedded into $L^2(\Omega)$. The estimate (1.2) is a sufficient condition for those.

Proof. We set

$$\lambda_1 = \inf_{\|\varphi\|_{L^2} = 1, \varphi \in H_{X,0}^1(\Omega)} \{\|X\varphi\|_{L^2}^2\}.$$

Suppose that $\lambda_1 = 0$. Then there exists $\{\varphi_j\} \subset H_{X,0}^1(\Omega)$ such that $\|X\varphi_j\|_{L^2(\Omega)} \to 0$ and $\|\varphi_j\|_{L^2(\Omega)} = 1$. By using (1.2), $H_{X,0}^1(\Omega)$ is compactly embedded into $L^2(\Omega)$. The variational calculus deduces that there exists $\varphi_0 \in H_{X,0}^1(\Omega)$, $\|\varphi_0\|_{L^2} = 1$, $\varphi_0 \ge 0$ satisfying

$$\triangle_X \varphi_0 = 0, \qquad \|X\varphi_0\|_{L^2(\Omega)} = 0.$$

Since $\triangle_X$ is hypoelliptic in Ω, we have $\varphi_0 \in C^\infty(\Omega)$ and

$$X_j\varphi_0(x) = 0, \qquad \forall x \in \Omega, \text{ and } j = 1, \dots, m.$$

This implies that φ_0 is constant along the integral paths of vector fields of $X_1, \dots, X_m$. Now the controllability of Proposition 2.1 deduces that φ_0 is constant on each connected component of Ω.

Since $\partial\Omega$ is smooth and non characteristic, by taking $x_0 \in \partial\Omega$, we may assume that X_1 transverse $\partial\Omega$ near x_0. Then $X_1\varphi_0(x) = 0$ implies that $\varphi_0(x) = 0$ near x_0, which shows that $\varphi_0 \equiv 0$ on Ω. This is impossible because $\|\varphi_0\|_{L^2} = 1$, so that we prove finally $\lambda_1 > 0$.

3. Weak solutions of linear Dirichlet problems

The regularity of weak solutions in Theorems 1.1 and 1.3 is based on the results of linear equations which we first study in this section.

We have now the following result in the infinitely degenerate case.

Theorem 3.1. *Suppose that the system of vector fields X verifies the estimate* (1.2) *on* $\tilde{\Omega}$, $\partial\Omega$ *is* C^∞ *and non characteristic. For* $f \in C^\infty(\bar{\Omega})$ *and* $g \in C^\infty(\partial\Omega)$, *consider the operators* $L = \sum_{j=1}^m X_j^* X_j + a(x)$ *with* $a \in C^\infty(\bar{\Omega})$, $a(x) \geq 0$, *and the linear Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u|_{\partial\Omega} = g. \end{cases} \tag{3.1}$$

Then the Dirichlet problem (3.1) *possesses a unique solution* $u \in C^\infty(\bar{\Omega})$.

Remark. 1) The assumption $a(x) \geq 0$ is used for the existence of weak sloutions and the uniqueness of solutions, but it is not necessary for the regularity of solutions.

2) For the regularity of functions f and g, we can consider in Sobolev space. Precisely, if we suppose that $f \in H^s(\Omega)$, $g \in H^{s+3/2}(\partial\Omega)$ for some $s > -3/2$, we can prove, with a very small modification, that the solution of Theorem 3.1 is in $H^s(\Omega)$.

3) If the boundary $\partial\Omega$ possesses the characteristic points, the problem of regularity up to the boundary is very complicated and it is still an open problem in the general case.

Existence of weak solution: Since $g \in C^\infty(\partial\Omega)$, there exists $\tilde{g} \in C^\infty(\bar{\Omega})$ such that $\tilde{g}|_{\partial\Omega} = g$. We consider the following homogeneous Dirichlet problems

$$\begin{cases} Lv = f - L\tilde{g} = \tilde{f} & \text{in } \Omega \\ v|_{\partial\Omega} = 0. \end{cases}$$

It follows from the Poincaré inequality (2.7) that L is positive in $H^1_{X,0}(\Omega)$, and hence the Lax-Milgram theorem gives the existence of weak solution $v \in H^1_{X,0}(\Omega)$ for any $\tilde{f} \in L^2(\Omega)$. It is easy to see that $u = v + \tilde{g}$ is a weak solution of the Dirichlet problem (3.1).

C^∞ **regularity:** The estimate (1.2) implies that L is hypoelliptic on the interior of Ω, so that $u \in C^\infty(\Omega)$. As for the C^∞ regularity of weak solution up to the boundary, its proof is the same as the one of Theorem 1.1, so we send it to the section 5.

Uniqueness of solution: If u is a weak solution of Dirichlet problem $Lu = 0$ in Ω and $u|_{\partial\Omega} = 0$, then above regularity results show $u \in C^\infty(\bar{\Omega})$. We get $u \equiv 0$ in Ω by the maximum principle and the controllability as in the proof of Lemma 2.1.

The nonlinear equation (1.7) is quite different from (1.3) whoes nonlinear term $F(x,u)$ is smooth with respect to u, so that the nonlinear composition theorem is applicable to get the regularity of $F(x,u(x))$ from those of u. But in equation (1.7), the nonlinear term is $\tilde{F}(u) = au \log|u| + bu$, which is smooth if $u \neq 0$ but only continous near $u = 0$. Remark that if $u \in L^{\infty}(\Omega)$, we have $\tilde{F}(u) \in L^{\infty}(\Omega)$.

To prove the regularity of weak solution for nonlinear Dirichlet problem (1.7) we consider, in the first step, the Dirichlet problem (3.1) with $f \in L^{\infty}(\Omega)$, keeping $\tilde{F}(u) \in L^{\infty}(\Omega)$ in mind.

Theorem 3.2. *Suppose that $\partial\Omega$ is C^{∞} and non characteristic for X. If $f \in L^{\infty}(\Omega)$, $g \in C^2(\partial\Omega)$, then the Dirichlet problem* (3.1) *possesses a unique solution $u \in C^0(\bar{\Omega})$.*

Without loss of generality, we suppose that $g = 0$. The existence of weak solution and the uniqueness are the same as Theorem 3.1 since $L^{\infty}(\Omega) \subset L^2(\Omega)$. We prove now the continuity of weak solution. We shall use the so-called approximation method for f. By using Theorem 3.1, the Green function $G(x,y)$ of L exists and has the following properties (see also [1]):

$$LG(x,\cdot) = \delta_x, \qquad G(x,y) \geq 0, \qquad G \in C^{\infty}(\Omega \times \Omega \backslash \{(x,x); x \in \Omega\}),$$

$$G(x,y) = G(y,x), \qquad G(x,\cdot)|_{\partial\Omega} = 0, \qquad v(x) = \int_{\Omega} G(x,y)dy \in C^{\infty}(\bar{\Omega}).$$

In fact, $v = \int_{\Omega} G(x,y)dy$ is the solution of problem (3.1) with $f = 1$, $g = 0$, so that we have $v \in C^{\infty}(\bar{\Omega})$. We prove now the continuity of weak solution by the following two propositions.

Proposition 3.1. *If $f \in L^{\infty}(\Omega)$, then $v(x) = Gf(x) = \int_{\Omega} G(x,y)f(y)dy$ is a weak solution of equation $Lv = f$ and $v \in L^{\infty}(\Omega)$.*

Proof. For $\varphi \in C_0^{\infty}(\Omega)$, we have

$$\langle L(Gf), \varphi \rangle = \langle Gf, L\varphi \rangle = \int_{\Omega} \left(\int_{\Omega} G(x,y)f(y)dy \right) L\varphi(x)dx.$$

Since $L\varphi \in C_0^{\infty}(\Omega)$, there exist, $\psi \in C_0^{\infty}(\Omega)$, $0 \leq \psi \leq 1$, $\psi(x) = 1$ for $x \in \operatorname{Supp} \varphi$. Then we have

$$\int_{\Omega} \left(\int_{\Omega} |G(x,y)f(y)L\varphi(x)|dx \right) dy \leq \|f\|_{L^{\infty}} \|L\varphi\|_{L^{\infty}} \int_{\Omega} \left(\int_{\Omega} G(x,y)\psi(x)dx \right) dy.$$

By C^{∞} regularity result of Theorem 3.1 we get $G\psi \in C^{\infty}(\bar{\Omega})$, so that we have proved

$$G(x,y)f(y)L\varphi(x) \in L^1(\Omega \times \Omega).$$

Then Fubini theorem implies that

$$\langle LGf, \varphi \rangle = \int_{\Omega} \left(\int_{\Omega} G(x, y) L\varphi(x) dx \right) f(y) dy = \langle f, \varphi \rangle.$$

Therefore in $\mathscr{D}'(\Omega)$ we have $LGf = f$ and

$$\|Gf\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \|G\|_{L^\infty(\Omega, L^1(\Omega))}.$$

We prove now the continuity of $v = Gf$ on $\bar{\Omega}$ for $f \in L^\infty(\Omega)$. Suppose that

$$K_0 \subset\subset \cdots \subset\subset K_j \subset\subset K_{j+1} \subset \cdots \subset\subset \Omega$$

is a sequence of compact subset of Ω with the property dis $(K_j, \partial K_{j+1}) = \alpha_0 2^{-j} > 0$. Let $\chi_j \in C_0^\infty(\mathring{K}_{j+1})$, $\chi_j(x) = 1$ for $x \in K_j$ and ε_j is a mollifier of radius $\alpha_0 2^{-j-1}$. Then for $f \in L^\infty(\Omega)$, we have

$$f_j = \varepsilon_j * (\chi_j f) \in C_0^\infty(\Omega), \qquad \|f_j\|_{L^\infty} \leq \|f\|_{L^\infty}, \qquad \text{Supp } f_j \subset K_{j+2},$$

and $\lim f_j = f$ in $L^\infty(\Omega)$. We have

Proposition 3.2. *The sequence $\{v_j = Gf_j\}$ is equi-continuous and uniformly bounded on $\bar{\Omega}$.*

End of proof for Theorem 3.2. From this proposition, the weak solution of equation $Lu = f$ is the uniform limit of Gf_j by the Ascoli theorem. Furthermore, the weak solution u is continuous on $\bar{\Omega}$ and $u|_{\partial\Omega} = 0$.

Proof of Proposition 3.2. Since

$$\|Gf_j\|_{L^\infty(\Omega)} \leq \|f_j\|_{L^\infty(\Omega)} \|G\|_{L^\infty(\Omega, L^1(\Omega))} \leq \|f\|_{L^\infty(\Omega)} \|G\|_{L^\infty(\Omega, L^1(\Omega))},$$

$\{v_j\}$ is uniformly bounded on $\bar{\Omega}$. For the equi-continuity, we first have

$$|Gf_j(x) - Gf_j(x')| \leq \|f\|_{L^\infty(\Omega)} \int_{\Omega} |G(x, y) - G(x', y)| dy.$$

Claim: for any $\delta > 0$, and $x' \in \bar{\Omega}$, there exists $\eta > 0$ such that

$$\int_{\Omega} |G(x, y) - G(x', y)| dy \leq \delta, \qquad \forall x, x' \in \bar{\Omega}, |x - x'| < \eta.$$

Then the compactness of $\bar{\Omega}$ implies the equi-continuous of $\{v_j = Gf_j\}$ on $\bar{\Omega}$.

We prove now the preceding claim. Since $G(x', \cdot) \in L^1(\Omega)$, there exists $\eta_1 > 0$ such that

$$\int_{B(x', \eta_1) \cap \Omega} G(x', y) dy \leq \int_{B(x', 2\eta_1) \cap \Omega} G(x', y) \varphi(y) dy \leq \int_{B(x', 2\eta_1) \cap \Omega} G(x', y) dy \leq \frac{\delta}{4},$$

where $\varphi \in C_0^\infty(B(x', 2\eta_1))$, $0 \le \varphi \le 1$, $\varphi(x) = 1$, $\forall x \in B(x', \eta_1)$. Since $u_1(x) = G\varphi$ is the solution of problem $Lu = \varphi$, $u|_{\partial\Omega} = 0$, we have that

$$u_1(x) = \int_\Omega G(x, y)\varphi(y)dy = \int_{B(x', 2\eta_1)\cap\Omega} G(x, y)\varphi(y)dy \in C^\infty(\bar{\Omega}).$$

Then there exists $\eta_1 \ge 2\eta_2 > 0$ such that

$$|u_1(x) - u_1(x')| \le \frac{\delta}{4}, \qquad \forall x \in \bar{\Omega},\ |x - x'| < \eta_2.$$

Consequently for any $x \in \bar{\Omega}$, $|x - x'| < \eta_2 \le \eta_1/2$,

$$\int_{B(x', \eta_1)\cap\Omega} G(x, y)dy \le \int_{B(x', 2\eta_1)\cap\Omega} G(x, y)\varphi(y)dy \le \frac{\delta}{2},$$

and

$$\begin{aligned}\int_\Omega |G(x, y) - G(x', y)|dy &\le \int_{B(x', \eta_1)\cap\Omega} |G(x, y) - G(x', y)|dy \\ &\quad + \int_{\Omega\setminus B(x', \eta_1)} |G(x, y) - G(x', y)|dy \\ &\le \frac{3\delta}{4} + \int_{\Omega\setminus B(x', \eta_1)} |G(x, y) - G(x', y)|dy,\end{aligned}$$

but $G(\cdot,\cdot)$ is uniformly continuous on $B(x', \eta_1/2) \times (\bar{\Omega}\setminus B(x', \eta_1))$, then there exist $0 < \eta \le \eta_2$ such that

$$|G(x, y) - G(x', y)| \le \frac{\delta}{4|\Omega|}, \qquad \forall |x - x'| < \eta,\ \forall y \in \bar{\Omega}\setminus B(x', \eta_1).$$

We have proved the proposition.

From this Theorem 3.2, we have the continuity of weak solution u for nonlinear problems (1.7). In fact we have proved $u \in L^\infty(\Omega)$ in [19], and so $f = au\log|u| + bu \in L^\infty(\Omega)$. Consequently u and f are also continuous in $\bar{\Omega}$. To prove $u(x) > 0$ in interior of Ω, we need now the following strong maximum principle for the continuous weak solution.

Lemma 3.1. *Suppose that $\partial\Omega$ is C^∞ and non characteristic for X. If $u \in C(\bar{\Omega})$ is a weak solution of Dirichlet problem* (3.1) *with $g \in C^2(\partial\Omega)$ and $f \in C(\bar{\Omega})$. Let $\omega = \{x \in \Omega; f(x) \le 0\}$. Then u can not take its positive maximum on interior points of ω unless it is constant on the connected component of those points in the interior of ω.*

Proof. As in the proof of Proposition 3.1 of [1], suppose that F is the set of positive maximum points of u in $\mathring{\omega}$, $F \neq \emptyset$ and $F \neq \mathring{\omega}$. Then there exist $x_0, x_1 \in \mathring{\omega}$, $\rho > 0$ such that

$$\bar{B}(x_0, \rho) \cap F = \{x_1\}, \qquad B(x_0, 2\rho) \subset \mathring{\omega}.$$

For $k > 0$, we set $v(x) = e^{-k|x-x_0|^2} - e^{-k\rho^2}$. For k big enough, there exist c_1, $C_1 > 0$ such that $Lv(x) \leq -C_1 < 0$ for any $x \in B(x_1, c_1\rho)$. For $\lambda > 0$, set now $w = u + \lambda v$. Then we have that $m_0 = w(x_1) = u(x_1) = \max_{\mathring{\omega}} u > 0$. Choose $\lambda > 0$ small enough such that $w(x) \leq m_0 - \delta_0$ for all $x \in \partial B(x_1, c_1\rho)$ and for some $\delta_0 > 0$. Then we have that $w(x_1) = m_0$ and

$$Lw = f + Lv = \tilde{f} \leq -C_1 < 0, \quad \text{on } B(x_1, c_1\rho), \qquad \text{and}$$

$$w(x) \leq m_0 - \delta_0, \quad \forall x \in \partial B(x_1, c_1\rho).$$

But we can not use directly Theorem 1.5 to get the contradiction, since w is only continuous. Consequently we need to use the approximation method again.

We extend $\tilde{f}$ as a continuous function on $\boldsymbol{R}^n$ and take $\{\tilde{f}_j\}_{j=1}^{\infty} \subset C^{\infty}(\boldsymbol{R}^n)$ such that $\|\tilde{f}_j - \tilde{f}\|_{L^{\infty}(\Omega)} \to 0$. For a $\tilde{g} \in C^2(\boldsymbol{R}^n)$ with $\tilde{g}|_{\partial\Omega} = g + v|_{\partial\Omega}$, take $\{\tilde{g}_j\}_{j=1}^{\infty} \subset C^{\infty}(\boldsymbol{R}^n)$ such that $\|\tilde{g}_j - \tilde{g}\|_{C^2} \to 0$. If w_j is a solution to (3.1) with $f = \tilde{f}_j$, $g = \tilde{g}_j|_{\partial\Omega}$, then $w_j \in C^{\infty}(\bar{\Omega})$. By using the Green function we see that

$$\|w_j - w\|_{L^{\infty}(\Omega)} \leq \|\tilde{g}_j - \tilde{g}\|_{L^{\infty}(\Omega)} + C(\|\tilde{f}_j - \tilde{f}\|_{L^{\infty}(\Omega)} + \|L\tilde{g}_j - L\tilde{g}\|_{L^{\infty}(\Omega)}).$$

Thus there exists j_0 big enough such that for any $j \geq j_0$,

$$\|w_j - w\|_{L^{\infty}(\Omega)} \leq \delta_0/4, \qquad \|\tilde{f}_j - \tilde{f}\|_{L^{\infty}(\Omega)} \leq C_1/2.$$

Therefore we get finally

$$Lw_{j_0}(x) = \tilde{f}_{j_0}(x) \leq -C_1/2 < 0, \qquad \forall x \in B(x_1, c_1\rho),$$

and

$$w_{j_0}(x) \leq m_0 - \frac{3}{4}\delta_0, \quad \forall x \in \partial B(x_1, c_1\rho); \qquad w_{j_0}(x_1) \geq m_0 - \frac{1}{4}\delta_0.$$

This is impossible from the maximum principle of Theorem 1.5. We have proved Lemma 3.1.

From Theorem 3.2 and Lemma 3.1, we have proved the continuity and positivity for the weak solution of Dirichlet problem (1.7). We state those results by the following theorem.

Theorem 3.3. *Under the hypothesis of Theorem* 1.3, *the weak solutions of problems* (1.7) *belong to* $C^0(\bar{\Omega})$ *and* $u(x) > 0$ *for all* $x \in \Omega$.

In fact, if $u(x_0) = 0$ for an $x_0 \in \Omega$ then, by using the continuity of u, there exists $r > 0$ such that $f(x) = au(x) \log u(x) + bu(x) \geq 0$ for any $x \in B(x_0, r) \subset \Omega$. Applying Lemma 3.1 to $1 - u$ we have that $u(x) = 0$ in $\bar{B}(x_0, r)$. Repeating this method we finally obtain $u = 0$ in Ω, which is impossible since the solution is non-trivial.

4. Littlewood-Paley theory for logarithmic Sobolev spaces

Let $\ell > 0$, and define the following logarithmic Sobolev's space:

$$H_\ell^{\log}(\boldsymbol{R}^n) = \{u \in L^2(\boldsymbol{R}^n); (\log\langle\xi\rangle)^\ell \hat{u}(\xi) \in L^2(\boldsymbol{R}^n)\},$$

where $\langle\xi\rangle = (e^2 + |\xi|^2)^{1/2}$. We study now the Littlewood-Paley decomposition for this function space as in [2, 21].

Let $\mathscr{C}_0 = \{\xi \in \boldsymbol{R}^n; e < \langle\xi\rangle < e^3\}$, $\mathscr{C}_k = e^k \mathscr{C}_0$, $k \in \boldsymbol{N}$, $\mathscr{C}_{-1} = \{\xi \in \boldsymbol{R}^n; \langle\xi\rangle < e^2\}$, there exist $\psi \in C_0^\infty(]0, e^2[)$, $\varphi \in C_0^\infty(]e, e^3[)$ such that

$$\psi(\langle\xi\rangle) + \sum_{j=0}^{\infty} \varphi(e^{-j}\langle\xi\rangle) = 1, \qquad \forall \xi \in \boldsymbol{R}^n.$$

For $f \in L^2(\boldsymbol{R}^n)$, we set

$$\Delta_{-1} f = \psi(\Lambda) f, \qquad \Delta_j f = \varphi(e^{-j}\Lambda) f, \quad j \in \boldsymbol{N}.$$

Then $f = \sum \Delta_j f$ in $L^2(\boldsymbol{R}^n)$, and we have the following characterization for function space $H_\ell^{\log}(\boldsymbol{R}^n)$.

Lemma 4.1. *For $\ell > 0$, we have that*

1) *if $u \in H_\ell^{\log}(\boldsymbol{R}^n)$, then*

$$\|\Delta_j u\|_{L^2(\boldsymbol{R}^n)} \leq c_j j^{-\ell}, \qquad \|\{c_j\}\|_{\ell^2} \leq \|u\|_{H_\ell^{\log}(\boldsymbol{R}^n)}.$$

2) *if $u \in L^2(\boldsymbol{R}^n)$, and*

$$\|\Delta_j u\|_{L^2(\boldsymbol{R}^n)} \leq c_j j^{-\ell}, \qquad \{c_j\} \in \ell^2,$$

then $u \in H_\ell^{\log}(\boldsymbol{R}^n)$, and for any $S \geq 1$

$$S^{2\ell} \|(\log \Lambda)^\ell u\|_{L^2(\boldsymbol{R}^n)}^2 \leq C_1 \ell^{2\ell} \|u\|_{L^2(\boldsymbol{R}^n)}^2 + C_2^S S^{2\ell} \|\{c_j\}\|_{\ell^2}^2,$$

with C_1, C_2 independent of S, ℓ and u.

Proof. 1) For $u \in H_\ell^{\log}(\boldsymbol{R}^n)$, we have

$$\|\Delta_j u\|_{L^2}^2 = \int \varphi(e^{-j}\langle\xi\rangle)^2 |\hat{u}(\xi)|^2 d\xi \leq j^{-2\ell} \int_{\mathscr{C}_j} (\log\langle\xi\rangle)^{2\ell} \varphi(e^{-j}\langle\xi\rangle)^2 |\hat{u}(\xi)|^2 d\xi.$$

We set

$$c_j^2 = \int_{\mathscr{C}_j} (\log\langle\xi\rangle)^{2\ell}\varphi(e^{-j}\langle\xi\rangle)^2|\hat{u}(\xi)|^2 d\xi.$$

Then the fact $\psi^2(\xi) + \sum_{j=0}^{\infty}\varphi(e^{-j}\langle\xi\rangle)^2 \le 1$ implies that

$$\sum_{j=-1}^{\infty} c_j^2 \le \int_{\boldsymbol{R}^n} (\log\langle\xi\rangle)^{2\ell} \sum_{j=-1}^{\infty} \varphi(e^{-j}\langle\xi\rangle)^2|\hat{u}(\xi)|^2 d\xi \le \|u\|^2_{H_\ell^{\log}(\boldsymbol{R}^n)}.$$

2) For $S > 0$, we have

$$\begin{aligned} S^{2\ell}\|(\log\Lambda)^{\ell}u\|^2_{L^2} &\le 3\sum_{S(j+3)\le\ell} (S(j+3))^{2\ell}\int\varphi(e^{-j}\langle\xi\rangle)^2|\hat{u}(\xi)|^2 d\xi \\ &\quad + 3\sum_{S(j+3)>\ell}(S(j+3))^{2\ell}\|\Delta_j u\|^2_{L^2} \\ &\le 3\ell^{2\ell}\|u\|^2_{L^2} + 3S^{2\ell}\sum_{S(j+3)>\ell}(j+3)^{2\ell}j^{-2\ell}c_j^2 \\ &\le 3\ell^{2\ell}\|u\|^2_{L^2} + 3S^{2\ell}\sum_j (1+3/j)^{2S(j+3)}c_j^2 \\ &\le 3\ell^{2\ell}\|u\|^2_{L^2} + 3(e^6 2^6)^S S^{2\ell}\|\{c_j\}\|^2_{\ell^2}. \end{aligned}$$

As in the classical case, for the second part in the preceding lemma, we have more general results

Lemma 4.2. *Suppose that* $\{u_k\}_{k\in\boldsymbol{N}}$ *is a sequence of* $L^2(\boldsymbol{R}^n)$, *with* $\operatorname{Supp}\hat{u}_k \subset B(0, Ke^k)$ *and for* $\ell > 1/2$,

$$\|u_k\|_{L^2(\boldsymbol{R}^n)} \le c_k k^{-\ell}, \qquad \{c_k\} \in \ell^2.$$

Then $u = \sum_k u_k \in H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)$ *and for any* $S \ge 1$,

$$\begin{aligned} S^{2\ell-1}\|(\log\Lambda)^{\ell-1/2}u\|^2_{L^2(\boldsymbol{R}^n)} &\le C_1(\ell-1/2)^{2\ell-1}\|u\|^2_{L^2(\boldsymbol{R}^n)} \\ &\quad + C_2^S S^{2\ell-1}(2\ell-1)\|\{c_k\}\|^2_{\ell^2}, \end{aligned}$$

with C_1, C_2 *independent of* S, ℓ *and* u.

Remark. We have a loss of $1/2$ for the index because of the logarithmic sum.

Proof. Since $\ell > 1/2$, we have that $u = \sum_k u_k$ converges in $L^2(\boldsymbol{R}^n)$, in fact,

$$\|u\|_{L^2} \le \sum_k \|u_k\|_{L^2} \le \sum_k c_k k^{-\ell} \le \|\{c_k\}\|_{\ell^2}\left(\sum_k k^{-2\ell}\right)^{1/2}.$$

We suppose now $S = 1$, since the general case of S is similar as Lemma 4.1. We set

$$u = \sum_{j=-1}^{\infty} \Delta_j u = \sum_{j=-1}^{\infty} v_j = \sum_{j=-1}^{\infty} \sum_k \Delta_j u_k.$$

Then

$$\begin{aligned} \|u\|^2_{H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)} &\le 2\left\| \sum_{j+3\le \ell-1/2} \Delta_j u \right\|^2_{H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)} + 2\left\| \sum_{j+3>\ell-1/2} \Delta_j u \right\|^2_{H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)} \\ &\le 2(\ell-1/2)^{2\ell-1}\|u\|^2_{L^2} + 2\sum_{j+3>\ell-1/2} (j+3)^{2\ell-1}\|\Delta_j u\|^2_{L^2}. \end{aligned}$$

On the other hand, there exists $N_1 > 0$ (depending only on K) such that for any $j > k + N_1$, $\mathscr{C}_j \cap B(0, Ke^k) = \emptyset$, then $\Delta_j u_k = 0$. We have $v_j = \sum_{k\ge j-N_1} \Delta_j u_k$, and

$$\begin{aligned} \|\Delta_j u\|^2_{L^2} = \int \left| \sum_{k\ge j-N_1} \Delta_j u_k \right|^2 dx &\le \left(\sum_{k\ge j-N_1} k^{-2\ell} \right) \left(\sum_{k\ge j-N_1} \int k^{2\ell} |\Delta_j u_k|^2 dx \right) \\ &\le (2\ell-1)(j-N_1)^{-2\ell+1} \sum_{k\ge j-N_1} k^{2\ell} \|\Delta_j u_k\|^2_{L^2}. \end{aligned}$$

Setting now $\tilde{c}_j^2 = \sum_{k\ge j-N_1} k^{2\ell}\|\Delta_j u_k\|^2_{L^2}$, we have

$$\sum_j \tilde{c}_j^2 \le \sum_k k^{2\ell}\|u_k\|^2_{L^2} \le \sum_k c_k^2.$$

Finally, for $j + 3 > \ell - 1/2$,

$$\left(\frac{j+3}{j-N_1}\right)^{2\ell-1} \le \left(\frac{j+3}{j-N_1}\right)^{2(j+3)} \le e^{2(N_1+3)}(N_1+4)^{2(N_1+3)} \le C_2.$$

We have proved the lemma.

Lemma 4.3. *Suppose that* $\{u_k\}$ *is a sequence in* $C^\infty(\boldsymbol{R}^n)$ *and for* $\ell > 1/2$ *there exists a function* $v \in H^{\log}_\ell(\boldsymbol{R}^n)$ *satisfying the following: For any* $\alpha \in \boldsymbol{N}^n$, *there exist* $B_{|\alpha|} \ge 0$ *such that*

$$\|D^\alpha u_k\|_{L^2(\boldsymbol{R}^n)} \le B_{|\alpha|} e^{k|\alpha|} \|\Delta_k v\|_{L^2(\boldsymbol{R}^n)}.$$

Then $u = \sum_k u_k \in H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)$ *and for any* $S \ge 1$,

$$S^{2\ell-1}\|u\|^2_{H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)} \le C_S((\ell-1/2)^{2\ell-1}\|v\|^2_{L^2(\boldsymbol{R}^n)} + S^{2\ell-1}(2\ell-1)\|v\|^2_{H^{\log}_\ell(\boldsymbol{R}^n)}),$$

with C_S *depending only on* B_0, $B_{[S]+2}$ *and* C_1, C_2 *the constants in Lemmas* 4.1 *and* 4.2.

Proof. As in the lemma 4.2, we have $u=\sum_k u_k \in L^2$. We decompose,

$$u_k = u_k^1 + u_k^2 = \psi(e^{-k-1}\Lambda)u_k + (1-\psi(e^{-k-1}\Lambda))u_k.$$

Then $u^1 = \sum u_k^1$ satisfies the hypothesis of Lemma 4.2, we have for $S \geq 1$,

$$S^{2\ell-1}\|u^1\|^2_{H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)} \leq C_1(\ell-1/2)^{2\ell-1}B_0^2\|v\|^2_{L^2} + C_2^S B_0^2 S^{2\ell-1}(2\ell-1)\|v\|^2_{H^{\log}_{\ell}(\boldsymbol{R}^n)}.$$

We study now $u^2=\sum u_k^2$, with the conditions

$$\operatorname{Supp} u_k^2 \subset \{\xi \in \boldsymbol{R}^n; \langle\xi\rangle \geq e^k\}, \qquad \|D^\alpha u_k^2\|_{L^2} \leq B_\alpha e^{k|\alpha|}\|\Delta_k v\|_{L^2}.$$

For $k \geq p+3$, $\mathscr{C}_p \cap \{\xi \in \boldsymbol{R}^n; \langle\xi\rangle \geq e^k\} = \varnothing$, we have $\Delta_p u^2 = \sum_{k\leq p+2}\Delta_p u_k^2$. Then

$$\begin{aligned}
\|\Delta_p u^2\|^2_{L^2} &\leq \left(\sum_{k\leq p+2} e^{2k}\right)\left(\sum_{k\leq p+2} e^{-2k}\|\Delta_p u_k^2\|^2_{L^2}\right)\\
&\leq 2e^{2(p+2)}\sum_{k\leq p+2} e^{-2k}\|\Delta_p u_k^2\|^2_{L^2}\\
&\leq 2e^4 p^{-2\ell+1}\sum_{k\leq p+2} e^{-2k}\|\langle D\rangle(\log\Lambda)^{\ell-1/2}\Delta_p u_k^2\|^2_{L^2}.
\end{aligned}$$

Set now $\tilde{c}_p^2 = \sum_{k\leq p+2} e^{-2k}\|\langle D\rangle(\log\Lambda)^{\ell-1/2}\Delta_p u_k^2\|^2_{L^2}$. We have

$$\sum_{p=-1}^{\infty}\tilde{c}_p^2 \leq \sum_k e^{-2k}\|\langle D\rangle(\log\Lambda)^{\ell-1/2}u_k^2\|^2_{L^2}.$$

By Lemma 4.1, we have

$$S^{2\ell-1}\|(\log\Lambda)^{\ell-1/2}(u^2)\|^2_{L^2} \leq C_1(\ell-1/2)^{2\ell-1}\|u\|^2_{L^2} + C_2^S S^{2\ell-1}\|\{\tilde{c}_p\}\|^2_{\ell^2}.$$

We study now $\|\{\tilde{c}_p\}\|_{\ell^2}$. For simplicity of the notation, we replace $\ell-1/2$ by ℓ in what follows,

$$\begin{aligned}
&\|\langle D\rangle(\log\Lambda)^{\ell}u_k^2\|^2_{L^2}\\
&\quad = \int\langle\xi\rangle^{-2([S]+1)}\langle\xi\rangle^{2[S]+4}(\log\langle\xi\rangle)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2|\hat{u}_k(\xi)|^2 d\xi,
\end{aligned}$$

and if $([S]+1)(k+2)\geq\ell$,

$$\begin{aligned}
&\langle\xi\rangle^{-2([S]+1)}(\log\langle\xi\rangle)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2\\
&\quad\leq e^{-2([S]+1)(k+2)}(k+2)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2;
\end{aligned}$$

if $([S]+1)(k+2) < \ell$,

$$\begin{aligned}&\langle\xi\rangle^{-2([S]+1)}(\log\langle\xi\rangle)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2\\&\quad\le e^{-2([S]+1)(k+2)}\left(\frac{\ell}{[S]+1}\right)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2.\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{p=-1}^{\infty}\tilde{c}_p^2 \le &\sum_{([S]+1)(k+2)<\ell} e^{-2k}e^{-2([S]+1)(k+2)}\left(\frac{\ell}{[S]+1}\right)^{2\ell}\|u_k\|^2_{H^{[S]+2}}\\&+\sum_{([S]+1)(k+2)\ge\ell} e^{-4([S]+1)}e^{-2k([S]+2)}\left(1+\frac{2}{k}\right)^{2\ell}k^{2\ell}\|u_k\|^2_{H^{[S]+2}},\end{aligned}$$

where $H^{[S]+2}$ is classical Sobolev space on $\boldsymbol{R}^n$. From the hypothesis of the lemma,

$$\|u_k\|_{H^{[S]+2}} \le B_{[S]+2}e^{k([S]+2)}\|\Delta_k v\|_{L^2},$$

we have

$$\sum_{p=-1}^{\infty}\tilde{c}_p^2 \le B^2_{[S]+2}(S^{-2\ell}\ell^{2\ell}\|v\|^2_{L^2}+\|v\|^2_{H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)}).$$

We have proved the lemma with the constant C_S depending on B_0, $B_{[S]+2}$ and C_1, C_2.

We study now the non-linear composition for the function of space $H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)$. We have the following result.

Theorem 4.1. *Suppose that* $F \in C^\infty(\boldsymbol{R})$, $F(0)=0$, *and* $u \in H^{\log}_{\ell}(\boldsymbol{R}^n)\cap L^\infty(\boldsymbol{R}^n)$ *a real function for* $\ell > 1/2$. *Then* $F(u) \in H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)\cap L^\infty(\boldsymbol{R}^n)$ *and for any* $S \ge 1$

$$S^{2\ell-1}\|F(u)\|^2_{H^{\log}_{\ell-1/2}(\boldsymbol{R}^n)} \le C_S\left(\left(\ell-\frac{1}{2}\right)^{2\ell-1}\|u\|^2_{L^2(\boldsymbol{R}^n)}+S^{2\ell-1}(2\ell-1)\|u\|^2_{H^{\log}_{\ell}(\boldsymbol{R}^n)}\right),$$

with C_S *depending only on* $\sup_{|t|\le\|u\|_{L^\infty}}|F^{(j)}(t)|$ *and* $\|u\|^j_{L^\infty}$ *for* $j=0,1,\dots,[S]+2$.

Remark. This theorem is still true for the vector value function $u=(u_1,\dots,u_m)$ and $F(t_1,\dots,t_m)\in C^\infty(\boldsymbol{R}^m)$.

Proof. We have firstly

$$\|F(u)\|_{L^2} = \|F(u)-F(0)\|_{L^2} \le (\sup_{|t|\le\|u\|_{L^\infty}})|F'(t)|\,\|u\|_{L^2}.$$

We denote, for $k \geq 1$, $S_k u = \sum_{j=-1}^{k-2} \Delta_j u$, then for $u \in H_\ell^{\log}(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$, we have $F(u) = \lim_{k\to+\infty} F(S_k u)$ in $L^2(\boldsymbol{R}^n)$, so that

$$F(u) = F(S_1 u) + \sum_{k=2}^{\infty} (F(S_k u) - F(S_{k-1}u)) = \sum_{k=1}^{\infty} f_k$$

with $f_1 = F(S_1 u)$ and for $k > 1$

$$f_k = \int_0^1 F'(S_{k-1}u + t\Delta_k u) dt \Delta_k u.$$

Since for any $\alpha \in \boldsymbol{N}^n$,

$$\|D^\alpha(S_{k-1}u + t\Delta_k u)\|_{L^\infty} \leq C_{|\alpha|} e^{k|\alpha|} \|u\|_{L^\infty}, \qquad \|D^\alpha \Delta_k u\|_{L^2} \leq e^{(k+3)|\alpha|} \|\Delta_k u\|_{L^2},$$

the Faà-di-Bruno formula implies that

$$\|D^\alpha f_k\|_{L^2} \leq B_{|\alpha|} e^{k|\alpha|} \|\Delta_k u\|_{L^2}$$

with $B_{|\alpha|}$ depending only on $\sup_{|t|\leq\|u\|_{L^\infty}} |F^{(j)}(t)|$ and $\|u\|_{L^\infty}^j$ for $j = 0, 1, \ldots, |\alpha| + 2$.

Then $\sum_k f_k$ satisfies the hypothesis of Lemma 4.3, and so we have proved the theorem.

To study the regularity up to the boundary for nonlinear problems, we introduce the following tangential logarithmic Sobolev spaces (see [26]): For $\ell > 0$, we set

$$H_{0,\ell}^{\log}(\boldsymbol{R}^n) = \{u \in L^2(\boldsymbol{R}^n); (\log\langle(\xi', 0)\rangle)^\ell \hat{u}(\xi) \in L^2(\boldsymbol{R}^n)\},$$

and

$$H_{0,\ell}^{\log}(\boldsymbol{R}_+^n) = \{u \in L^2(\boldsymbol{R}_+^n); (\log\langle(\xi', 0)\rangle)^\ell \mathscr{F}_{x'} u(\xi', x_n) \in L^2(\boldsymbol{R}_+^n)\},$$

where $\xi = (\xi', \xi_n) \in \boldsymbol{R}^{n-1} \times \boldsymbol{R}$, $\boldsymbol{R}_+^n = \{(x', x_n); x' \in \boldsymbol{R}^{n-1}, x_n > 0\}$. We have

$$H_{0,\ell}^{\log}(\boldsymbol{R}^n)|_{\boldsymbol{R}_+^n} = H_{0,\ell}^{\log}(\boldsymbol{R}_+^n).$$

We use now the tangential Littlewood-Paley decomposition:

$$\Delta'_{-1} f = \psi(\Lambda') f, \qquad \Delta'_j f = \varphi(e^{-j}\Lambda') f, \quad j \in \boldsymbol{N},$$

where $\mathscr{F}(\varphi(\Lambda')f) = \varphi(\langle(\xi', 0)\rangle)\hat{f}$, and the function spaces $H_{0,\ell}^{\log}(\boldsymbol{R}_+^n)$ is characterized by

$$\sum j^{2\ell} \|\Delta'_j u\|_{L^2(\boldsymbol{R}_+^n)}^2 < +\infty.$$

We have the similar results as Lemmas 4.1–4.3 and Theorem 4.1 for the tangential function spaces.

5. Nonlinear hypoellipticity

We consider now our nonlinear function $t \log t$ and weak solution $u \in H^1_{X,0}(\Omega)$ of problem (1.7). From Theorem 3.3, we have that $u \in C^0(\bar{\Omega})$ and $u(x) > 0$ for all $x \in \Omega$. Take $\alpha, \beta \in C_0^\infty(\Omega)$ with $\alpha \subset\subset \beta$. Let N_0 be a fixed positive integer and suppose that

$$\|D^\lambda \alpha\|_{L^\infty} \le C_\lambda N_0^{|\lambda|}, \qquad u(x) \ge 2c_0 > 0, \quad \forall x \in \operatorname{Supp} \beta.$$

Let $\chi \in C_0^\infty(\boldsymbol{R})$, $0 \le \chi \le 1$ and $\chi(t) = 1$, $t \ge c_0$, $\chi(t) = 0$, $t \le c_0/2$. We have

$$G(t) = \chi(t) t \log|t| \in C^\infty(\boldsymbol{R}).$$

Then, for $u \in C^0(\bar{\Omega})$, $u(x) > 0$, $x \in \Omega$ the solution of problem (1.7), we have

$$\alpha u \log u = \alpha(\beta u) \log(\beta u) = \alpha G(\beta u).$$

For the nonlinear term in equation (1.3), we have $\alpha F(x, u) = \alpha F(x, \beta u)$ for any $\alpha \subset\subset \beta$. Therefore in the interior of Ω, we have the same smooth nonlinear term in two cases of Theorems 1.1 and 1.3. By using Theorem 4.1 and its remark, we have the following estimate: If $\beta u \in H_\ell^{\log}(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$ for some $\ell > 1/2$, then for any $S \ge 1$, we have

$$\tag{5.1} \begin{aligned} \|(\log \Lambda^S)^{\ell - 1/2}(\alpha F(x, u))\|_{L^2}^2 \le A_S^2 (&\ell^{2\ell - 1} N_0^{2(S+2)} \|\beta u\|_{L^2}^2 \\ &+ (2\ell - 1) \|(\log \Lambda^S)^\ell (\beta u)\|_{L^2}^2), \end{aligned}$$

where A_S depends on S, $\|u\|_{L^\infty}$ and $\|\alpha(x) F(x, t)\|_{C^{[S]+2}(\Omega \times [-\|u\|_{L^\infty}, \|u\|_{L^\infty}])}$, but not on ℓ, N_0.

By interpolation, the estimate (1.2) implies that: For any small $\varepsilon > 0$, any $N > 0$, there exists $C_{\varepsilon, N} > 0$ such that

$$\tag{5.2} \|\log \Lambda v\|_{L^2}^2 \le \varepsilon \sum_{p=1}^m \|X_p v\|_{L^2}^2 + C_{\varepsilon, N} \|v\|_{H^{-N}}^2, \qquad \forall v \in C_0^\infty(\tilde{\Omega}),$$

where H^{-N} is classical Sobolev space. For small $\delta > 0$, we set $\Lambda_\delta = (1 - \delta \triangle)^{-1}$, then this is a uniformly bounded family of operators on $H^m(\boldsymbol{R}^n)$ for any $m \in \boldsymbol{R}$, and $\Lambda_\delta(\alpha u) \in H^2(\boldsymbol{R}^n)$ if $u \in L^2_{\mathrm{loc}}(\Omega)$. We prove now the following proposition.

Proposition 5.1. *Suppose that the system of vector fields X satisfies the logarithmic regularity estimate* (1.2), *and $u \in H^1_{X,\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ is a weak solution of equation* (1.3). *Then for any $\alpha \in C_0^\infty(\Omega)$ and any $\ell \in \boldsymbol{N}$, $S \ge 1$, we have*

$$\tag{5.3} \|(\log \Lambda^S)^\ell \Lambda_\delta(\alpha u)\|_{L^2(\boldsymbol{R}^n)} \le (M_0 \ell)^\ell \ell^{ms} R_S,$$

where M_0 *depends only on* Supp α*,* m_S *depends only on* S*,* R_S *depends on* A_S *of* (5.1) *and* $\|u\|_{L^2(\Omega)}$*. Furthermore the constant* M_0*,* m_S *and* R_S *are independent of small* $\delta > 0$ *and* $\ell \in \boldsymbol{N}$*.*

Proof of first part of Theorem 1.1. **Nonlinear interior regularity**
By using the estimate (5.3) with $S = 4eM_0$, we have

$$\begin{aligned}\|\langle D\rangle^2 \Lambda_\delta(\alpha u)\|_{L^2} &\le \sum_{\ell=0}^{\infty} \|(\log \Lambda^2)^\ell \Lambda_\delta(\alpha u)\|_{L^2}(\ell!)^{-1} \\ &\le \sum_{\ell=0}^{\infty} \|(\log \Lambda^S)^\ell \Lambda_\delta(\alpha u)\|_{L^2}\left(\frac{2}{S}\right)^\ell (\ell!)^{-1} \\ &\le R_S \sum_{\ell=1}^{\infty}\left(\frac{1}{2}\right)^\ell \ell^{m_S} + \|\alpha u\|_{L^2} < +\infty,\end{aligned}$$

where we have used the estimate $\ell^\ell \le e^\ell \ell!$. Since R_S, m_S independent of δ, we have proved $\alpha u \in H^2(\boldsymbol{R}^n)$. Now $\Lambda_\delta(\alpha u) \in H^4$, the similar calculus as above give that $\alpha u \in H^4(\boldsymbol{R}^n)$ if we take $S = 2 \times 4eM_0$ in (5.3). By recurrence we get that $\alpha u \in H^m(\boldsymbol{R}^n)$ for any $m \in \boldsymbol{N}$. It follows from the Sobolev embedding theorem that $\alpha u \in C^\infty(\boldsymbol{R}^n)$. Since $\alpha \in C_0^\infty(\Omega)$ is arbitrary, we have proved $u \in C^\infty(\Omega)$.

Proof of Proposition 5.1. For $\ell \ge 1$ fixed, we choose the functions of $C_0^\infty(\Omega)$ as in [15, 16],

$$\alpha = \alpha_\ell \subset\subset \alpha_{\ell-1} \subset\subset \cdots \subset\subset \alpha_1 \subset\subset \alpha_0 = \beta,$$

such that

$$\|D^\lambda \alpha_j\|_{L^\infty} \le C_\lambda \ell^{|\lambda|}, \qquad \forall \lambda \in \boldsymbol{N}^n \tag{5.4}$$

with C_λ depending only on a_0. For the proof of Proposition 5.1, we prove the following estimate: for any $1 \le j \le \ell$, and any $j \le k \le \ell$, we have

$$\|(\log \Lambda^S)^j \Lambda_\delta(\alpha_k u)\|_{L^2} \le (M_0 \ell)^j \ell^{m_S} R_S \tag{5.5}$$

with the constant as in Proposition 5.1.

We need also the following two results about pseudo-differential calculus, whose proofs will be given in the next section.

First result is about the pseudo-differential operators as a regularlizer.

Proposition 5.2. *For any* $m, m' \in \boldsymbol{N}$*, we have*

$$\|(\alpha_k - 1)(\log \Lambda^S)^j \Lambda_\delta(\alpha_{k+1} u)\|_{H^m}^2 \le C_{S,m,m'}(j!\ell^{3m+2m'+2S+3n+4})^2 \|\beta u\|_{H^{-m'}}^2,$$

with $C_{S,m,m'}$ *independent of* ℓ*,* j *and* δ*, and*

$$\|\alpha_k(\log \Lambda^S)^j \Lambda_\delta(\alpha_{k+1}u)\|^2_{H^{-s}} \le C_S(j!\ell^{2S+3n+7})^2\|\beta u\|^2_{L^2},$$

with C_S *independent of* ℓ, j *and* δ.

For the commutators, we have

Proposition 5.3. *Let* X *be vector fields,* $1 \le j \le \ell$, $j \le k \le \ell$, *we have*

$$\|[X, \alpha_k(\log \Lambda^S)^j \Lambda_\delta \alpha_{k+1}]\alpha_k u\|^2_{L^2} \le C_S(\ell^2 |||u|||^2_{j,k,S} + (j!)^2\ell^{10(S+n+2)}\|\beta u\|^2_{L^2}),$$

and

$$\|[X, [X, \alpha_k(\log \Lambda^S)^j \Lambda_\delta \alpha_{k+1}]]\alpha_k u\|^2_{L^2} \le C_S(\ell^4 |||u|||^2_{j,k,S} + (j!)^2\ell^{10(S+n+2)}\|\beta u\|^2_{L^2}),$$

with C_S *independent of* j, k, ℓ *and* δ, *where*

$$|||u|||^2_{j,k,S} = \sum_{0 \le j' \le \min\{j, S+2\}} \left(\frac{j!}{(j-j')!}\right)^2 \|(\log \Lambda^S)^{j-j'}\Lambda_\delta \alpha_{k-j'}u\|^2_{L^2}.$$

We prove now (5.5) by induction on j.

1) For $j = 1$, $1 \le k \le \ell - 1$, take $\alpha_{k+1}\Lambda_\delta \alpha_k^2 \Lambda_\delta(\alpha_{k+1}u) \in H^1_0(\Omega)$ as test function in (1.3),

$$\sum_{p=1}^m \int_\Omega (X_p u)X_p(\alpha_{k+1}\Lambda_\delta \alpha_k^2 \Lambda_\delta(\alpha_{k+1}u))dx$$

$$= \int_\Omega \alpha_{k+1}(F(x,u) - X_0 u)(\Lambda_\delta \alpha_k^2 \Lambda_\delta(\alpha_{k+1}u))dx.$$

Then it follows from (5.4), the Cauchy-Schwarz inequality and the fact $|\langle X_0 v, v\rangle| \le C\|v\|^2_{L^2}$ for reals functions and real vector X_0, that

$$\sum_{p=1}^m \|X_p \alpha_k \Lambda_\delta(\alpha_{k+1}u)\|^2_{L^2} \le C_1\|\alpha_{k+1}u\|^2_{L^2} + C_2\ell^2\|\alpha_k u\|^2_{L^2},$$

where C_1 and C_2 are the constants in (5.1) and (5.4). On the other hand, (1.2) gives that

$$\|\log \Lambda(\alpha_k \Lambda_\delta(\alpha_{k+1}u))\|^2_{L^2} \le \varepsilon\|X(\alpha_k \Lambda_\delta(\alpha_{k+1}u))\|^2_{L^2} + C_\varepsilon\|\alpha_{k+1}u\|^2_{L^2}.$$

We have for any $S \ge 1$,

$$\begin{aligned}\|\log(\Lambda^S)\Lambda_\delta(\alpha_{k+1}u)\|^2_{L^2} &\le S^2\varepsilon\|X(\alpha_k \Lambda_\delta(\alpha_{k+1}u))\|^2_{L^2} + S^2 C_\varepsilon\|\alpha_{k+1}u\|^2_{L^2}\\ &\quad + \|\log(\Lambda^S)(\alpha_k - 1)\Lambda_\delta(\alpha_{k+1}u)\|^2_{L^2}\\ &\le S^2\varepsilon(C_1\|\alpha_{k+1}u\|^2_{L^2} + C_2\ell^2\|\alpha_k u\|^2_{L^2})\\ &\quad + S^2 C_\varepsilon\|\alpha_{k+1}u\|^2_{L^2} + \|\log(\Lambda^S)(\alpha_k - 1)\Lambda_\delta(\alpha_{k+1}u)\|^2_{L^2}.\end{aligned}$$

For the last term of right hand side, Proposition 5.2 gives

$$\|\log(\Lambda^S)(\alpha_k - 1)\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 \leq C_S \ell^{4S+6n+8}\|\beta u\|_{L^2}^2.$$

We have proved (5.5) for $j = 1$ if we choose $\varepsilon > 0$ small such that $\varepsilon S^2 \leq 1$ and

$$M_0^2 \geq C_1 + C_2 + C_3 + 1, \qquad R_S^2 \geq (SC_{1/S} + C_S)\|\beta u\|_{L^2}^2,$$

$$2m_S \geq 10(S + n + 2).$$

2) Suppose now that there exists a $j \leq \ell - 1$ such that (5.5) is true for any $p \leq j$. We shall prove (5.5) for $j + 1$. Firstly, take $\delta \to 0$, we have for any $p \leq j$ and $p \leq k \leq \ell$

$$\|(\log \Lambda^S)^p(\alpha_k u)\|_{L^2} \leq (M_0\ell)^p \ell^{m_S} R_S, \tag{5.6}$$

and

$$\sum_{0 \leq j' \leq \min\{j, S+2\}} \left(\frac{j!}{(j-j')!}\right)^2 \|(\log \Lambda^S)^{j-j'}\Lambda_\delta \alpha_{k-j'}u\|_{L^2}^2 \leq C_S(M_0\ell)^{2j}\ell^{2m_S}R_S^2. \tag{5.7}$$

For $j \leq k \leq \ell - 1$, set

$$v = \alpha_{k+1}\Lambda_\delta(\log \Lambda^S)^j \alpha_k^2(\log \Lambda^S)^j \Lambda_\delta(\alpha_{k+1}u),$$

then $v \in H_0^1(\Omega)$, using v as test function in (1.3),

$$\int_\Omega \left(\sum_{p=1}^m X_p^* X_p u\right) v \, dx = \int_\Omega (F(x,u) - X_0 u) v \, dx.$$

By using integration by parts, the Cauchy-Schwarz inequality and the fact that $|\langle X_0 v, v\rangle| \leq C\|v\|_{L^2}^2$, we have that

$$\begin{aligned}
&\sum_{p=1}^m \|X_p \alpha_k (\log \Lambda^S)^j \Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 \\
&\leq \frac{1}{2}\|\log \Lambda^S \alpha_k (\log \Lambda^S)^j \Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 + 8\|(\log \Lambda^S)^{j-1}\Lambda_\delta(\alpha_{k+1}F(x,u))\|_{L^2}^2 \\
&\quad + 8\sum_{p=0}^m \|[\alpha_k(\log \Lambda^S)^j \Lambda_\delta \alpha_{k+1}, X_p](\alpha_k u)\|_{L^2}^2 \\
&\quad + 12\sum_{p=0}^m \|[\alpha_k(\log \Lambda^S)^j \Lambda_\delta \alpha_{k+1}, X_p^*](\alpha_k u)\|_{L^2}^2 \\
&\quad + \frac{8}{\ell^2}\sum_{p=0}^m \|[[\alpha_k(\log \Lambda^S)^j \Lambda_\delta \alpha_{k+1}, X_p^*], X_p](\alpha_k u)\|_{L^2}^2 + \ell^2\|(\log \Lambda^S)^j \Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2
\end{aligned}$$

$$+ 2\|\beta F(x,u)\|_{L^2}\|(\log \Lambda^S)^j(\alpha_k - 1)\alpha_k(\log \Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}$$

$$= (\mathbf{1}) + (\mathbf{2}) + \cdots + (\mathbf{7}).$$

By the induction hypothesis, for any $j \le k \le \ell$, estimate (5.1) gives

$$\begin{aligned}(\mathbf{2}) &= 8\|(\log \Lambda^S)^{j-1}\Lambda_\delta(\alpha_{k+1}F(x,\alpha_k u))\|_{L^2}^2\\ &\le A_S^2(\ell^{2j-2}\ell^{2S+4}\|\alpha_k u\|_{L^2}^2 + 2(j-1)\|(\log \Lambda^S)^j(\alpha_k u)\|_{L^2}^2)\\ &\le A_S^2(\ell^{2S+4}\ell^{2(j-1)}\|\beta u\|_{L^2}^2 + 2(j-1)(M_0\ell)^{2j}\ell^{2m_S}R_S^2).\end{aligned}$$

The first estimate of Proposition 5.3 and (5.7) give

$$(\mathbf{3}) + (\mathbf{4}) \le C_S(\ell^2(M_0\ell)^{2j}\ell^{2m_S}R_S^2 + \ell^{2j}\ell^{10(S+n+2)}\|\beta u\|_{L^2}^2).$$

The second estimate of Proposition 5.3 yields

$$(\mathbf{5}) \le C_S(\ell^2(M_0\ell)^{2j}\ell^{2m_S}R_S^2 + \ell^{2j}\ell^{10(S+n+2)}\|\beta u\|_{L^2}^2),$$

and by Proposition 5.2 we have

$$(\mathbf{7}) \le C_S\ell^{2j}\ell^{10(S+n+2)}\|\beta u\|_{L^2}^2.$$

The estimation of term (**6**) follows from the induction hypothesis. If $m_S \ge 5(S+n+2)$ then we have finally

$$\begin{aligned}\sum_{p=1}^m \|X_p\alpha_k(\log \Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 &\le \frac{1}{2}\|\log(\Lambda^S)\alpha_k(\log \Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2\\ &\quad + \tilde{C}_S(M_0\ell)^{2j+2}\ell^{2m_S}R_S^2.\end{aligned}$$

Using (5.2) with $N = S$ and Proposition 5.2, we get

$$\begin{aligned}&\|\log \Lambda\alpha_k(\log \Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2\\ &\quad\le \varepsilon\sum_{p=1}^m \|X_p\alpha_k(\log \Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 + C_{\varepsilon,S}\|\alpha_k(\log \Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{H^{-S}}^2\\ &\quad\le \frac{\varepsilon}{2}\|\log(\Lambda^S)\alpha_k(\log \Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 + \varepsilon\tilde{C}_S(M_0\ell)^{2j+2}\ell^{2m_S}R_S^2\\ &\qquad + C_{\varepsilon,S}(j!)^2\ell^{10(S+n+2)}\|\beta u\|_{L^2}^2.\end{aligned}$$

By using the first inequality of Proposition 5.2, we have

$$\|\log(\Lambda^S)(\alpha_k - 1)(\log \Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 \le C_S((j+1)!)^2\ell^{10(S+n+2)}\|\beta u\|_{L^2}^2.$$

In view of $m_S \ge 5(S+n+2)$ again, we get

$$\begin{aligned}
&\|(\log \Lambda^S)^{j+1}\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2\\
&\quad\le \frac{\varepsilon S^2}{2}\|(\log \Lambda^S)^{j+1}\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 + \varepsilon S^2 \tilde{C}_S (M_0\ell)^{2j+2}\ell^{2ms}R_S^2\\
&\qquad + S^2 C_{\varepsilon,S}(j!)^2\ell^{2ms}\|\beta u\|_{L^2}^2 + \left(1+\frac{\varepsilon S^2}{2}\right)C_S((j+1)!)^2\ell^{2ms}\|\beta u\|_{L^2}^2.
\end{aligned}$$

Choose $\varepsilon > 0$ small enough such that $\varepsilon S^2 \le 1$, $\varepsilon S^2\tilde{C}_S \le 1/4$, we get

$$\begin{aligned}
\|(\log \Lambda^S)^{j+1}\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2 &\le \frac{1}{2}(M_0\ell)^{2j+2}\ell^{2ms}R_S^2 + 2S^2 C_{\varepsilon,S}(j!)^2\ell^{2ms}\|\beta u\|_{L^2}^2\\
&\quad + 3C_S((j+1)!)^2\ell^{2ms}\|\beta u\|_{L^2}^2.
\end{aligned}$$

We have proved (5.5) if we take

$$R_S^2 \ge 2(2S^2 C_{\varepsilon,S+1} + 3C_S)\|\beta u\|_{L^2}^2.$$

Regularity up to the boundary

Fix a $x_0 \in \partial\Omega$ and take a sufficiently small neighbourhood V_0 of x_0 in $\tilde{\Omega}$. We use the standard process of localization and a C^∞ change of variable to flatten out the boundary part $\partial\Omega \cap V_0$. Without loss of generality, we suppose that $g = 0$. If $u \in L^\infty(\Omega) \cap H_X^1(\Omega)$ is a weak solution of equation (1.3), we have already the interior regularity $u \in C^\infty(\Omega)$. On the domain $\Omega \cap V_0$, after straighten (we keep the same notation for the solution u), we have the following equation (see [6, 9, 26]):

$$\begin{cases} \partial_{x_n}^2(\alpha u) - \sum_{j=1}^{m-1} Y_j^* Y_j(\alpha u) = \partial_{x_n}(a_0\beta u) + Y_0(\beta u) + \tilde{F}(x, \beta u), & \text{in } \boldsymbol{R}_+^n \\ \beta u(x', 0) = 0, \quad \text{for } x' \in \boldsymbol{R}^{n-1} \end{cases} \tag{5.8}$$

where $\alpha, \beta, a_0 \in C_0^\infty(\overline{\boldsymbol{R}_+^n})$, $\alpha \subset\subset \beta$ with $\operatorname{Supp}\beta$ a neighborhood of 0 in $\boldsymbol{R}^n$, and $Y_j = \sum_{k=1}^{n-1} a_{jk}(x', x_n)\partial_{x_k}$, $j = 0, 1, \dots, m-1$ are the tangential vector fields. We have that the system of vector fields $Y = (\partial_{x_n}, Y_1, \dots, Y_{m-1})$ satisfies still the logarithmic regularity estimates (1.1) or (1.2) on a neighborhood $\mathcal{O} \subset \boldsymbol{R}^n$ of 0. Remark that we have $\beta u \in L^\infty(\boldsymbol{R}_+^n) \cap H_{Y,0}^1(\boldsymbol{R}_+^n)$.

Let $\Lambda' = (e + |D'|^2)^{1/2}$ with $D' = (D_{x_1}, \dots, D_{x_{n-1}})$. On account of (1.2), for any small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|(\log \Lambda')v\|_{L^2(\boldsymbol{R}^n)}^2 \le \varepsilon\left(\sum_{j=1}^{m-1}\|Y_j v\|_{L^2(\boldsymbol{R}^n)}^2 + \|\partial_{x_n}v\|_{L^2(\boldsymbol{R}^n)}^2\right) + C_\varepsilon\|v\|_{L^2(\boldsymbol{R}^n)}^2, \tag{5.9}$$

for all $v \in C_0^\infty(\mathcal{O} \cap \boldsymbol{R}_+^n)$. This is true for $v \in H_{Y,0}^1(\mathcal{O} \cap \boldsymbol{R}_+^n)$.

Firstly for the nonlinear term we have the similar result as in (5.1): Suppose that $\beta u \in H_{0,\ell}^{\log}(\boldsymbol{R}_+^n) \cap L^\infty(\boldsymbol{R}_+^n)$ for some $\ell > 1/2$, then for any $S \ge 1$, we have

$$(5.10)\qquad \|(\log \Lambda'^S)^{\ell-1/2}(\alpha\tilde{F}(x,\beta u))\|^2_{L^2(\boldsymbol{R}^n_+)} \le A_S^2(\ell^{2\ell-1}\|\beta u\|^2_{L^2(\boldsymbol{R}^n_+)} + (2\ell-1)\|(\log \Lambda'^S)^{\ell}(\beta u)\|^2_{L^2(\boldsymbol{R}^n_+)}),$$

where A_S depends on S, $\|u\|_{L^\infty}$ and $\|\alpha(x)\tilde{F}(x,t)\|_{C^{[S]+2}((\overline{\boldsymbol{R}^n_+}\times[-\|u\|_{L^\infty},\|u\|_{L^\infty}])}$, but not on ℓ.

If the equation (1.3) is linear, namely, when we consider the regularity up to the boundary for Theorem 3.1, we use the following estimate: If $f\in C^\infty(\overline{\boldsymbol{R}^n_+})$, then for any $\ell\in\boldsymbol{N}$, any $S\ge 1$, and $\alpha\in C_0^\infty(\overline{\boldsymbol{R}^n_+})$,

$$\|(\log \Lambda'^S)^{\ell}(\alpha f)\|_{L^2(\boldsymbol{R}^n_+)} \le \ell!\|\Lambda'^S(\alpha f)\|_{L^2(\boldsymbol{R}^n_+)}.$$

For small $\delta>0$, we set $\Lambda'_\delta=(1-\delta\triangle_{x'})^{-1}$, with $\triangle_{x'}=\sum_{j=1}^{n-1}\partial^2_{x_j}$, this is a tangential regularization operators. As for Proposition 5.1, we have that for any $\ell\in\boldsymbol{N}$, and any $S\ge 1$,

$$(5.11)\qquad \|(\log \Lambda'^S)^{\ell}\Lambda'_\delta(\alpha u)\|_{L^2(\boldsymbol{R}^n_+)} \le (M_0\ell)^{\ell}\ell^{ms}R_S,$$

with the same constants as in (5.3). By using the estimates (5.9) and (5.10), the proof of this estimate is exactly as that of Proposition 5.1, for example, in the step 2 of the proof for Proposition 5.1, we take here

$$v=\alpha_{k+1}\Lambda'_\delta(\log \Lambda'^S)^j\alpha_k^2(\log \Lambda'^S)^j\Lambda'_\delta(\alpha_{k+1}u),$$

as test function in (5.8). In fact, we have $v,\partial_{x_n}v,\Lambda'v\in L^2(\boldsymbol{R}^n_+)$ and $v(x',0)=0$, then $v\in H^1_0(\boldsymbol{R}^n_+)$. Moreover, $\alpha_{k+1}\Lambda'_\delta(\log \Lambda'^S)^j\alpha_k^2(\log \Lambda'^S)^j\Lambda'_\delta\alpha_{k+1}$ is a tangential pseudo-differential operators, thus all pseudo-differential calculus in the proof is tangential, and the integration by part for the variable x_n take only once.

Now the estimate (5.11) implies that $\Lambda'^m(\alpha u)\in L^2(\boldsymbol{R}^n_+)$ for any $m\in\boldsymbol{N}$ and any $\alpha\in C_0^\infty(\mathcal{O}\cap\overline{\boldsymbol{R}^n_+})$, and we have already $\partial_{x_n}(\alpha u)\in L^2(\boldsymbol{R}^n_+)$, so that we have $\alpha u\in H^1(\boldsymbol{R}^n_+)$. For $m\ge 2$, we have, by using the equation (5.8),

$$\partial^2_{x_n}(\alpha u)=\sum_{j=1}^{m-1}Y_j^*Y_j(\alpha u)+\partial_{x_n}(a_0\beta u)+Y_0(\beta u)+\tilde{F}(x,\beta u)\in L^2(\boldsymbol{R}^n_+),$$

then, we have $\alpha u\in H^2(\boldsymbol{R}^n_+)$. By induction we prove that $\alpha u\in H^m(\boldsymbol{R}^n_+)$ for any $m\in\boldsymbol{N}$. We have proved finally $\alpha u\in C^\infty(\overline{\boldsymbol{R}^n_+})$ by Sobolev embedding theorem. Take $\alpha=1$ near $0\in\boldsymbol{R}^n$, we have proved $u\in C^\infty(\tilde{\mathcal{O}}\cap\overline{\boldsymbol{R}^n_+})$ for $\tilde{\mathcal{O}}$ a neighborhood of 0 in $\boldsymbol{R}^n$. Thus we get the C^∞ regularity of solution up to the boundary.

6. Pseudo-differential calculus for the symbol of logarithmic type

To prove Propositions 5.2 and 5.3, we recall two elementary lemmas about the pseudo-differential calculus (see for example [13]). For a symbol $p(x,\xi)\in S^m_{1,0}$ and $k\in\boldsymbol{N}$, we set

$$|p|_k^{(m)} = \sup_{x,\xi\in\boldsymbol{R}^n} \sup_{|\alpha+\beta|\le k} |p_{(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{|\alpha|-m}, \qquad (p_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta p(x,\xi).)$$

Lemma 6.1. *Let* $p(x,\xi)\in S_{1,0}^{m_1}$ *and* $q(x,\xi)\in S_{1,0}^{m_2}$. *For* $0\le\theta\le 1$ *set*

$$r_\theta(x,\xi) = Os - \iint e^{-iy\cdot\eta} p(x,\xi+\theta\eta)q(x+y,\xi)\frac{dyd\eta}{(2\pi)^n},$$

where $Os-\iint$ *denotes the oscillatory integral* (*see* §6 *of* [13] *Chapter* 1). *Then* $r_\theta(x,\xi)\in S_{1,0}^{m_1+m_2}$, *more precisely, for any* $k\in\boldsymbol{N}$ *we have*

$$|r_\theta|_k^{(m_1+m_2)} \le C_{k,n}|p|_{k+n+1}^{(m_1)}|q|_{k+n+1+|m_1|}^{(m_2)}, \tag{6.1}$$

where $C_{k,n}$ *is a positive constant depending only on* k *and* n.

The lemma is only a special case of Lemma 2.4 of [13] Chapter 2, except for the precise numbers of suffices given in the right hand sides of (6.1). In view of this lemma, it is easy to get the following lemma, by using the usual asymtotic formula for the product of pseudodifferential operators.

Lemma 6.2. *Let* $p(x,\xi)\in S_{1,0}^{m_1}$ *and* $q(x,\xi)\in S_{1,0}^{m_2}$. *Then for any* $N\in\boldsymbol{N}$ *we have*

$$[P,Q] = \sum_{0<|\alpha+\beta|<N} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} Q_{(\beta)}^{(\alpha)}P_{(\alpha)}^{(\beta)} + R_N,$$

where $P_{(\alpha)}^{(\beta)} = p_{(\alpha)}^{(\beta)}(x,D)$. *Here* $\sigma(R_N) = r_N(x,\xi)$ *belongs to* $S_{1,0}^{m_1+m_2-N}$ *and satisfy*

$$|r_N|_k^{(m_1+m_2-N)} \le C_0|p|_{k+N+|m_2|+n+1}^{(m_1)}|q|_{k+N+|m_1|+n+1}^{(m_2)} \tag{6.2}$$

with a constant $C_0>0$ *independent of* $p(x,\xi)$ *and* $q(x,\xi)$.

Proposition 6.1. *For any multi-index* $\lambda\ne 0$ *there exists a* $C_\lambda>0$ *such that for any real* $S\ge 1$ *and any* $j\in\boldsymbol{N}$ *we have*

$$|\partial_\xi^\lambda(\log\langle\xi\rangle^S)^j)| \le C_\lambda S^{|\lambda|}\langle\xi\rangle^{S-|\lambda|}j!,$$

and hence $(\log\langle\xi\rangle^S)^j)/j!$ *belongs to* $S_{1,0}^S$ *uniformly with respect to* j.

Proof. The Faà-di-Bruno formula shows that for any $\lambda\ne 0$,

$$(\log^j\langle\xi\rangle)^{(\lambda)} = \sum_{1\le j'\le\min\{|\lambda|,j\}} \frac{j!}{(j-j')!}(\log^{j-j'}\langle\xi\rangle)\omega_{j',\lambda}(\xi),$$

where $\omega_{j',\lambda}(\xi)$ belong to $S_{1,0}^{-|\lambda|}$ uniformly with respect to j. Multiplying S^j by both sides, we get

$$
\begin{aligned}
|(\log^j\langle\xi\rangle^S)^{(\lambda)}| &\le C_\lambda \sum_{1\le j'\le\min\{|\lambda|,j\}} \frac{j!}{(j-j')!}\frac{\log^{j-j'}\langle\xi\rangle^S}{\langle\xi\rangle^S} S^{j'}\langle\xi\rangle^{S-|\lambda|} \\
&\le C_\lambda S^{|\lambda|}\langle\xi\rangle^{S-|\lambda|} \sum_{1\le j'\le\min\{|\lambda|,j\}} \frac{j!}{(j-j')!}(j-j')! \\
&\le C_\lambda |\lambda| S^{|\lambda|}\langle\xi\rangle^{S-|\lambda|} j!.
\end{aligned}
$$

The proof of the above proposition and the Leibniz formula yield.

Corollary 6.1. *For any λ it follows that*

$$
((\log\Lambda^S)^j\Lambda_\delta)^{(\lambda)} = \sum_{0\le j'\le\min\{|\lambda|,j\}} \frac{j!}{(j-j')!}(\log\Lambda^S)^{j-j'}\Lambda_\delta(S^{j'}\tilde{\omega}^\delta_{j',\lambda}(D)),
$$

where $\{\tilde{\omega}^\delta_{j',\lambda}(\xi); 0<\delta<1\}$ is a bounded set of $S^{-|\lambda|}_{1,0}$. Furthermore, if we put $\tilde{\Lambda}^S_{j,\delta} = (\log\Lambda^S)^j\Lambda_\delta/j!$ then for any $k\in\mathbf{N}$ we see

$$
|\sigma(\tilde{\Lambda}^S_{j,\delta})|^{(S)}_k \le C_k S^k \tag{6.3}
$$

with a constant C_k independent of j, δ and S, which implies that those symbols belong to a bounded set of $S^S_{1,0}$ uniformly with respect to j and δ.

Proof of Proposition 5.2. In view of Corollary 6.1, it follows from Lemma 6.2 that for any $N\in\mathbf{N}$,

$$
\begin{aligned}
&(\alpha_k-1)(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)/j! \\
&\quad = (\alpha_k-1)\sum_{|\lambda|<N}\frac{1}{\lambda!}(\alpha_{k+1})_{(\lambda)}(\tilde{\Lambda}^S_{j,\delta})^{(\lambda)}\beta u + (\alpha_k-1)R_N\beta u,
\end{aligned}
$$

where the first term of the right hand side vanishes, and it follows from (5.4) that

$$
\frac{1}{j!}\|(\alpha_k-1)(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{H^m} \le C_m\ell^m\|(\Lambda^m R_N\Lambda^{m'})\Lambda^{-m'}\beta u\|_{L^2}.
$$

If we choose the smallest N such that $N\ge S+m+m'$, then $\Lambda^m R_N\Lambda^{m'}$ is a L^2 bounded operator and it follows from the Calderón-Vaillancourt theorem that

$$
\frac{1}{j!}\|(\alpha_k-1)(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{H^m} \le C_0C_m\ell^m|\sigma(\Lambda^m R_N\Lambda^{m'})|^{(0)}_{n+1}\|\beta u\|_{H^{-m'}},
$$

where C_0 depends only on n. By means of (6.1) and (6.2) we have

$$
\begin{aligned}
|\sigma(\Lambda^m R_N\Lambda^{m'})|^{(0)}_{n+1} &\le C_{m,m'}|\sigma(R_N)|^{(S-N)}_{n+1+m+m'+n+1} \\
&\le C'_{m,m'}|\sigma(\tilde{\Lambda}^S_{j,\delta})|^{(S)}_{m+m'+3(n+1)+N}|\alpha_{k+1}|^{(0)}_{m+m'+3(n+1)+N+S}.
\end{aligned}
$$

In view of (6.3) and (5.4), we obtain the first estimate of Proposition 5.2, that is,

$$(6.4)\qquad \|(\alpha_k - 1)\tilde{\Lambda}^S_{j,\delta}(\alpha_{k+1}u)\|_{H^m} \le CS^{2m+2m'+3n+4}\ell^{3m+2m'+2S+3n+4}\|\beta u\|_{H^{-m'}}.$$

By means of the Leibniz formula we have $|\sigma(\alpha_k\tilde{\Lambda}^S_{j,\delta})|^{(S)}_{k_0} \le C_S\ell^{k_0}$, and it follows from (6.1) that

$$|\sigma(\Lambda^{-S}\alpha_k\tilde{\Lambda}^S_{j,\delta})|^{(0)}_{n+1} \le C_S\ell^{S+2n+2},$$

which yields the second estimate of Proposition 5.2.

Proof of Proposition 5.3. As to the first estimate it suffices to show

$$(6.5)\qquad \|[X, \tilde{\Lambda}^S_{j,\delta}\alpha_{k+1}]\alpha_k u\|_{L^2} \le C_S(\ell|||u|||_{j,k,S}/j! + \ell^{5(S+n+2)}\|\beta u\|_{L^2}),$$

because of (6.4) with $(m, m') = (1,0), (0,1)$. If we write

$$\tilde{\Lambda}^S_{j,\delta}\alpha_{k+1} = \sum_{0\le|\lambda|\le[S]+1}\frac{1}{\lambda!}(\alpha_{k+1})_{(\lambda)}(\tilde{\Lambda}^S_{j,\delta})^{(\lambda)} + R_{[S]+2},$$

then we have

$$\|[X, R_{[S]+2}]\alpha_k u\|_{L^2} \le C_S\ell^{5(S+n+2)}\|\beta u\|_{L^2},$$

similarly as in the proof of Proposition 5.2. By using Lemma 6.2 again we have

$$[X, \tilde{\Lambda}^S_{j,\delta}\alpha_{k+1}]\alpha_k u \equiv -\sum_{\substack{0\le|\lambda|\le[S]+1\\ 0<|\gamma+\gamma'|\le[S]+1-|\lambda|}}\frac{(-1)^{|\gamma|}}{\lambda!\gamma!\gamma'!}X^{(\gamma)}_{(\gamma')}(\alpha_{k+1})_{(\lambda+\gamma)}(\tilde{\Lambda}^S_{j,\delta})^{(\lambda+\gamma')}(\alpha_k u),$$

where we denote by $\equiv$, neglecting terms which can be estimated by the second term of the right hand side of (6.5). In view of $|\lambda + \gamma' + \gamma| \ge \max\{1, |\lambda + \gamma|\}$, it follows from (5.4) and the first formula of Corollary 6.1 that

$$\begin{aligned}\|[X, \tilde{\Lambda}^S_{j,\delta}\alpha_{k+1}]\alpha_k u\|_{L^2} \le C_S \sum_{j'=0}^{\min\{[S]+1,j\}}\ell\sum_{p=0}^{[S]+1}\left\|\left(\frac{\ell}{\Lambda}\right)^p\tilde{\Lambda}^S_{j',\delta}\alpha_{k-j'}u\right\|_{L^2}\\ + \ell^{5(S+n+2)}\|\beta u\|_{L^2},\end{aligned}$$

because of $\alpha_k = (\alpha_k - 1)\alpha_{k-j'}\beta + \alpha_{k-j'}$ for $j' \ge 1$ and the formula similar to (6.4). Noting $(\ell/\langle\xi\rangle)^p \le 1 + (\ell/\langle\xi\rangle)^{[S]+1}$ we obtain (6.5). For the proof of the second estimate it suffices to show

$$(6.6)\qquad \|[X, [X, \tilde{\Lambda}^S_{j,\delta}\alpha_{k+1}]]\alpha_k u\|_{L^2} \le C_S(\ell^2|||u|||_{j,k,S}/j! + \ell^{5(S+n+2)}\|\beta u\|_{L^2})$$

on account of (6.4). Note that

$$
\begin{aligned}
[X,[X,\tilde{\Lambda}^S_{j,\delta}\alpha_{k+1}]]\alpha_k u &\equiv \sum_{0<|\lambda+\gamma+\gamma'|\le[S]+2} C_{\lambda,\gamma,\gamma'}[(\alpha_{k+1})_{(\lambda+\gamma)}X^{(\gamma)}_{(\gamma')}(\tilde{\Lambda}^S_{j,\delta})^{(\lambda+\gamma')},X](\alpha_k u)\\
&\equiv \sum_{0<|\lambda+\gamma+\gamma'|\le[S]+2}\ \sum_{\substack{0<|\kappa+\kappa'|\le\\ [S]+2-|\lambda+\gamma+\gamma'|}}\ \sum_{\substack{\kappa=\kappa_1+\kappa_2\\ \kappa'=\kappa_1'+\kappa_2'}} C_{\lambda,\gamma,\gamma',\kappa_1,\kappa_2,\kappa_1',\kappa_2'}(\alpha_{k+1})_{(\lambda+\gamma+\kappa_1)}\\
&\qquad\times X^{(\kappa)}_{(\kappa')}X^{(\gamma+\kappa_1')}_{(\gamma'+\kappa_2)}(\tilde{\Lambda}^S_{j,\delta})^{(\lambda+\gamma'+\kappa_2')}(\alpha_k u).
\end{aligned}
$$

Since it follows that $|\lambda+\gamma+\gamma'+\kappa+\kappa'|\ge\max\{2,|\lambda+\gamma+\kappa_1)|\}$ we obtain (6.6) by the same way as to (6.5).

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