LOCAL EXISTENCE WITH PHYSICAL VACUUM BOUNDARY CONDITION TO EULER EQUATIONS WITH DAMPING

CHAO-JIANG XU AND TONG YANG

Abstract *In this paper, we consider the local existence of solutions to Euler equations with linear damping under the assumption of physical vacuum boundary condition. By using the transformation introduced in* [13] *to capture the singularity of the boundary, we prove a local existence theorem on a perturbation of a planar wave solution by using Littlewood-Paley theory and justifies the transformation introduced in* [13] *in a rigorous setting.*

Key words Euler equations, physical vacuum boundary condition, Littlewood-Paley theory, local existence.

A.M.S. Classification 35L67, 35L65, 35L05.

1. Introduction

In this paper, we are interested in the time evolution of a gas connecting to vacuum with physical boundary condition. By assuming that the governed equations for the gas dynamics are Euler equations with linear damping, cf. [16] for physical interpretation, one can see that the system fails to be strictly hyperbolic at the vacuum boundary because the characteristics of different families coincide. As discussed in the previous works, cf. [5, 11, 12, 13], the canonical vacuum boundary behavior is the case when the space derivative of the enthalpy is bounded but not zero. In this case, the pressure has its non-zero finite effect on the evolution of the vacuum boundary. However, for this canonical (physical) case, the system becomes singular in the sense that it can not be symmetrizable with regular coefficients so that the local existence theory for the classical hyperbolic systems can not be applied. Furthermore, the linearized equation at the boundary gives a Keyldish type equation for which general local existence theory is still not known. Notice that this linearized equation is quite different from the one considered in [18] for weakly hyperbolic equation which is of Tricomi type. To capture this singularity in the nonlinear settting, a transformation was introduced in [13] and some local existence results for bounded domain were also discussed. The transformed equation is a second order nonlinear wave equation of an unknown function $\acute{A}(y; t)$ with coefficients as functions of $y^{-1}\acute{A}(y; t)$ and $\acute{A}(0; t) \equiv 0$. Along the vacuum boundary, the physical boundary condition implies that the coefficients are functions of $\acute{A}_y(0; t)$ which are bounded and away from zero. Hence, the wave equation has no singularity or degeneracy. However, its coefficients have the above special form so that the local existence theory developed for the classical nonlinear wave equation can not be applied directly, [8, 9]. There are other works on this system with vacuum, please refer to [6, 10] ect. and reference therein.

TO APPEAR IN “J. DIFF. EQU.”

Even though a transformation to capture the singularity in the physical boundary condition at vacuum interface is introduced in [13], the energy method presented there may not give a rigorous proof of the existence theory, especially in the general setting. It is because the coefficients in the reduced wave equation which are power functions of $y^{-1}Á$ correspond to the fractional differentiations of $Á$. Under this consideration, we think the application of Littlewood-Paley theory based on Fourier theory is more appropriate. Therefore, as the first step in this direction, in this paper we will study the local existence of solutions satisfying the physical boundary condition when the initial data is a small perturbation of a planar wave solution where the enthalpy is linear in the space variable, [11]. By applying the Littlewood-Paley theory, we obtain the solution local in time with the prescribed physical boundary condition.

Precisely, we consider the one dimensional compressible Euler equations for isentropic flow with damping in Eulerian coordinates

$$\begin{aligned} ½_t + (½u)_x &= 0; \\ ½u_t + ½uu_x + p(½)_x &= -½u; \end{aligned} \tag{1.1}$$

where $½$, u and $p(½)$ are density, velocity and pressure respectively. And the linear frictional coefficient is normalized to 1. When the initial density function contains vacuum, the vacuum boundary Γ is defined as

$$\Gamma = cl\{(x;t) \mid ½(x;t) > 0\} \cap cl\{(x;t) \mid ½(x;t) = 0\}:$$

Since the second equation in (1.1) can be rewritten as

$$u_t + uu_x + i_x = -u;$$

with i being the enthalpy, one can see that the term i_x represents the effect of the pressure on the particle path, in particular, on the vacuum boundary. It is shown in [12, 15, 19] that there is no global existence of regular solutions satisfying $i_x \equiv 0$ along the vacuum boundary. That is, in general, i is not C^1 crossing the vacuum boundary. Hence, the canonical behavior of the vacuum boundary should satisfy the condition $i_x \neq 0$ and is bounded. This special feature of the solution can be illustrated by the stationary solutions and some self-similar solutions, also for different physical systems, such as Euler-Poisson equations for gaseous stars and Navier-Stokes equations, cf. [5, 11, 13, 17]. Notice that the charateristics of Euler equations is $u \pm c$, with $c = \sqrt{p_\rho(½)}$. And for isentropic polytropy gas, $i = \frac{c^2}{\gamma-1}$, where $° > 1$ is the adiabatic constant. Hence the characteristics are singular with infinite space derivative at the vacuum boundary if physical boundary condition is assumed. This singularity yields the smooth reflection of the characteristic curves on the vacuum boundary and then causes analytical difficulty.

Another way to view the canonical boundary condition comes from the study of porous media equation. It is known that the Euler equations with linear damping behave like the porous media equation at least away from vacuum when $t \to \infty$, cf.[7] and some corresponding results in the weak sense with vacuum which will not be discussed here. For the porous media equation, the free boundary of the support of the solution has a canonical behavior which would be the same as or similar to the one described above for Euler equations with damping. However, there is still no satisfatory results on the change of solution behavior along the vacuum boundary even though the corresponding waiting time problem for porous media equation is well understood, cf.[1].

In this paper, we will concentrate on the Euler equations with linear damping when the initial data is a small perturbation of a planar wave in one dimensional space. Since our concern is the behavior of the solution related to vacuum and any shock wave vanishes at vacuum [14], it is reasonable to consider our problem without shock waves. In fact, any shock wave appears initially or in finite time will decay to zero exponentially in time because of the dissipation from the linear damping. By using the special property of the one dimensional gas dynamics, we can rewrite the system (1.1) by using Lagrangian coordinates to make all the particle paths, in particular the vacuum boundary, as straight lines. (1.1) in Lagrangian coordinates takes the form

$$\begin{aligned} &v_t - u_\xi = 0; \\ &u_t + p(v)_\xi = -u; \end{aligned} \tag{1.2}$$

where $v = \frac{1}{\rho}$ is the specific volume and $» = \int_0^x ½(y; t)dy$. Moreover, we assume that the pressure function satisfies the °-law, i.e., $p(v) = ¾^2 v^{-\gamma}$, $° > 1$. Notice that the physical singularity, $i_x \neq 0$ but bounded, along the vacuum boundary in Eulerian coordinates corresponds to $0 < |p_\xi(v)| < \infty$ in the Lagrangian coordinates.

In order to capture this singularity in the solution and symmetrize the system (1.2), the following coordinate transformation was introduced in [13],

$$» = y^{\frac{2\gamma}{\gamma-1}}:$$

Here, we assume that the initial density function $½_0(x) = 0$ for $x < 0$ in the Eulerian coordinates. Then the system (1.2) can be rewritten as

$$\begin{aligned} &Á(v)_t + ¹u_y = 0; \\ &u_t + ¹Á(v)_y = -u; \quad y > 0; \quad t > 0; \end{aligned} \tag{1.3}$$

where $Á(v) = \frac{2\sqrt{\gamma}\sigma}{\gamma-1} v^{-\frac{\gamma-1}{2}}$, and

$$¹ = \frac{(° - 1)¾}{\sqrt{°}} (vy^{\frac{2}{\gamma-1}})^{-\frac{\gamma+1}{2}} = \cdot (y^{-1}Á)^{\frac{\gamma+1}{\gamma-1}};$$

for some positive constant $\cdot$. Without any ambiguity and up to a scaling, $\cdot$ can be chosen to be 1 and we still denote the independent variable by y for simplicity of notation. Notice that near the vacuum boundary, both $Á(v)_y$ and $¹$ are bounded away from zero under the physical boundary condition.

Therefore, the vacuum problem considered can be formulated into the following boundary value problem:

$$(¹^{-1}w_t)_t - (¹w_y)_y + ¹^{-1}w_t = 0; \tag{1.4}$$

$$(w; w_t)|_{t=0} = (w_0; w_1); \tag{1.5}$$

$$w(0; t) = 0; \tag{1.6}$$

$$0 < C_1 \leq y^{-1}w(y; t) \leq C_2; \tag{1.7}$$

with $¹ = (y^{-1}w(y; t))^{\alpha}; ® > 1;$ and compatibilities conditions $@_y^{2\ell} w_0(0) = 0; ` = 0; 1; 2; \cdots :$

It is easy to see that the above equation has a special linear unbounded solution for $y \geq 0$ given by $w(y; t) = a_0 y$ with constant $a_0 > 0$. This solution is also obtained in [11] together with other self-similar solutions with physical boundary condition. To justify the above transformation for local existence purpose, we will consider the local existence of solution when the initial data is a small perturbation to the above special solution.

That is, the initial data is assumed to be

$$w_0(y) = y(a_0 + v_0(y)); \; w_1(y) = y v_1(y);$$

for some constant $a_0 > 0$. And the solution is of following type

$$w(y; t) = y(a_0 + v(y; t)):$$

Then the problem on (1.4)-(1.7) in this setting becomes

$$v_{tt} - ({}^{1}{}^2 v_y)_y + \text{®}(a_0 + v)^{3\alpha - 1} v_y^2 - \frac{\text{®} + 2}{y}\, {}^{1}{}^2 v_y - \frac{\text{®} v_t^2}{a_0 + v} + v_t = 0; \tag{1.8}$$

$$(v; v_t)|_{t=0} = (v_0; v_1) \in H^s(\mathbb{R}_+) \times H^{s-1}(\mathbb{R}_+); \tag{1.9}$$

$$v(0; t) = 0; \quad t > 0; \tag{1.10}$$

$$\|v\|_{L^\infty([0,T]\times\mathbb{R}_+)} \leq \frac{1}{2} a_0; \tag{1.11}$$

with ${}^{1} = (a_0 + v(y; t))^\alpha$ and $\text{®} > 1$.

For this problem, we have the following main theorem in this paper.

Theorem 1.1. *Suppose that, for some $b_0 > 0$, we have*

$$\textit{Supp}\, v_0; \; \textit{Supp}\, v_1 \subset [b_0; +\infty[; \; \|v_0\|_{L^\infty(\mathbb{R}_+)} \leq \frac{1}{4} a_0; \tag{1.12}$$

and $s > \frac{3}{2}$. Then there exists $0 < T < a_0^{-\alpha} b_0$ such that the problems (1.8)-(1.11) has a unique solution

$$v \in C([0; T]; H^s(\mathbb{R}_+)) \cap C^{0,1}([0; T]; H^{s-1}(\mathbb{R}_+)):$$

Notice that the case when $b_0 = 0$ is more difficult and will not be discussed here. Since the solution is regular up to the vacuum boundary and the density function in positive except on the vacuum boundary, the result in Theorem 1.1 can be reduced straightforwardly to the solution to the Equations (1.1).

Notice that here the initial perturbation is in a compact subset in $(0; \infty)$ and the local time existence is proved before the perturbation influence the propagation of the boundary. Therefore, it is interesting and important to consider how the behavior of the boundary changes in later time due to the perturbation. But this is not in the scope of this paper and will be pursued by the authors in the future. For this, the transformation introduced in [13] could still be useful. Furthermore, the physical boundary condition holds also for multi-dimensional space by considering the stationary solutions, [5]. Hence, the evolution of the vacuum interface in multi-dimensional space can also be considered with more difficulty because there is no Lagrangian coordinates to fix the vacuum interface.

The rest of the paper is arranged as follows. In Section 2, we shall briefly include the Littlewood-Paley theory for the proof of local existence. The proof of Theorem

1.1 is given in Section 3 where a linearized system is analyzed to yield a sequence of solutions being convergent to the one in Theorem 1.1.

2. Littlewood-Paley theory

In this section, we will recall some elementary properties of Littlewood -Paley theory for the Sobolev spaces, for the details please refer to [2, 3, 4]. Set

$$H^s(\mathbb{R}^d) = \{f \in \mathcal{S}'; (1+|»|^2)^{s/2}\hat{f} \in L^2(\mathbb{R}^d)\};$$

with the norm $\|f\|_{H^s} = \|(1+|»|^2)^{s/2}\hat{f}\|_{L^2}$. We consider now a dyadic decomposition of $\mathbb{R}^d$. For $K > 1$ a fixed constant, and $p \in \mathbb{N}_+$, we set

$$\mathcal{C}_p = \{» \in \mathbb{R}^d; K^{-1}2^p \le |»| \le K2^{p+1}\}; \tag{2.1}$$

and $\mathcal{C}_{-1} = B(0;K) = \{» \in \mathbb{R}^d; |»| \le K\}$, then $\{\mathcal{C}_p\}_{-1}^{+\infty}$ is a uniformly finite recover of $\mathbb{R}^d$, that means, if $|p-q| \ge N_1 = 2(1+2\log_2 K)+2$, we have $\mathcal{C}_q \cap \mathcal{C}_p = \emptyset$.

We can also construct two functions $';\tilde{A} \in C_0^\infty(\mathbb{R}^d)$, with Supp$\tilde{A} \subset \mathcal{C}_{-1};$ Supp $' \subset \mathcal{C}_0$, such that for any $» \in \mathbb{R}^d$ and N_0,

$$\tilde{A}(») + \sum_{p=0}^{\infty} '(2^{-p}») = 1; \quad \tilde{A}(») + \sum_{p=0}^{N_0-1} '(2^{-p}») = \tilde{A}(2^{-N_0}»):$$

Then one can define the following operators of localization in Fourier space, for $u \in \mathcal{S}'(\mathbb{R}^d)$,

$$\Delta_p u = u_p = \mathcal{F}^{-1}('(2^{-p}\cdot)\hat{u}(\cdot)) = 2^{pd}\int_{\mathbb{R}^d} f(2^p y)u(x-y)dy; \text{ for } p \in \mathbb{N}$$

and

$$\Delta_{-1}u = u_{-1} = \mathcal{F}^{-1}(\tilde{A}(\cdot)\hat{u}(\cdot));$$

where $\hat{u} = \mathcal{F}(u)$ denotes the Fourier transformation of u, and $f = \mathcal{F}^{-1}(')$. It is evident that $u_p \in \mathcal{S}'$ for any $u \in \mathcal{S}'$, Supp$\hat{u}_p \subset \mathcal{C}_p$, and $u = \sum_{p=-1}^{\infty} u_p$, in sense of $\mathcal{S}'$.

Since Supp$\hat{u}_p \subset \mathcal{C}_p$, Paley-Wienner-Schwartz theorem implies that $u_p \in C^\infty$ and the Sobolev space can be characterized as follows.

Lemma 2.1. *For $s > 0$, the following properties are equivalent.*

(a) $u \in H^s(\mathbb{R}^d)$;

(b) $u = \sum_{p=-1}^{\infty} u_p$ in $\mathcal{S}'$, Supp $\hat{u}_p \subset \mathcal{C}_p$ and $\|u_p\|_{L^2} \le c_p 2^{-ps}$; $\{c_p\} \in `^2$;

(c) $u = \sum_{p=-1}^{\infty} u_p$ in $\mathcal{S}'$, Supp $\hat{u}_p \subset B(0;K_1 2^p)$ and $\|u_p\|_{L^2} \le c_p 2^{-ps}$; $\{c_p\} \in `^2$;

(d) $u = \sum_{p=-1}^{\infty} u_p$ in $\mathcal{S}'$, $u_p \in C^\infty$ and for any $® \in \mathbb{N}^d; |®| \le [s]+1$,

$$\|D^\alpha u_p\|_{L^2} \le c_{p,\alpha}2^{-p(s-|\alpha|)}; \quad \{c_{p,\alpha}\}_{p\in\mathbb{N}} \in `^2:$$

Remark : The equivalence of (a) and (b) holds for all $s \in \mathbb{R}$.

For the L^∞ estimate, we need the following lemma.

Lemma 2.2. *Suppose that $a \in L^\infty(\mathbb{R}^d)$; Supp $\hat{a} \subset B(0;R)$, then $a \in C^\infty(\mathbb{R}^d)$, and for any $® \in \mathbb{N}^d$ there exist $C(d;®) > 0$ such that*

$$\|D^\alpha a\|_{L^\infty} \le C(d;®)R^{|\alpha|}\|a\|_{L^\infty}: \tag{2.2}$$

For some N_0 large enough, $B(0; 4K2^{-N_0})$ is a very small ball. Set

$$\mathcal{C}_0' = \mathcal{C}_0 + B(0; 4K2^{-N_0});$$

then $\{\mathcal{C}_p'\} = \{2^p\mathcal{C}_0'\}$ has the same properties as $\{\mathcal{C}_p\}$. We define

$$S_q u = \sum_{-1\le p\le q-N_0} u_p; \;\; T_u v = \sum_q (S_q u)v_q; \;\; R(u; v) = \sum_{|p-q|<N_0} u_p v_q:$$

Then, we have

$$uv = T_u v + T_v u + R(u; v);$$

and the following lemma.

Lemma 2.3. *(a), For any* $a \in L^\infty$, *for any* $s \in \mathbb{R}$, *the maps* $T_a : H^s \to H^s$ *is continuous and*

$$\|T_a\|_{\mathcal{L}(H^s,H^s)} \le C_s\|a\|_{L^\infty}: \tag{2.3}$$

(b) If $u \in H^{s_1}; v \in H^{s_2}; s_1 + s_2 > 0$, *we have*

$$\|R(u; v)\|_{H^{s_1+s_2-d/2}} \le C\|u\|_{H^{s_1}}\|v\|_{H^{s_2}}: \tag{2.4}$$

(c) If $s \ge 0$, *then* $H^s(\mathbb{R}^d) \cap L^\infty$ *is an algebra, and for any* $u; v \in H^s(\mathbb{R}^d) \cap L^\infty$, *we have*

$$\|uv\|_{H^s} \le C(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}); \tag{2.5}$$

where C *depends only on* $d; s$.

A more general case of lemma 2.3 is the following.

Lemma 2.4. *Let* $F \in C^\infty(\mathbb{R}^1); F(0) = 0$. *If* $f \in H^s(\mathbb{R}^d)\cap L^\infty; s \ge 0$, *is a real function, then the composition* $F(f) \in H^s(\mathbb{R}^n)$ *and*

$$\|F(f)\|_{H^s} \le C(F; s; \|f\|_{L^\infty})\|f\|_{H^s};$$

with

$$C(F; s; \|f\|_{L^\infty}) = C_d \sum_{j=1}^{[s]+2} Sup_{0\le t\le\|f\|_{L^\infty}} |F^{(j)}(t)|\|f\|_{L^\infty}^{j-1}:$$

For later use, we also need the following estimate.

Lemma 2.5. *Let* $a; b \in H^s(\mathbb{R}^d)$ *with* $s > 1 + d{=}2$, *then for* $k \in \mathbb{N}$, *we have*

$$\|[\Delta_k; a]@_y b\|_{H^{s-1}(\mathbb{R}^d)} \le C_s\|a\|_{H^s(\mathbb{R}^d)}\|b\|_{H^{s-1}(\mathbb{R}^d)} \tag{2.6}$$

and

$$\|[\Delta_k; a]@_y b\|_{H^s(\mathbb{R}^d)} \le C_s'\|a\|_{H^s(\mathbb{R}^d)}\|b\|_{H^s(\mathbb{R}^d)}: \tag{2.7}$$

Proof : We prove only (2.6), following the notations of lemma 2.3, we have

$$\begin{aligned} &[\Delta_k; a]@_y b = \Delta_k(a@_y b) - a\Delta_k(@_y b) \\ &= \Delta_k(T_a@_y b + T_{\partial_y b}a + R(a; @_y b)) - (T_a\Delta_k(@_y b) + T_{\Delta_k\partial_y b}a + R(a; \Delta_k(@_y b))): \end{aligned}$$

Since $@_y b \in H^{s-1} \subset C^{s-1-d/2} \subset L^\infty$, (a) and (b) of lemma 2.3 give

$$\|T_{\partial_y b}a + R(a; @_y b) + T_{\Delta_k\partial_y b}a + R(a; \Delta_k(@_y b))\|_{H^{s-1}(\mathbb{R}^d)} \le C_s\|a\|_{H^s(\mathbb{R}^d))}\|b\|_{H^{s-1}(\mathbb{R}^d))}:$$

On the other hand, there exists N_0 such that

$$\Delta_k(T_a\partial_y b) - T_a\Delta_k(\partial_y b) = \sum_{|k'-k|\le N_0} (\Delta_k(S_{k'}(a)\Delta_{k'}\partial_y b) - S_{k'}(a)\Delta_{k'}(\Delta_k(\partial_y b))),$$

and

$$\begin{aligned}&\Delta_k(S_{k'}(a)\Delta_{k'}\partial_y b) - S_{k'}(a)\Delta_{k'}(\Delta_k(\partial_y b))(x)\\ &= 2^{dk}\int f(2^k(y-x))(S_{k'}(a)(y) - S_{k'}(a)(x))\Delta_{k'}\partial_y b(y)dy \quad .\end{aligned}$$

Hence

$$\begin{aligned}&\|\Delta_k(S_{k'}(a)\Delta_{k'}\partial_y b) - S_{k'}(a)\Delta_{k'}(\Delta_k(\partial_y b))\|_{L^2}\\ &\le 2^{-k}\|tf(t)\|_{L^1}\|\nabla a\|_{L^\infty}\|\Delta_{k'}\partial_y b\|_{L^2} \le C_{N_0}\|a\|_{H^s}\|b\|_{H^{s-1}}.\end{aligned}$$

This completes the proof of the lemma.

3. Proof of the Theorem

In this section, we are going to prove the local existence of solution stated in Theorem 1.1. The proof is based on the study of a linearized problem. We want to construct a convergent sequence of solutions to the linearized problem and show that the limit is the solution to the nonlinear problem (1.8)- (1.11) with the property stated in Theorem 1.1.

Under the hypothesis of theorem 1.1, we study now the sequence of functions $\{v^n\}_{n\in\mathbb{N}}$ defined inductively as follows.

$$v^1 = v_0, \tag{3.1}$$

$$v^{n+1}_{tt} - ((\imath^n)^2 v^{n+1}_y)_y = f^n, \tag{3.2}$$

$$(v^{n+1}, v^{n+1}_t)|_{t=0} = (v_0, v_1), \tag{3.3}$$

with

$$\begin{aligned}\imath^n(y,t) &= (a_0 + v^n(y,t))^\alpha,\\ f^n(y,t) &= -\alpha(a_0+v^n)^{3\alpha-1}(v^n)^2_y + \frac{\alpha+2}{y}(\imath^n)^2 v^n_y + \frac{\alpha v^n_t}{a_0+v^n}v^n_t - v^n_t.\end{aligned}$$

For $0 < T_1 < a_0^{-\alpha}b_0$ and

$$M_0 = B_1\left(\|v_0\|^2_{H^s} + \|v_1\|^2_{H^{s-1}}\right)^{1/2},$$

with $B_1 = 2(8/a_0)^\alpha$ if $a_0 \le 2$, and $B_1 = 2(2a_0)^\alpha$ if $a_0 > 2$, we define

$$\begin{aligned}X_{s,T_1} = \Big\{ & v \mid v \in C^0([0,T_1]; H^s(\mathbb{R}_+)) \cap C^{0,1}([0,T_1]; H^{s-1}(\mathbb{R}_+)),\\ & |||v|||_{X_{s,T_1}} = \left(\|v\|^2_{L^\infty([0,T_1];H^s(\mathbb{R}_+))} + \|v_t\|^2_{L^\infty([0,T_1];H^{s-1}(\mathbb{R}_+))}\right)^{1/2} \le M_0,\\ & \|v\|_{L^\infty(\mathbb{R}_+\times[0,T_1])} \le \frac{1}{2}a_0,\ \text{Supp } v \subset \{(y,t) \in \mathbb{R}_+ \times [0,T_1],\ y + a_0^\alpha t \ge b_0\}\Big\}.\end{aligned}$$

We will prove the following theorem.

Theorem 3.1. *(a) For any* $s > 3/2, 0 < \varepsilon < b_0, 0 < T_1 \leq a_0^{-\alpha}(b_0 - \varepsilon)$, *if* $v^n \in X_{s,T_1}$, *then the Cauchy problem (3.2)-(3.3) has a solution*

$$v^{n+1} \in C^0([0,T_1]; H^s(\mathbb{R}_+)) \cap C^{0,1}([0,T_1]; H^{s-1}(\mathbb{R}_+)),$$

with

$$\textit{Supp } v^{n+1} \subset \{(y,t) \in [\varepsilon,+\infty[\times[0,T_1];\ y + a_0^\alpha t \geq b_0\}.$$

(b) For any $s > 3/2$, *there exists* $0 < T_1 < a_0^{-\alpha} b_0$, *such that if* $v^n \in X_{s,T_1}$, *then the solution* v^{n+1} *of Cauchy problem (3.2)-(3.3) belongs to* X_{s,T_1}, *that means that the sequence* $\{v^n\}$ *is well-defined and uniformly bounded in* X_{s,T_1}.

(c) There exists $0 < T_2 \leq T_1$ *such that the sequence* $\{v^n\}$ *is a Cauchy sequence in* X_{s-1,T_2}.

Proof. First for part (a), since $0 < T_1 \leq a_0^{-\alpha}(b_0 - \varepsilon)$, if $v^n \in X_{s,T_1}$, then

$$\text{Supp } v^n \subset \{(y,t) \in [\varepsilon,+\infty[\times[0,T_1];\ y + a_0^\alpha t \geq b_0\}.$$

We have that

$$\mu^n \in C^0([0,T_1]; H^s(\mathbb{R})) \cap C^{0,1}([0,T_1]; H^{s-1}(\mathbb{R})),$$

and

$$f^n \in C^0([0,T_1]; H^{s-1}(\mathbb{R})) \cap C^{0,1}([0,T_1]; H^{s-2}(\mathbb{R})).$$

Thus, the existence theorem for linear Cauchy problem gives the existence of solution to (3.2) and (3.3), cf. [9]

$$v^{n+1} \in C^0([0,T_1]; H^s(\mathbb{R})) \cap C^{0,1}([0,T_1]; H^{s-1}(\mathbb{R})).$$

Moreover for $v^n \in X_{s,T_1}$, we have $\mu^n(y,t) = a_0^\alpha, f^n(y,t) = 0, v_0 = v_1 = 0$ for all $(y,t) \in \mathbb{R}_+ \times [0,T_1];\ y + a_0^\alpha t \leq b_0$, so that in this domain, v^{n+1} is the solution of problem

$$v_{tt}^{n+1} - a_0^{2\alpha} v_{yy}^{n+1} = 0;\ (v,v_t)|_{t=0} = (0,0).$$

Then $v^{n+1} = 0$ in this domain and this gives part (a).

We now turn to part (b). For $0 < \varepsilon < b_0$, take $\chi \in C^\infty(\mathbb{R}), \chi(y) = \frac{1}{y}$ if $y \geq \varepsilon, \chi(y) = \frac{2}{\varepsilon}$ if $y \leq \varepsilon/2$. We suppose always $0 < T_1 \leq a_0^\alpha(b_0 - \varepsilon)$. For $\bar{v} \in X_{s,T_1}$, we set

$$\begin{aligned} \mu(y,t) &= (a_0 + \bar{v}(y,t))^\alpha, \\ \bar{f}(y,t) &= -\alpha(a_0 + \bar{v})^{3\alpha-1}(\bar{v})_y^2 + \frac{\alpha+2}{y}(\mu)^2 \bar{v}_y + \frac{\alpha \bar{v}_t^2}{a_0 + \bar{v}} - \bar{v}_t. \end{aligned}$$

Remark that $\chi(y)\bar{v}_y = \frac{1}{y}\bar{v}_y$, since $\bar{v}_y(y,t) = 0$ if $y \leq \varepsilon$. Then by using Theorem 2.4, we have

$$\begin{aligned} \mu &\in C^0([0,T_1]; H^s(\mathbb{R})) \cap C^{0,1}([0,T_1]; H^{s-1}(\mathbb{R})), \\ \bar{f} &\in C^0([0,T_1]; H^{s-1}(\mathbb{R})) \cap C^{0,1}([0,T_1]; H^{s-2}(\mathbb{R})). \end{aligned}$$

And

$$\|\bar{f}\|_{L^\infty([0,T_1];H^{s-1}(\mathbb{R}))} \leq \frac{B_2}{\varepsilon} M_0^{[s]+1};\ \|\mu\|_{L^\infty([0,T_1];H^s(\mathbb{R}))} \leq B_2 M_0^{[s]}, \tag{3.4}$$

with constant B_2 depends only on α, a_0, b_0 and s. We consider now the following linear problems

$$v_{tt} - ((\gamma)^2 v_y)_y = \bar{f}, \tag{3.5}$$

$$(v, v_t)|_{t=0} = (v_0, v_1), \tag{3.6}$$

In fact, part (b) is equivalent to the following claim.

Claim: Suppose that

$$\operatorname{Supp} v_0, \operatorname{Supp} v_1 \subset [b_0, +\infty[, \quad \|v_0\|_{L^\infty(\mathbb{R}_+)} \le \frac{1}{4} a_0,$$

and $s > \frac{3}{2}$. There exists $T_1 > 0$ depending on a_0, b_0, s, M_0, such that, for any $\bar{v} \in X_{s,T_1}$, the solution v of problem (3.5)-(3.6) is also in X_{s,T_1}.

For the above claim, we only need to prove the following estimate for the solution v of problem (3.5)-(3.6).

$$\|v\|^2_{L^\infty([0,T_1];H^s(\mathbb{R}))} + \|v_t\|^2_{L^\infty([0,T_1];H^{s-1}(\mathbb{R}))} \le M_0^2. \tag{3.7}$$

By using Sobolev embedding theorem, Lipschitz estimate and L^∞ boundedness of v_0, we get immediately, for $T_1 \le a_0/(4M_0C_s)$,

$$\|v\|_{L^\infty(\mathbb{R}\times[0,T_1])} \le \frac{1}{2} a_0.$$

To apply the Lemma 2.1, we need the following estimate,

$$\|\Delta_k v_t\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))} + \|\Delta_k v_y\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))} \le c_k^2 2^{-2k(s-1)}, \tag{3.8}$$

with $\sum c_k^2 \le M_0^2$ for $k \in \mathbb{N}$.

Since $v_{tt} \in L^\infty([0, T_1]; H^{s-2}(\mathbb{R}))$, by applying Δ_k to the equation (3.5) and integrating its product with $\Delta_k v_t$ over (y, t) in $\mathbb{R} \times [0, t]$, we have,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} |\Delta_k v_t|^2(y, t)dy + \frac{1}{2}\int_{\mathbb{R}} \gamma^2 |\Delta_k v_y|^2(y, t)dy = \frac{1}{2}\int_{\mathbb{R}} |\Delta_k v_1|^2(y)dy \\
&+\frac{1}{2}\int_{\mathbb{R}} \gamma^2 |\Delta_k (v_0)_y|^2(y)dy + \int_0^t\int_{\mathbb{R}} \Delta_k(\bar{f})\Delta_k(v_t)dydt \\
&+\int_0^t\int_{\mathbb{R}} \gamma\gamma_t |\Delta_k(v_y)|^2 dydt - \int_0^t\int_{\mathbb{R}} [\Delta_k, \gamma^2] v_y \Delta_k(v_{ty})dydt.
\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\|\Delta_k v_t\|^2_{L^2(\mathbb{R})}(t) + \|\Delta_k v_y\|^2_{L^2(\mathbb{R})}(t) \le \frac{1}{4} B_1^2 (\|\Delta_k v_1\|^2_{L^2(\mathbb{R})} + \|\Delta_k (v_0)_y\|^2_{L^2(\mathbb{R})}) \\
&+T_1^2 B_1^2 \|\Delta_k(\bar{f})\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))} + T_1^2 B_1^2 2^k \|\widetilde{\Delta}_k([\Delta_k, \gamma^2]v_y)\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))} \\
&+\frac{1}{2}\|\Delta_k(v)_t\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))} + \frac{1}{4} T_1 B_1^2 \|\gamma\gamma_t\|_{L^\infty(\mathbb{R}\times[0,T_1])} \|\Delta_k v_y\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))},
\end{aligned}$$

where $\widetilde{\Delta}_k = \sum_{|k'-k|\le N_1} \Delta_{k'}$, and $\widetilde{\Delta}_k \circ \Delta_k = \Delta_k$. We have

$$\|\gamma\gamma_t\|_{L^\infty(\mathbb{R}\times[0,T_1])} \le \alpha(2a_0)^{2\alpha-1}\|v_t\|_{L^\infty(\mathbb{R}\times[0,T_1])} \le \alpha(2a_0)^{2\alpha-1} C_s \|v_t\|_{L^\infty([0,T_1];H^{s-1}(\mathbb{R}))},$$

where $s-1>1{=}2$. By choosing $0<T_1$ small enough satisfying

$$T_1B_1^2\circledR(2a_0)^{2\alpha-1}C_s\|v_t\|_{L^\infty([0,T_1];H^{s-1}(\mathbb{R}))}\le T_1B_1^2\circledR(2a_0)^{2\alpha-1}C_sM_0\le 2;$$

we have

$$\|\Delta_kv_t\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))}+\|\Delta_kv_y\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))}\le\frac{1}{2}B_1^2(\|\Delta_kv_1\|^2_{L^2(\mathbb{R})}+\|\Delta_k(v_0)_y\|^2_{L^2(\mathbb{R})})$$
$$+2T_1^2B_1^2\|\Delta_k(\bar f)\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))}+2T_1^2B_1^22^{2k}\|\widetilde\Delta_k([\Delta_k;\imath^2]v_y)\|^2_{L^\infty([0,T_1];L^2(\mathbb{R}))}:$$

Hence, (2.7) and (3.4) yield

$$\begin{aligned}\|\Delta_k(\bar f)\|_{L^\infty([0,T_1];L^2(\mathbb{R}))}&\le c_k2^{-k(s-1)}\|\bar f\|_{L^\infty([0,T_1];H^{s-1}(\mathbb{R}))}\\&\le\frac{B_2}{\text{"}}M_0^{[s]+1}c_k2^{-k(s-1)};\\\|\widetilde\Delta_k([\Delta_k;\imath^2]v_y)\|_{L^\infty([0,T_1];L^2(\mathbb{R}))}&\le c_k2^{-ks}\|\imath^2\|_{L^\infty([0,T_1];H^s(\mathbb{R}))}\|v\|_{L^\infty([0,T_1];H^s(\mathbb{R}))}\\&\le B_2M_0^{[s]}c_k2^{-ks}\|v\|_{L^\infty([0,T_1];H^s(\mathbb{R}))};\end{aligned}$$

with $\sum c_k^2\le1$. By choosing $0<T_1B_1B_2M_0^{[s]}\le\sqrt{2}{=}4$ in the above estimate , we complete the proof of the claim and then obtain the part (b).

Finally, we want to prove part (c) of Theorem 3.1. Let $\{v^n\}$ be a sequence of functions defined by (3.1)-(3.3), we prove that there exists $0<T_2\le T_1$ such that it is a Cauchy sequence in $C^0([0;T_2];H^{s-1}(\mathbb{R}))\cap C^{0,1}([0;T_2];H^{s-2}(\mathbb{R}))$. In fact we will prove the following estimate, for any $n\in\mathbb{N}$,

$$\|v^{n+1}-v^n\|^2_{L^\infty([0,T_2];H^{s-1}(\mathbb{R}))}+\|v_t^{n+1}-v_t^n\|^2_{L^\infty([0,T_2];H^{s-2}(\mathbb{R}))}\le2^{-n}M_0^2: \tag{3.9}$$

Set $u^{n+1}=v^{n+1}-v^n;n\in\mathbb{N}$, we have

$$\begin{aligned}&u_{tt}^{n+1}-((\imath^n)^2u_y^{n+1})_y=(f^n-f^{n-1})-(((\imath^n)^2-(\imath^{n-1})^2)u_y^n)_y\\&(u^{n+1};u_t^{n+1})|_{t=0}=(0;0):\end{aligned}$$

where $v^{n+1};v^n;v^{n-1}\in X_{s,T_1}$, and

$$\begin{aligned}&(\imath^n)^2-(\imath^{n-1})^2=b_1(v^n;v^{n-1})u^n;\\&f^n-f^{n-1}=b_2(v^n;v^{n-1};@v^n;@v^{n-1})u^n\\&+b_3(v^n;v^{n-1};@v^n;@v^{n-1})u_t^n+b_4(v^n;v^{n-1};@v^n;@v^{n-1})u_y^n;\end{aligned}$$

with

$$\|b_j\|_{L^\infty([0,T_1];H^{s-1}(\mathbb{R}))}\le A(\text{"};M_0);\ \ j=1;\cdots;4:$$

For $t\in[0;T];0<T\le T_1$,

$$\left|\int_0^t\int_{\mathbb{R}}\Delta_k(f^n-f^{n-1})\Delta_ku_t^{n+1}dydt\right|\le TA(\text{"};M_0)c_k2^{-k(s-2)}\|\Delta_ku_t^{n+1}\|_{L^\infty([0,T];L^2(\mathbb{R}))}$$
$$\times\left(\|u^n\|_{L^\infty([0,T];H^{s-1}(\mathbb{R}))}+\|u_t^n\|_{L^\infty([0,T];H^{s-2}(\mathbb{R}))}\right):$$

and

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}} \Delta_k(((1^n)^2-(1^{n-1})^2)v_y^n)_y\Delta_k u_t^{n+1}dydt\right|\\
&\leq \left|\int_0^t\int_{\mathbb{R}} \Delta_k((1^n)^2-(1^{n-1})^2)v_y^n\Delta_k u_{ty}^{n+1}dydt\right|\\
&\leq TA('';M_0)c_k2^{-k(s-2)}\|\Delta_k u_t^{n+1}\|_{L^\infty([0,T];L^2(\mathbb{R}))}\|u^n\|_{L^\infty([0,T];H^{s-1}(\mathbb{R}))}:
\end{aligned}$$

By using (2.6), we have

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}} \left([\Delta_k;(1^n)^2]u_y^{n+1}\right)\Delta_k u_{ty}^{n+1}dydt\right|\\
&\leq TB(M_0)c_k2^{-k(s-2)}\|\Delta_k u_t^{n+1}\|_{L^\infty([0,T];L^2(\mathbb{R}))}\|u^{n+1}\|_{L^\infty([0,T];H^{s-1}(\mathbb{R}))}:
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\|\Delta_k u_t^{n+1}\|^2_{L^\infty([0,T];L^2)}+\|\Delta_k u_y^{n+1}\|^2_{L^\infty([0,T];L^2)}\leq\\
&4T^2A('';M_0)^2c_k^22^{-2k(s-2)}(\|u_t^n\|^2_{L^\infty([0,T];H^{s-2}(\mathbb{R}))}+\|u_y^n\|^2_{L^\infty([0,T];H^{s-2}(\mathbb{R}))})\\
&+4T^2B(M_0)^2c_k^22^{-2k(s-2)}\|u^{n+1}\|^2_{L^\infty([0,T];H^{s-1}(\mathbb{R}))}:
\end{aligned}$$

By multipling this inequality by $2^{2k(s-2)}$ and summing over k, we have for $4T^2B(M_0)^2\leq 1=2$,

$$\begin{aligned}
&\|u_t^{n+1}\|^2_{L^\infty([0,T];H^{s-2}(\mathbb{R}))}+\|u_y^{n+1}\|^2_{L^\infty([0,T];H^{s-2}(\mathbb{R}))}\\
&\leq 8T^2A('';M_0)^2(\|u_t^n\|^2_{L^\infty([0,T];H^{s-2}(\mathbb{R}))}+\|u_y^n\|^2_{L^\infty([0,T];H^{s-2}(\mathbb{R}))}):
\end{aligned}$$

Now by choosing $8T^2A('';M_0)^2\leq 1=2$, we have (3.9). This completes the proof of Theorem 3.1.

Now the proof for Theorem 1.1 can be stated as a consequence of Theorem 3.1 as follows. Since the sequence $\{v^n\}$ is a Cauchy sequence in X_{s-1,T_2} and bounded in X_{s,T_2}, it is also the Cauchy sequence in X_{s',T_2} for all $s'<s$ by interpolation. Then the limit v is in $C^0([0;T_2];H^s(\mathbb{R}_+))\cap C^{0,1}([0;T_2];H^{s-1}(\mathbb{R}_+))$. Since $s>3=2$, v is a solution of equation (1.8) and this yields Theorem 1.1.

Acknowledgment: The research was supported CityU Direct Allocation Grant # 7100198. And the authors would like to thank the referee's insightful idea on improving the theorem in this paper.

References

[1] D. G. Aronson, L. A. Caffarelli and S. Kamin, How an initial stationary interface begins to move in porous media flow, SIAM J. Math., Anal., 14(1983), 639-658.

[2] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales de l'École Normale Supérieure*, 14, 1981, pages 209–246.

[3] J.-Y. Chemin, Fluides parfaits incompressibles, *Astérisque*, 230 (1995).

[4] J.-Y. Chemin, B. Desjardins, I. Gallagher, E. Grenier, Fluids with anisotropic viscosity. *Special issue for R. Temam's 60th birthday. M2AN Math. Model. Numer. Anal.* 34, (2000), no. 2, 315-335

[5] Yinbin Deng, Tai-Ping Liu, Tong Yang and Zheng-an Yao, Solutions with vacuum of Euler-Poisson equations, Arch. Rat. Mech. Anal., 164(2002), 3, 261-285.

[6] M. Grassin and D. Serre, Global smooth solutions to Euler equations for an isentropic perfect gas, C.R. Acad. Sci. Paris Sér. I Math, 325(1997), No. 7, 721-726.

[7] L. Hsiao and T.-P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Comm. Math. Phys.143 (1992), 599-605.

[8] L. Hormander, Lectures on Nonlinear Hyperbolic Differential Equations, 1997, Springer-Verlag.

[9] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Rational. Mech. Anal., 58 (1975), 181-205.

[10] L.W. Lin, On the vacuum state for the Equations of Isentropic Gas Dynamics, J. of Math. Anal. Appl., Vol. 121, No. 2 (1987) pp. 406–425.

[11] T.-P. Liu, Compressible flow with damping and vacuum, Japan J. Appl. Math., Vol. 13, No. 1, 25-32.

[12] T.-P. Liu and T. Yang, Compressible Euler equations with vacuum, Journal of Differential Equations, Vol. 140, No. 2, 1997, 223-237.

[13] T.-P. Liu and T. Yang, Compressible flow with vacuum and physical singularity, Methods and Applications of Analysis, Vol. 7, No. 3, 495-510, 2000.

[14] T.-P. Liu and J. Smoller, On the vacuum state for isentropic gas dynamics equations, Advances in Math., 1(1980), 345-359.

[15] T. Makino, Blowing up solutions of the Euler-Poisson equation for the evolution of gaseous stars, Transport Theory and Statistical Physics 21 (1992), 615–624.

[16] T. Nishida, Nonlinear hyperbolic equations and related topics in fluid dynamics, Publ. Math. D'Orsay, (1978), 46-53.

[17] M. Okada, Free boundary value problems for the equation of one-dimensional motion of viscous gas, Japan J. Appl. Math., 6, No.1, 161-177, 1989.

[18] O.A. Oleinik, On the Cauchy problem for weakly hyperbolic equations, Comm. Pure Appl. Math., Vol., XXIII, (1970), 569-586.

[19] Z. Xin, Blow-up of smooth solutions to the compressible Navier-Stokes equations with compact density, Comm. Pure Appl. math., 51 (1998), 229-240.

CHAO-JIANG XU, UNIVERSITÉ DE ROUEN, LABORATOIRE DE MATHÉMATIQUES, 76821 MONT-SAINT-AIGNAN, FRANCE
E-mail address: `Chao-Jiang.Xu@univ-rouen.fr`

TONG YANG, DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, KOWLOON, HONG KONG
E-mail address: `matyang@cityu.edu.hk`