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The dirichlet problems for a class of semilinear sub-elliptic equations[1]

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1. Introduction

In this work, we study the following semilinear Dirichlet problem:

$$\begin{cases} \sum_{j=1}^{m} X_j^* X_j u + cu = f(x,u), & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $X = \{X_1, \ldots, X_m\}$ is a system of real smooth vector fields defined in an open domain $M \subset \mathbb{R}^n$, $n \geq 2$, Ω is a bounded open subdomain of M with $\partial\Omega$ smooth, $c(x) \geq c_0 > 0$. X_j^* denote the adjoint of X_j. We assume that the system of vector fields $X = \{X_1, \ldots, X_m\}$ satisfies the following Hörmander's condition:

$X_1, \ldots, X_m$ together with their commutators $X_\alpha = [X_{\alpha_1}, \ldots [X_{\alpha_{s-1}}, X_{\alpha_s}] \ldots]$ up to some fixed length r span the tangent space at each point of M.

We have the following theorem:

Theorem 1. *Assume that the system of vector fields* $X = \{X_1, \ldots, X_m\}$ *satisfies the Hörmander's condition,* $\partial\Omega$ *is smooth and non-characteristic for the system* $X_1, \ldots, X_m$; $f \in C^\infty(\overline{\Omega} \times \mathbb{R}), \partial_u f(x,u) \leq 0, \varphi \in C^\infty(\partial\Omega)$. *Then there exists a solution* $u \in C^\infty(\overline{\Omega})$ *of Dirichlet problem* (1).

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Since Eq. (1) is subelliptic, we call Eq. (1) semilinear subelliptic. Hörmander's condition permits us to get some properties of Hörmander's operators $H=\sum_{j=1}^m X_j^* X_j + c$ similar to those of the Laplacian (see [1, 3, 4, 9, 11]). Using these properties, we have proved the interior regularities for quasilinear second-order subelliptic equation of the form $\sum_{ij=1}^m A_{ij}(x,u,Xu)X_iX_ju + B(x,u,XU)=0$, and the existence of weak solution for variational problems (see [13, 15]). The results of this work is about the existence and C^∞ regularities up to the boundary for the semilinear degenerate elliptic Dirichlet problems.

2. Function spaces and preliminary lemmas

We define now the metric on M associated with X as in [9, 15].

Definition 1. Let $C(\delta)$ be a class of absolutely continuous mappings $\phi:[0,1]\to M$ which almost everywhere satisfy the differential equation

$$\phi'(t)=\sum_{|J|\le r} a_J(t)X_J(\phi(t)) \tag{2}$$

with $|a_J(t)|<\delta^{|J|}$; then we define

$$\rho(x,y)=\inf\{\delta>0 \mid \exists\phi\in C(\delta) \text{ with } \phi(0)=x,\ \phi(1)=y\}. \tag{3}$$

Then, ρ is a local metric on M, and for any small compact subset $K\subset M$, there exists a constant $C>0$ such that

$$C^{-1}|x-y|\le\rho(x,y)\le C|x-y|^{1/r}$$

for any $x,y\in K$. We can define a family of balls by this metric.

$$B(x,\varepsilon)=\{y\in M;\rho(x,y)<\varepsilon\}$$

for $x\in M$, and $\varepsilon>0$ small enough. Denote by $B_E(x,\varepsilon)$ the Euclidean ball. Then, for all compact $K\subset M$, there exists constants $C_1>0$, $C_2>0$, and $\varepsilon_0>0$ such that

$$B_E(x,C_1\varepsilon^r)\subset B(x,\varepsilon)\subset B_E(x,C_2\varepsilon)$$

for all $x\in K$ and $0<\varepsilon\le\varepsilon_0$.

We introduce now a class of "non-isotropic" Hölder continuous functions. For $1>\alpha>0$, we define $(S^0(\Omega)=C^0(\Omega)\cap L^\infty(\Omega))$

$$S^\alpha(\Omega)=\left\{f\in S^0(\Omega);[f]^X_{\alpha,\Omega}=\sup_{x,y\in\Omega}\frac{|f(x)-f(y)|}{\rho(x,y)^\alpha}<+\infty\right\} \tag{4}$$

and for $k\in\mathbb{N}, 1>\alpha\ge 0$, we define

$$S^{k,\alpha}(\Omega)=\{u\in S^\alpha(\Omega);X^Ju\in S^\alpha(\Omega),\forall|J|\le k\}. \tag{5}$$

Set

$$[u]^X_{k,0,\Omega} = \sup_{|J|=k} \sup_{x\in\Omega} |X^J u(x)|$$

and

$$[u]^X_{k,\alpha,\Omega} = \sup_{|J|=k} [X^J u(x)]^X_{k,\alpha,\Omega}.$$

The norms on $S^{k,\alpha}(\Omega)$ are given by

$$\|u\|_{S^{k,\alpha}(\Omega)} = \sum_{j=0}^{k} [u]^X_{j,0,\Omega} + [u]^X_{k,\alpha,\Omega}. \tag{6}$$

Then the space $S^{k,\alpha}(\Omega)$ is a Banach space (see [15]).

As for the classical Hölder space, we also have the interpolation inequalities in the space $S^{k,\alpha}(\Omega)$. For $j+\beta<k+\alpha$, $j,k\in\mathbb{N}$, $0\le\alpha,\beta\le 1, u\in S^{k,\alpha}(\Omega)$, and any $\varepsilon>0$, we have

$$\|u\|_{S^{j,\beta}(\Omega)} \le \varepsilon\|u\|_{S^{k,\alpha}(\Omega)} + C(\varepsilon, j, k, \Omega, r)\|u\|_{L^\infty(\Omega)}. \tag{7}$$

This implies the following compactness results.

Lemma 1. *Let K be a bounded subset of $S^{k,\alpha}(\Omega)$, $k+\alpha>0$. If $j+\beta<k+\alpha$, then K is precompact in $S^{j,\beta}(\Omega)$.*

3. Schauder estimates for the Hörmander operators

We study in this section the following linear Dirichlet problem:

$$Hu = f \quad \text{in } \Omega; \qquad u=\varphi \quad \text{on } \partial\Omega. \tag{8}$$

with $c(x)\ge c_0>0$. From the subellipticity of Hörmander's operators H, we have (see [1, 3, 6])

Lemma 2. *Assume that the system of vector fields $X_1,\ldots,X_m$ satisfies the Hörmander's condition, and $\partial\Omega$ is non-characteristic for operators H. Then the linear Dirichlet problem* (8) *possess a unique solution $u\in C^\infty(\overline{\Omega})$.*

By [1], there exists Green's kernel $G(x,y)$ for H. From [11, 15] we have

Lemma 3. *For $n\ge 2, K\subset\subset\Omega$, and $(x,y)\in K\times K$, we have*

$$|X^J G(x,y)| \le C_J\rho(x,y)^{2-|J|}|B(x,\rho(x,y))|^{-1}, \tag{9}$$

where the differential is taken in x or y. And for any $\alpha>0$, there exists a constant C such that

$$\int_{B(x,\delta)} \rho(x,y)^\alpha |B(x,\rho(x,y))|^{-1} \le C\delta^\alpha.$$

We shall use the inequality (9) to prove the Schauder estimate of Hörmander operators in the "non-isotropic" Hölder spaces $S^{k,\alpha}$. Firstly, we have the weak maximum principle

Lemma 4. *If* $u\in S^2(\Omega)\cap C^0(\overline{\Omega})$ *is a solution of Dirichlet problem* (9), $c(x)\geq c_0>0$. *Then we have*

$$\|u\|_{L^\infty(\Omega)}\leq c_0^{-1}\|f\|_{L^\infty(\Omega)}. \tag{10}$$

This is the theorem of Bony [1]. We also have the strong maximum principle of Bony.

Lemma 5. *Assume that* $X_1,\dots,X_m$ *satisfies the Hörmander's condition,* $u\in S^2(\Omega)\cap C^0(\overline{\Omega})$ *verifies* $Hu\leq 0$ *in* Ω, *and* $u\leq 0$ *on* $\partial\Omega$. *Then* $u\leq 0$ *in* Ω.

We now prove the estimate of $\|Xu\|_{L^\infty}$, and Schauder-type estimate for operators H in the interior of Ω.

Lemma 6. *Let* $u\in S^2(\Omega)\cap C^0(\overline{\Omega}), u|_{\partial\Omega}=0$, *then for all* $K\subset\subset\Omega$, *there exists a constant* C *such that, for* $1\leq k\leq m$,

$$\max_K |X_k u|\leq C\sup_\Omega |Hu|. \tag{11}$$

And if $u\in S^{2,\alpha}(\Omega)\cap C^0(\overline{\Omega}), u|_{\partial\Omega}=0$, $\alpha>0$, *then*

$$\|u\|_{S^{2,\alpha}(K)}\leq C\|Hu\|_{S^\alpha(\widetilde{K})}, \tag{12}$$

where $K\subset\subset\widetilde{K}\subset\subset\Omega$.

Proof. For $\varepsilon>0$ small enough, we denote by $K_\varepsilon=\{y\in\Omega;\rho(x,y)<\varepsilon,x\in K\}$. Take $\varphi\in C_0^\infty(\Omega),\varphi(x)=1$, for $x\in K_\varepsilon$. Using the Green's kernel of the Dirichlet problem, we have $H(\varphi u)\in C_0^0(\Omega)$, and for $x\in K$

$$u(x)=\int_\Omega G(x,y)H(\varphi u)(y)\,\mathrm{d}y.$$

Since

$$\begin{aligned} H(\varphi u)&=\sum_{j=1}^m X_j^*X_j(\varphi u)+c(\varphi u)\\ &=\varphi H(u)+\sum_{j=1}^m (X_j\varphi X_j^*u+X_j^*\varphi X_ju)+u\sum_{j=1}^m X_j^*X_j(\varphi), \end{aligned}$$

we have, for $x \in K$,

$$u(x) = \int_\Omega \varphi(y)G(x,y)H(u)(y)\,\mathrm{d}y + \int_\Omega u(y)G(x,y)\sum_{j=1}^m X_j^* X_j(\varphi)(y)\,\mathrm{d}y$$
$$+ \int_\Omega G(x,y)\sum_{j=1}^m (X_j\varphi X_j^* u + X_j^*\varphi X_j u)(y)\,\mathrm{d}y.$$

Since $\operatorname{supp}\varphi \subset\subset \Omega$, integrating by parts, we have

$$\begin{aligned} X_k u(x) = & \int_\Omega X_k^x(\varphi(y)G(x,y))H(u)(y)\,\mathrm{d}y \\ & + \int_\Omega u(y)X_k^x G(x,y)\sum_{j=1}^m X_j^* X_j(\varphi)(y)\,\mathrm{d}y \\ & + \int_\Omega \sum_{j=1}^m [X_k^x X_j^y(G(x,y)X_j\varphi(y)) \\ & + X_k^x X_j^{*y}(G(x,y)X_j^*\varphi(y))]u(y)\,\mathrm{d}y \\ = & \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Using Lemma 3, we have for $x \in K$,

$$\begin{aligned} d|\mathrm{I}| &\le C\sup_\Omega |H(u)| \int_\Omega \rho(x,y)|B(x,\rho(x,y))|^{-1}\,\mathrm{d}y \\ &\le \widetilde{C}\sup_\Omega |H(u)|, \\ |\mathrm{II}| &\le C\max_\Omega |u| \int_\Omega \rho(x,y)|B(x,\rho(x,y))|^{-1}\,\mathrm{d}y \\ &\le \widetilde{C}\sup_\Omega |u|, \\ |\mathrm{III}| &\le C\max_\Omega |u| \int_{\Omega\setminus K_\varepsilon} |B(x,\rho(x,y))|^{-1}\,\mathrm{d}y \\ &\le C\max_\Omega |u| \int_{\Omega\setminus K_\varepsilon} |B(x,\varepsilon)|^{-1}\,\mathrm{d}y \le \widetilde{C}\sup_\Omega |u|. \end{aligned}$$

which give the estimates

$$\max_K |X_k u| \le C\left\{\sup_\Omega |H(u)| + \max_\Omega |u|\right\}.$$

By Lemma 4, we obtain the first part of lemma. The second part is just the results in [15].

4. Existence of solutions

We now prove the Theorem 1 in two steps:
(a) Assume that f is bounded

$$|f(x,u)| \leq N; \tag{13}$$

we prove that for the semilinear Dirichlet problem (1) there exists a solution.

We can assume that $\varphi = 0$, since if w is the solution of the following linear Dirichlet problems,

$$Hw = 0 \quad \text{in } \Omega; \qquad w = \varphi \quad \text{on } \partial\Omega,$$

then Lemma 2 gives that $w \in C^\infty(\overline{\Omega})$. Set $u = v + w$, then the Dirichlet problem (1) is equivalent to the following homogeneous semilinear Dirichlet problem:

$$Hv = f(x, v + w) \quad \text{in } \Omega; \quad v = 0 \quad \text{on } \partial\Omega. \tag{14}$$

Let v be a solution of following linear problem:

$$Hv = N \quad \text{in } \Omega; \qquad v = 0 \quad \text{on } \partial\Omega,$$

then, by Lemmas 2 and 5, we have $v \geq 0$ in Ω and $v \in C^\infty(\overline{\Omega})$. Take

$$k = \inf_{-\max v \leq u \leq \max v} \left(\frac{\partial f(x,u)}{\partial u} \right),$$

then $k \leq 0$, and for all $-\max v \leq u \leq w \leq \max v$ we have

$$f(x,u) - f(x,w) - k(u - w) \leq 0.$$

The solution of problem (14) will be the limit of u_l, where $u_0 = v$, and u_l is the solutions of following problems:

$$L[u_l] \equiv H(u_l) - ku_l = f(x, u_{l-1}) - ku_{l-1}; \quad u_l|_{\partial\Omega} = 0. \tag{15}$$

From Lemma 2, there exists a solution of Eq. (15) and $u_l \in C^\infty(\overline{\Omega})$ for all $l \in \mathbb{N}$. Note that

$$L[u_1] = f(x,v) - kv \leq N - kv = L[v].$$

Using Lemma 5, we have $u_1 \leq v$. And

$$Hu_1 = k(u_1 - v) + f(x,v) \geq f(x,u_1) \geq -N = H(-v),$$

which give $u_1 \geq -v$, so that

$$-v \leq u_1 \leq v.$$

We will prove the following estimates by induction:

$$-v \leq u_l \leq u_{l-1} \leq v \quad (l = 1, 2, \ldots). \tag{16}$$

Assume that Eq. (16) is true for l, then

$$L[u_{l+1} - u_l] = f(x,u_l) - f(x,u_{l-1}) - k(u_l - u_{l-1}) \le 0.$$

By Lemma 5, we have $u_{l+1} \le u_l$. On the other hand,

$$H(u_{l+1}) = k(u_{l+1} - u_l) + f(x,u_l) \ge f(x,u_{l+1}) \ge -N = H(-v),$$

which gives $-v \le u_{l+1}$. We have proved that Eq. (16) is also true for $l+1$. So Eq. (16) is true for all $l \in \mathbb{N}$.

We have proved that $\max_\Omega |u_l| \le \max_\Omega |v| = M < +\infty$; then $\{|u_l|\}$ is bounded in $C^0(\overline{\Omega})$, and

$$|H(u_l)| = |k(u_l - u_{l-1}) + f(x,u_{l-1})| \le 2|k|M + N = \widetilde{M}.$$

From Lemma 7, for $K \subset\subset \widetilde{K} \subset\subset \Omega$, we have

$$\max_{\widetilde{K}} |X_j u_l| \le C\widetilde{M},$$

where $C, \widetilde{M}$ are independent of l. On the other hand,

$$\begin{aligned}\max_{\widetilde{K}} |X_j f(x,u_l)| &= \max_{\widetilde{K}} \left| \frac{\partial f(x,u_l)}{\partial u} X_j(u_l) \right| \\ &\le \max_{x\in\Omega, |u_l| \le M} \left| \frac{\partial f(x,u_l)}{\partial u} \right| \max_{\widetilde{K}} |X_j(u_l)| \\ &\le \widetilde{C}\widetilde{M}.\end{aligned}$$

Then $\{H(u_l) = k(u_l - u_{l-1}) + f(x,u_{l-1})\}$ is bounded in $S^1(\widetilde{K}) \subset S^\alpha(\widetilde{K})$, for any $1 > \alpha > 0$. Using the Schauder estimate (12), and the interpolation Lemma 1, there exists a subsequence $\{u_{l_j}\}$ which is convergence in $S^2(K)$, where K is any compact of Ω. Let $u_{l_j} \to u$, $j \to +\infty$, then $-v \le u \le v$. Set $u = 0$ in $\partial\Omega$, then $u \in S^2(\Omega) \cap C^0(\overline{\Omega})$. Take $l_j \to \infty$ in Eq. (15), then u is a solution of problem (1). We have finished the proof step (a).

(b) We now consider general case, and assume that $f(x,u) = f_1(x,u) + f_2(x,u)$, where $|f_2| \le N$, and $\partial_u f_1 \le 0$. Using Taylor formula, Eq. (1) can be written in the following form:

$$\begin{cases} L[u] \equiv H(u) - \dfrac{\partial f_1}{\partial u}(x,\widetilde{u})u = f_1(x,0) + f_2(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\widetilde{u}(x)$ is between 0 and $u(x)$. By Lemma 4, we have

$$|u| \le c_0^{-1} \max |f_1(x,0) + f_2(x,u)| \le M_1.$$

We define now a function $U(u)$ on $\mathbb{R}$, $U(u) = u$, if $|u| \le M_1$, and for all $u \in \mathbb{R}$, $|U(u)| \le 2M_1$, $\partial_u U(u) \ge 0$. Denoting $\widetilde{f}_1(x,u) = f_1(x,U(u))$, we now consider the following modified problem

$$H(u) = \widetilde{f}_1(x,u) + f_2(x,u) = F(x,u), \quad u|_{\partial\Omega} = 0. \tag{17}$$

Then $\partial_u \widetilde{f}_1 \le 0$ and $\widetilde{f}_1(x,0) = f_1(x,0)$. Since the bound of the solution of problem (1) is also M_1, the solution of problem (17) is also the solution of problem (1). But for function F, there exists a constant N_1 such that

$$|F(x,u)| \le \sup_{|u|\le 2M_1} |f_1(x,u) + f_2(x,u)| \le N_1.$$

Then we have now reduced the proof of step (b) to step (a), so there exists a solution of problem (17).

5. Regularity up to the boundary and uniqueness of solutions

The uniqueness of solution of the Dirichlet problem (1) is immediate. Assume that $u,\, v \in S^2(\Omega) \cap C^0(\overline{\Omega})$ are two solutions of problems (1), then $w = u - v$ satisfies the following problem

$$H(w) = f(x,u) - f(x,v); \quad w|_{\partial\Omega} = 0.$$

Since $f(x,u(x)) - f(x,v(x)) = \partial_u f(x,\widetilde{u}(x))w(x)$, and $\partial_u f(x,u) \le 0$ for all u. We have

$$\sum_{j=1}^{m} X_j^* X_j w + \widetilde{c} w = 0; \quad w|_{\partial\Omega} = 0.$$

with $\widetilde{c}(x) = c(x) - \partial_u f(x,\widetilde{u}(x)) \ge c_0 > 0$. Then Lemma 4 implies that $w \equiv 0$.

For the regularities of solutions of Eq. (1), we have, in the interior of Ω, $u \in C^\infty(\Omega)$. So we prove only regularities of solution up to the boundary of Ω, This is a local problem. By changing variables, near the boundary as in [3, 12], we can take $\overline{\Omega}^+ \subset \bar{\mathbb{R}}^n_+ = \{(x', x_n);\ x' \in \mathbb{R}^{n-1}, x_n \ge 0\}$, $\overline{\Omega}^+ = \omega \times [0, T[$, ω is a open subset of $\mathbb{R}^{n-1}$. And using the hypothesis non-characteristic of the boundary, the Dirichlet problems (1) can be transformed in the form

$$\begin{cases} \widetilde{H}(u) \equiv \partial^2_{x_n} u + \sum_{j=1}^{m-1} Y_j^* Y_j u + cu = f(x,u), & \text{in} \quad \Omega^+, \\ u|_{x_n=0} = \varphi, & \text{on} \quad \omega. \end{cases} \tag{18}$$

where $\operatorname{supp} u$, $\operatorname{supp} f(x,u(x)) \subset \overline{\Omega}^+$, $\operatorname{supp} \varphi \subset \omega$. The vector fields $Y_1, \ldots, Y_{m-1}$ are tangential to $\omega \times \{0\}$. $\partial_{x_n}, Y_1, \ldots, Y_{m-1}$ also satisfies Hörmander's condition.

We now study tangential function spaces $H^{s,s'}(\bar{\mathbb{R}}^n_+)$ introduced by Hörmander (see [10, 12]). For $s, s' \in \mathbb{R}$ we set

$$H^{s,s'}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n);\ (1 + |\xi|^2)^{s/2}(1 + |\xi'|^2)^{s'/2}\widehat{u} \in L^2(\mathbb{R}^n)\},$$

where $\xi = (\xi', \xi_n) \in \mathbb{R}^n$ and $\xi' \in \mathbb{R}^{n-1}$. Then $H^{s,s'}(\bar{\mathbb{R}}^n_+)$ is the space of restriction to $\bar{\mathbb{R}}^n_+$ of elements in $H^{s,s'}(\mathbb{R}^n)$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a radial and positive function which is equal to one for $|\xi| \le \frac{1}{2}$ and vanishes for $|\xi| \ge 1$. We set $\widetilde{\psi}(\xi') = \psi(\xi', 0)$. Following Sablé-Tougeron [10] we set for $u \in \mathscr{S}'(\mathbb{R}^n)$ and $p, p' \in \mathbb{N}$:

$$\widehat{S_p u}(\xi) = \psi(2^{-p}\xi \hat{u}(\xi), \qquad \widehat{S'_{p'} u}(\xi) = \widetilde{\psi}(2^{-p'}\xi')\hat{u}(\xi),$$

and $S_{pp'} = S_p \circ S'_{p'}$. We shall also set $\Delta_p = S_{p+1} - S_p$, $\Delta'_{p'} = S'_{p'+1} - S'_{p'}$ and $\Delta_{pp'} = \Delta_p \circ \Delta'_{p'}$.

Using the method of paradifferential operators of Bony (see [2, 10, 12]), we have

Theorem 2. *Let $F \in C^\infty(\bar{\mathbb{R}}^n_+ \times \mathbb{R})$ which is real and with compact support with respect to the x variable in $\bar{\mathbb{R}}^n_+$. Let $u \in C^0(\bar{\mathbb{R}}^n_+) \cap H^{s,s'}(\bar{\mathbb{R}}^n_+)$, $s>0$, $s'>0$, supp $u \subset\subset \bar{\mathbb{R}}^n_+$. Then $f(x,u(x)) \in C^0(\bar{\mathbb{R}}^n_+) \cap H^{s,s'}(\bar{\mathbb{R}}^n_+)$.*

Proof. As in [8] we can reduce to the case where F does not depend on x. Moreover, taking restrictions to $\bar{\mathbb{R}}^n_+$, the above result will be a consequence of the same result in the spaces $H^{s,s'}(\mathbb{R}^n)$.

We have firstly $\lim_{p\to+\infty} \|S'_p u - u\|_{L^\infty} = 0$. So following [8] we can write

$$F(u) = F(S_0 u) + \sum_{p=0}^{+\infty} \{F(S_{p+1}u) - F(S_p u)\}. \tag{19}$$

Moreover, for every N, $S_N u \in C^{+\infty}(\mathbb{R}^n)$. Now

$$F(S_{p+1}u) - F(S_p u) = \Delta_p u \int_0^1 F'(S_p u + t\Delta_p u)\,\mathrm{d}t.$$

So we have to prove

$$F(u) = F(S_0 u) + \sum_{p=0}^{+\infty} \Delta_p u \int_0^1 F'(S_p u + t\Delta_p u)\,\mathrm{d}t \in H^{s,s'}(\mathbb{R}^n). \tag{20}$$

cutting again each term $\Delta_p u$ in the tangential variables, we get

$$\begin{aligned} F(u) &= \sum_{q\le p+3} \Delta_{pq} u \int_0^1 F'(S_p u + t\Delta_p u)\,\mathrm{d}t \\ &= \sum_{p\le q+3} a_{pq}. \end{aligned}$$

Indeed if $q \ge p+4$, $\{2^{p-1} \le |\xi| \le 2^{p+2}\} \cap \{2^{q-1} \le |\xi'| \le 2^{q+2}\} = \emptyset$. We shall prove that $\forall \alpha \in \mathbb{N}^n$, $\forall \alpha' \in \mathbb{N}^{n-1}$

$$\|\partial_x^\alpha \partial_{x'}^{\alpha} a_{pq}\|_{L^2} \le C_{pq} 2^{p(|\alpha|-s)} 2^{q(|\alpha'|-s')}, \qquad \{C_{pq}\} \in l^2. \tag{21}$$

First, since $u \in H^{s,s'}(\mathbb{R}^n)$ we have (see Proposition 1.1 in [10])

$$\|\partial_x^\alpha \partial_{x'}^{\alpha} \Delta_{pq} u\|_{L^2} \le C_{pq} 2^{p(|\alpha|-s)} 2^{q(|\alpha'|-s')}, \qquad \{C_{pq}\} \in l^2. \tag{22}$$

Using Lemma 3 in [8] and $u \in C_0^0(\bar{\mathbb{R}}^n_+)$ we get

$$\|\partial_x^\alpha \partial_{x'}^{\alpha} F'(S_p u + t\Delta_p u)\|_{L^\infty} \le C 2^{p(|\alpha|+|\alpha'|)}, \tag{23}$$

which implies (21) for $q \le p+3$.

Now Theorem 2 follows from the following lemma (see Lemma 2.3 in [10]).

Lemma 7. *Let* $\{a_{pq}\}_{q<p}$ *be a sequence of function in* $C^\infty(\mathbb{R}^n)$ *such that for all* $\alpha\in\mathbb{N}^n$, $\alpha'\in\mathbb{N}^{n-1}$,

$$\|\partial_x^\alpha\partial_{x'}^{\alpha}a_{pq}\|_{L^2}\le C_{pq}2^{p(|\alpha|-s)}2^{q(|\alpha'|-s')},\qquad \{C_{pq}\}\in l^2. \tag{24}$$

Then $g=\sum_{q<p}a_{pq}\in H^{s,s'}(\mathbb{R}^n)$.

As in [3, 12], using the Hörmander's condition for the system $\partial_{x_n}, Y_1,\dots,Y_{m-1}$, we can obtain the following tangential subelliptic estimates for the operators $\widetilde{H}$:

Lemma 8. *For every compact* K *in* $\bar{\mathbb{R}}^n_+$, *there exists a constant* $C>0$ *and* $\varepsilon>0$ *such that*

$$\|u\|^2_{0,\varepsilon}\le C\{|(\widetilde{H}u,u)|+\|u\|^2_{L^2}\},$$

for every $u\in E=\{u\in C^\infty(\bar{\mathbb{R}}^n_+);\ \operatorname{supp}u\subset K,\ u|_{x_n=0}=0\}$. *Then if* u, $\widetilde{H}u\in H^{s,s'}(\bar{\mathbb{R}}^n_+)$, s, $s'\in\mathbb{R}$, $u|_{x_n=0}=0$, *we have* $u\in H^{s,s'+\varepsilon}(\bar{\mathbb{R}}^n_+)$ (*see* [3, 12]).

We also work in the local version near the boundary, and study the problem (18). From the existence results of Section 4, for the semilinear Dirichlet problem (1) there exists a solution $u\in C^\infty(\mathbb{R}^n_+)\cap C^0_0(\bar{\mathbb{R}}^n_+)$, then $u, f(x,u)\in C^0_0(\bar{\mathbb{R}}^n_+)\subset L^2(\Omega)$. Lemma 9 implies that $u\in H^{0,\varepsilon}(\bar{\mathbb{R}}^n_+)$. Using Eq. (18) we have

$$\partial^2_{x_n}u=f(x,u)-\sum_{j=1}^{m-1}Y_j^*Y_ju\in H^{0,\varepsilon-2}(\bar{\mathbb{R}}^n_+), \tag{25}$$

then $u\in H^{2,\varepsilon-2}(\bar{\mathbb{R}}^n_+)\subset H^{\varepsilon/2,\varepsilon/2}(\bar{\mathbb{R}}^n_+)$. Theorem 2 gives that $f(x,u)\in H^{\varepsilon/2,\varepsilon/2}(\bar{\mathbb{R}}^n_+)$. Iterating this argument we prove that $u\in H^{\varepsilon/2,+\infty}(\bar{\mathbb{R}}^n_+)$, and $f(x,u)\in H^{\varepsilon/2,+\infty}(\bar{\mathbb{R}}^n_+)$. Now use Eq. (25) again, we have $\partial^2_{x_n}u\in H^{\varepsilon/2,+\infty}(\bar{\mathbb{R}}^n_+)$, from which we deduce $u\in H^{2+\varepsilon/2,+\infty}(\bar{\mathbb{R}}^n_+)$. Iterating this argument again we prove that $u\in H^{+\infty,+\infty}(\bar{\mathbb{R}}^n_+)\subset C^\infty(\bar{\mathbb{R}}^n_+)$. The proof of Theorem 1 is complete. □

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