Trace theorem in Sobolev spaces associated with Hörmander's vector fields

H. BAHOURI & J.-Y. CHEMIN & C.-J. XU [*]

1 Introduction

Let $\Omega \subset \mathbf{R}^d, d \geq 2$ an open domain, and $P = (P_1, \cdots, P_n)$ a smooth real vector fields system. P satisfies the Hörmander condition of order 2, if

$$\operatorname{rank}\{P_1, \cdots, P_n, [P_k, P_j], k, j = 1, \cdots, n\} = d.$$

For $k \in \mathbf{N}$, we define

$$H^k(\Omega, P) = \{f \in L^2(\Omega); P^J f \in L^2(\Omega); |J| \leq k\},$$

where $P^J = P_{j_1} \cdots P_{j_l}, J = (j_1, \cdots, j_l), |J| = l$.

Denote by $\mathcal{X}$ the C^∞-module of vector fields span by $\{P_1, \cdots, P_n\}$. Let $\Sigma \subset \Omega$ a smooth surface, Σ is non characteristic for the vector fields system P, if for all $x \in \Sigma, \mathcal{X}_x$ is not included in $T_x\Sigma$. $\mathcal{X}_\Sigma$ denote the subspace of $T\Sigma$ define by $\mathcal{X}_{\Sigma|_x} = T_x\Sigma \cap \mathcal{X}|_x$. We suppose that there is a generator system $P_\Sigma = \{P_1^\Sigma, \cdots, P_n^\Sigma\}$ of $\mathcal{X}_\Sigma$, then $N_\Sigma, P_1^\Sigma, \cdots, P_n^\Sigma$ satisfy also the Hörmander's condition of order 2. We define

$$H^k(\Sigma, P_\Sigma) = \{f \in L^2(\Sigma); P_\Sigma^J f \in L^2(\Sigma), |J| \leq k\}.$$

We have proved the following trace theorems:

Theorem 1 *Denote by γ_Σ the trace operator on Σ, then we have that*

$$\gamma_\Sigma : H^k(\Omega, P) \to [H^k(\Sigma, P_\Sigma) \cap H^{k/2}(\Sigma), L^2(\Sigma)]_{1/(2k)}.$$

is a continuous and surjective application, here $H^{k/2}(\Sigma)$ is the usuel Sobolev space.

We study now the characteristic case. We consider only the problems on the Heisenberg groups $\mathbf{H}_d$, that means that we study the vector fields system

$$P_{\mathbf{H}_d} = \{X_j = \partial_{x_j} + y_j\partial s, Y_j = \partial_{y_j} - x_j\partial s;\ \ j = 1, \cdots, d\}$$

on $\mathbf{R}^{2d+1}$. We take now $\Sigma_0 = \{(x, y, s) \in \mathbf{R}^{2d+1}; s = 0\}$, then, for the system of vector fields $P_{\mathbf{H}_n}$, $(0, 0, 0)$ is the only characteristic point, and $(x, y, 0)$ is not characteristic if $(x, y) \neq (0, 0)$, we have a family of radial vector fields on Σ_0

$$\mathcal{R} = \{x_k\partial_{x_j} + y_j\partial_{y_k}; x_k\partial_{y_j} - x_j\partial_{y_k}; y_k\partial_{x_j} - y_j\partial_{x_k}; j, k = 1, \cdots, d\},$$

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[*] Partial differential equations and their applications (Wuhan, 1999), 1–14, World Sci. Publishing, River Edge, NJ, 1999

which is the projections of $P_{\mathbf{H}_d}$ to $T^\star(\Sigma_0 \cap \{|(x,y)| \geq \varepsilon > 0\})$ parallel a transverse vector fields. We define the function space on Σ_0

$$T^4(\Sigma_0) = \{u \in L^2(\Sigma_0); \mathcal{R}^l u \in L^2, l \leq 4; |(x,y)|^2 \triangle_{x,y} u \in L^2\}.$$

We have proved the following theorem

Theorem 2 *Denote by γ_{Σ_0} the trace operator on Σ_0, then*

$$\gamma_{\Sigma_0} : H^1(\mathbf{H}_d, P_{\mathbf{H}_d}) \to T^{1/2}(\Sigma_0) = [T^4(\Sigma_0), L^2(\Sigma_0)]_{1/(8)}$$

is a continuous and surjectif application.

Remark: We have proved that for $u \in H^1(\mathbf{H}_d, P_{\mathbf{H}_d})$,

- if Σ is non characteristic, we have

$$\gamma_\Sigma(u) \in H^{1/4}(\Sigma).$$

- for Σ_0 we have

$$\gamma_{\Sigma_0}(u) \in L^2(\Sigma_0),\ |(x,y)|^{1/4} \triangle_{x,y}^{1/8} \gamma_{\Sigma_0}(u) \in L^2(\Sigma_0).$$

The theorem 2 is also true for $\Sigma_{s_0} = \{(x,y,s) \in \mathbf{R}^{2d+1}, s = s_0\}$, for all $s_0 \in \mathbf{R}$, since $H^1(\mathbf{H}_d, P_{\mathbf{H}_d})$ is invariant by translation in s .

2 Weyl-Hörmander calculus

In this section, we recall some definitions of Weyl-Hörmander's calculus. Weyl's quantization associates to $a \in \mathcal{S}(\mathbf{R}^{2n})$ is the operator a^w defined by

$$a^w u(x) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} e^{i\langle x-z,\zeta\rangle} a\left(\frac{x+z}{2}, \zeta\right) u(z) dz d\zeta.$$

The composition formula is $a^w \circ b^w = (a \# b)^w$.

Defintion 1 *Let g measurable map from $\mathbf{R}^{2n}$ to the set of positive defined quadratic form on $\mathbf{R}^{2n}$. The metric g is an Hörmander's metric iff:*

$$g_X(X-Y) \leq \frac{1}{C_0} \Rightarrow C_0^{-1} g_X \leq g_Y \leq C_0 g_X$$

$$g_X \leq g_X^\sigma \quad \textit{with} \quad g_X^\sigma(T) = \sup_{W \neq 0} \frac{[T,W]}{g_X(W)};$$

$\exists N_0, \forall (X,Y) \in \mathbf{R}^{2n} \times \mathbf{R}^{2n}$,*such that*

$$\left(\frac{g_X}{g_Y}\right)^{\pm} \leq C_0 (1 + g_Y^\sigma(X-Y))^{N_0}.$$

Let us denote by U_X the g_X-ball of center X and of radius $r < C_0^{-1}$, i.e.

$$U_X = \{Y \in \mathbf{R}^{2n} \ / \ g_X(Y - X) < r\}.$$

We define the function $\triangle(\cdot, \cdot)$.

$$\triangle(X, Y) = 1 + \max\{g_X^\sigma(U_X - U_Y), g_Y^\sigma(U_X - U_Y)\},$$

$$g_X^\sigma(U_X - U_Y) = \inf_{(X', Y') \in U_X \times U_Y} g_X^\sigma(X' - Y').$$

One of the key properties of function $\triangle(\cdot, \cdot)$, is the following lemma.

Lemma 1 *$\exists N_1$ such that*

$$\sup_{X \in \mathbf{R}^{2n}} \int_{Y \in \mathbf{R}^{2n}} \triangle(X, Y)^{-N_1} |g_Y|^{\frac{1}{2}} dY < \infty, \tag{1}$$

where $|g_Y|$ denotes the determinant of the quadratic form g_Y in any symplectic basis of $\mathbf{R}^{2n}$.

If a and b are smooth and compactly supported functions on $\mathbf{R}^{2n}$, there is no reason why $a \# b$ should be so. The acquired notion is the following.

Defintion 2 *Let γ be a strictly positive defined quadratic form on $\mathbf{R}^{2n}$ such that $\gamma^\sigma \geq \gamma$ and $Y \in \mathbf{R}^{2n}$. Let us define on $\mathcal{S}(\mathbf{R}^{2n})$ the following semi-norms*

$$\|a\|_{k, Conf(\gamma, Y)} = \sup_{X \in \mathbf{R}^{2n}, j \leq k, \gamma(T_j) \leq 1} (1 + \gamma^\sigma(X - B_\gamma(Y, r)))^{\frac{k}{2}} |\partial_{T_1} \cdots \partial_{T_j} a(X)|.$$

Let g be an Hömander's metric and $(a_Y)_{Y \in \mathbf{R}^{2n}} \subset \mathcal{S}(\mathbf{R}^{2n})$. This family is uniformly confined iff, $\forall k$,

$$\|(a_Y)\|_{k, Conf(g)} = \sup_{Y \in \mathbf{R}^{2n}} \|a_Y\|_{k, Conf(g_Y, Y)} < \infty.$$

The key estimate is the following "biconfinement estimations": If $a, b \in \mathcal{S}(\mathbf{R}^{2n})$. Then $\forall (k, N)$, $\exists \ell, C$ such that, $\forall (Y, Z) \in \mathbf{R}^{2n} \times \mathbf{R}^{2n}$, we have

$$\begin{aligned} & \|a \# b\|_{k, Conf(g_Y, Y)} + \|a \# b\|_{k, Conf(g_Z, Z)} \\ \leq \ & C \triangle(Y, Z)^{-N} \|a\|_{\ell, Conf(g_Y, Y)} \|b\|_{\ell, Conf(g_Z, Z)}. \end{aligned} \tag{2}$$

We suppose the metric g is strongly temperate, then there exists two uniformly confined families (φ_Y) and (ψ_Y) so that, for any $X \in \mathbf{R}^{2n}$,

$$\int_{Y \in \mathbf{R}^{2n}} \varphi_Y(X) |g_Y|^{\frac{1}{2}} dY = \int_{Y \in \mathbf{R}^{2n}} (\psi_Y \# \varphi_Y)(X) |g_Y|^{\frac{1}{2}} dY = 1. \tag{3}$$

Let us define the concepts of g-weight and of symbols associated to some g-weight.

Defintion 3 *Let g be an Hörmander's metric, a measurable function m defined on $\mathbf{R}^{2n}$ with value in $\mathbf{R}_+^\star$ is a g-weight iff*

$$\exists \widetilde{C}, \exists \widetilde{N} \ / \ \left(\frac{m(X)}{m(Y)} \right)^{\pm 1} \leq \widetilde{C} \triangle(X, Y)^{\widetilde{N}}.$$

Let m be a g-weight. $S(m, g)$ denote the set of all smooth functions a so that, for any integer k,

$$\|a\|_{k; S(m,g)} = \sup_{j \leq k, X \in \mathbf{R}^{2n}, g_X(T_j) \leq 1} \frac{|\partial_{T_1} \cdots \partial_{T_j} a(X)|}{m(X)} < \infty$$

where $\partial_T a$ denotes the map $\langle da, T \rangle$.

We introduce now a "Littlewood-Paley" definition of Sobolev spaces.

Defintion 4 *Let g an Hörmander's metric and m a g-weight. The space $H(m,g)$ is the set of tempered distributions u so that*

$$\|u\|_{H(m,g)} = \left(\int m(Y)^2 \|\varphi_Y^w u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY\right)^{\frac{1}{2}} < \infty.$$

Remark:

- $H(m,g) = \{u \in \mathcal{S}'(\mathbf{R}^n); \forall a \in S(m,g), a^w u \in L^2\}$.
- $H(1,g) = L^2$.
- the space $H(m,g)$ is "almost independent" of the metric g.
- $\exists \mathcal{M} \in S(m,g)$ such that $u \in H(m,g) \Leftrightarrow \mathcal{M}^w u \in L^2$ and

$$C^{-1}\|u\|_{H(m,g)} \leq \|\mathcal{M}^w u\|_{L^2} \leq C\|u\|_{H(m,g)}.$$

In all that follows, we denote $H(m,g)$ by $H(m)$.

3 A simple proof in the case of integer index

We prove now the continuity of trace operators for integer index. After an extension of vector fields system by a elliptic system, we suppose that $P = (P_j)_{1\leq j\leq n}$ is well defined on $\mathbf{R}^d$ with constants coefficients outset of a compact set, satisfying the Hörmander condition of rank 2. After a change of variables and projection parallel a transverse vector fields, we suppose

$$P_1 = \partial_{x_1} \quad \text{and} \quad P_j = \sum_{\ell=2}^{n} a_j^\ell(x_1, x')\partial_{x_\ell} \quad \text{for} \quad j = 2, \cdots, n,$$

and $\Sigma = \{(x_1, x') \in \mathbf{R}^d; x_1 = 0\}$.

We define a metric on $T^\star\mathbf{R}^d$ by

$$\widetilde{g}_{(x,\xi)}(dx^2, d\xi^2) = \langle\xi'\rangle dx^2 + \frac{d\xi^2}{\langle\xi'\rangle},$$

where $(x,\xi) = (x_1, x', \xi_1, \xi') \in T^\star\mathbf{R}^d$. Then the restriction of this metric to $T^\star\Sigma$ is the metric

$$g_{(x',\xi')}(dx'^2, d\xi'^2) = \langle\xi'\rangle dx'^2 + \frac{d\xi'^2}{\langle\xi'\rangle}.$$

For $a \in \mathcal{S}(T^\star\Sigma)$, a^{w_Σ} is the Weyl's quantification of a on $L^2(\Sigma)$, define by

$$a^{w_\Sigma}u(x) = \int_{T^\star\Sigma} e^{i\langle\xi, x-y\rangle} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

We have the following Lemma from [Cancelier-Chemin-Xu].

Lemma 2 *The function defined by*

$$M(x,\xi) = \left(\langle\xi_1\rangle + m(x_1, x', \xi')^2\right)^{\frac{1}{2}}$$

is a $\widetilde{g}$ weight on $T^\star\mathbf{R}^n$, where

$$m(x_1,x',\xi')^2=\sum_{j=2}^{n}P_j(x_1,x',\xi')^2+\langle\xi'\rangle.$$

In particulier for all $x_1\in\mathbf{R}$, the function $m(x_1,\cdot)$ is a g-weights on $T^\star\mathbf{R}^{d-1}$.

Moreover for all integer k, there exists a constente C such that

$$\int_{\mathbf{R}}\|u(t,\cdot)\|^2_{H(m^k(t,\cdot))}dt+\int_{\mathbf{R}}\|\partial_{x_1}u(t,\cdot)\|^2_{H(m^{k-1}(t,\cdot))}dt\le C\|u\|^2_{H^k(\mathbf{R}^d,P)}.$$

We have

$$H^k(\mathbf{R}^d,P)=H(M^k)$$

with equivalent norms. We have the following theorem

Theorem 3 *There exists $C>0$ such that for all $u\in H(M^k)$, we have*

$$\sup_{t\in\mathbf{R}}\|u(t,\cdot)\|_{H(m^{k-\frac{1}{2}}(t,\cdot))}\le C\|u\|_{H(M^k)}.$$

Proof: We suppose $u\in C_0^\infty(\mathbf{R}^d)$, consider for $Y=(y',\eta')\in T^\star\mathbf{R}^{d-1}$, $\{\theta_Y\}$ an unitary partition,

$$\mathcal{I}(u)_Y(t)=m^{2k-1}(t,Y)\|\theta_Y^w u(t,\cdot)\|^2_{L^2(\mathbf{R}^{d-1})}$$

then we have

$$\begin{aligned}\mathcal{I}(u)_Y(t) &= (2k-1)\int_{-\infty}^{t}\partial_t m(t',Y)m(t',Y)^{2k-2}\|\theta_Y^w u(t',\cdot)\|^2_{L^2(\mathbf{R}^{d-1})}dt'\\ &+ 2\int_{-\infty}^{t}m(t',Y)^{2k-1}(\theta_Y^w\partial_t u(t',\cdot)|\theta_Y^w\partial_t u(t',\cdot))^2_{L^2(\mathbf{R}^{d-1})}dt'.\end{aligned}$$

Then lemma 2 and Cauchy-Schwarz inequality implies that

$$\begin{aligned}\mathcal{I}(u)_Y(t) &\le C\int_{-\infty}^{t}m(t',Y)^{2k}\|\theta_Y^w u(t',\cdot)\|^2_{L^2(\mathbf{R}^{n-1})}dt'\\ &+ 2\left(\int_{-\infty}^{t}m(t',Y)^{2k-2}\|\theta_Y^w\partial_t u(t',\cdot)|^2_{L^2(\mathbf{R}^{n-1})}dt'\right)^{\frac{1}{2}}\\ &\times \left(\int_{-\infty}^{t}m(t',Y)^{2k}\|\theta_Y^w u(t',\cdot)|^2_{L^2(\mathbf{R}^{n-1})}dt'\right)^{\frac{1}{2}}.\end{aligned}$$

which deduce

$$\sup_{t\in\mathbf{R}}m^{2k-1}(t,Y)\|\theta_Y u(t,\cdot)\|^2_{L^2(\mathbf{R}^{n-1})}\in L^1(dY).$$

Moreover for $Y\in T^\star\mathbf{R}^{d-1}$, the application

$$t\to m^{2k-1}(t,Y)\theta_Y^w u(t,\cdot)$$

is continuous. The Lebesgue dominate convergence theorem conclus the proof for continuous of trace operators.

4 Trace theorem in characteristic case

For $(x,y,s)\in \mathbf{H}_d$, we set $\rho(x,y,s)=\Big((|x|^2+|y|^2)^2+s^2\Big)^{\frac{1}{4}}$. We suppose that $u\in H^1(\mathbf{H}_d,P_{\mathbf{H}_d})$ with Supp $u\subset\{\rho(x,y,s)\le 1\}$, and consider a function $\varphi\in\mathcal{D}(\{1/2<|t|<2\})$ satisfying

$$\forall 0<|t|\le 1\,,\ \sum_{p=0}^{\infty}\varphi(2^pt)=1.$$

We define the family $(\varphi_p)_{p\in\mathbf{N}}$ and $(u_p)_{p\in\mathbf{N}}$ by

$$\varphi_p(x,y,s)=\varphi(2^p\rho)\quad\text{and}\quad u_p(x,y,s)=(\varphi_pu)(2^{-p}x,2^{-p}y,2^{-2p}s).$$

We have $P(\varphi_pu)=uP\varphi_p+\varphi_pPu$, and $P\varphi_p=\varphi'(2^p\rho)2^pP\rho$, since $|P\rho|\le C$, we have

$$\|P(\varphi_pu)\|^2_{L^2}\le C2^{2p}\|\varphi'_pu\|^2_{L^2}+2\|\varphi_pPu\|^2_{L^2}.$$

on the Supp φ'_pu, we have $2^{2p}\sim\rho^{-2}$,

$$2^{2p}\|\varphi'_pu\|^2_{L^2}\le C\int\frac{|\varphi'_pu|^2}{\rho^2}dxdyds.$$

The support of functions φ'_pu et $\varphi'_{p'}u$ is disjoint if $|p-p'|$ is big enough, we get

$$\sum_{p=0}^{\infty}\|P(\varphi_pu)\|^2_{L^2}\le C\int\frac{|u|^2}{\rho^2}dxdyds+\|u\|^2_{\dot H^1(\mathbf{H}_d,P_{\mathbf{H}_d})}.$$

We need now the following Hardy inequality on $\mathbf{H}_d$.

Lemma 3 *There are a constente* $C>0$ *such that for all* u *de* $H^1(\mathbf{H}_d,P_{\mathbf{H}_d})$*, we have*

$$\int\frac{|u|^2}{\rho^2}dxdyds\le C\|u\|^2_{\dot H^1(\mathbf{H}_d,P_{\mathbf{H}_d})}.$$

The proof of this lemma is standard.

Now for all $u\in H^1(\mathbf{H}_d,P_{\mathbf{H}_n})$, we have

$$\sum_{p=0}^{\infty}\|P(\varphi_pu)\|^2_{L^2}\le C\|u\|^2_{\dot H^1(\mathbf{H}_n,P_{\mathbf{H}_n})}.$$

We applies now the dilation of parameter 2^p on $\mathbf{H}_d$. Since

$$P(u_p)(x,y,s)=2^{-p}(P(\varphi_pu))(2^{-p}x,2^{-p}y,2^{-2p}s).$$

We have

$$\|Pu_p\|^2_{L^2(\mathbf{R}^{2d+1})}=2^{2pd}\|P(\varphi_pu)\|^2_{L^2(\mathbf{R}^{2d+1})}.$$

Now Supp $u_p\subset\{1/2\le\rho(x,y,s)\le 1\}$ is independent of p, and $\Sigma_0\cap\text{Supp}u_p$ is not characteristic for the system $P_{\mathbf{H}_d}$. So we can apply the theorem 1 to each term u_p, which give that $\gamma_{\Sigma_0}(u_p)\in H(\widetilde m^{1/2})$ where $\widetilde m(x,y,\xi,\eta)$ is the weight defined as in lemma 2 by using the projection of $P_{\mathbf{H}_d}$ to $\Sigma_0\cap\{1/2\le|(x,y)|\le 1\}$ parallel a vector fields transverse. Then for $(x,y)\in\mathbf{R}^{2d}, 1/2<|(x,y)|<1,\widetilde m(x,y,\xi,\eta)$ is equivalent to

$$m(x,y,\xi,\eta)=\left(1+|(x,y)|^2|(\xi,\eta)|^2+\sum_{R\in\mathcal{R}}R^4(x,y,\xi,\eta)\right)^{1/4}.$$

Since for $R \in \mathcal{R}$,

$$\begin{array}{rcl} R(u_p)(x,y,s) & = & (R(\varphi_p u))(2^{-p}x, 2^{-p}y, 2^{-2p}s), \\ |(x,y)|^2 \triangle_{x,y}(u_p)(x,y,s) & = & |(2^{-p}x, 2^{-p}y)|(\triangle(\varphi_p u))(2^{-p}x, 2^{-p}y, 2^{-2p}s). \end{array}$$

$m^s(x,y,D_x,D_y)$ is also a pseudo-differential operator of degree 0 by dilatation. That means for all $s \in \mathbf{R}$

$$\|\gamma_{\Sigma_0}(u_p)\|^2_{H(m^s)} = 2^{2pd}\|\gamma_{\Sigma_0}(\varphi_p u)\|^2_{H(m^s)}.$$

We have then proved

$$\sum_{p=0}^{\infty} \|\gamma_{\Sigma_0}(\varphi_p u)\|^2_{H(m^{1/2})} \leq C\|u\|^2_{\dot{H}^1(\mathbf{H}_d, P_{\mathbf{H}_d})}.$$

Use again the 0 degree property of operator $m^{1/2}(x,y,D_x,D_y)$, we get

$$\|\gamma(u)\|^2_{H(m^{1/2})} \leq C\|u\|^2_{H^1(\mathbf{H}_d, P_{\mathbf{H}_d})}.$$

By interpolation theory, we have

$$[T^4(\Sigma_0), L^2(\Sigma_0)]_{1/(8)} = H(m^{1/2}).$$

We have proved the continuity of trace operators

We prove now the surjectivity of trace operators γ_{Σ_0}, let $v \in H(m^{1/2})$, suppose also Supp $v \subset \{(x,y) \in \mathbf{R}^{2d}, |(x,y)| \leq 1\}$ and set

$$\widetilde{\varphi}_p(x,y) = \varphi(2^p|(x,y)|), \ v_p(x,y) = (\widetilde{\varphi}_p v)(2^{-p}x, 2^{-p}y),$$

since Supp$v_p \subset \{1/2 \leq |(x,y)| \leq 1\}$, and on $\{1/2 \leq |(x,y)| \leq 1\}$, $m^{1/2}$ is equivalent to $\widetilde{m}^{1/2}$, so we can use the relievement part of Theorem 1 in the non characteristic case, which implies that there exists $u_p \in H^1(\mathbf{H}_d, P_{\mathbf{H}_d})$ such that $\gamma(u_p) = v_p$,

$$\|u_p\|_{H^1(\mathbf{H}_d, P_{\mathbf{H}_d})} \leq C\|v_p\|_{H(m^{1/2})},$$

and Supp $u_p \cap$ Supp $u_{p'} = \emptyset$, if $|p - p'|$ is big enough. Then almost orthogonality implies that

$$u = \sum_{p=0}^{\infty} u_p \in H^1(\mathbf{H}_d, P_{\mathbf{H}_d}),$$

and same argument as before to get

$$\begin{aligned} \|u\|^2_{H^1(\mathbf{H}_d, P_{\mathbf{H}_d})} &\leq C\sum_{p=0}^{\infty} \|u_p\|^2_{H^1(\mathbf{H}_d, P_{\mathbf{H}_d})} \\ &\leq C\sum_{p=0}^{\infty} \|v_p\|^2_{H(m^{1/2})} \leq C\|v\|^2_{H(m^{1/2})}. \end{aligned}$$

Which prove the theorem 2.

5 Trace theorem in Weyl-Hörmander calculus

In this section, we prove the trace theorem in the non characteristic case for all index $s > 1/2$.

Theorem 4 *Let $\widetilde{m}(x_1, x', \xi')$ a $g_{\frac{1}{2},\frac{1}{2}}$-weight on $T^\star \mathbf{R}^d$ satisfying $\langle \xi' \rangle^{\frac{1}{2}} \leq \widetilde{m}(x, \xi') \leq \langle \xi' \rangle$. Then M define by*

$$M(x, \xi)^2 = \xi_1^2 + \widetilde{m}(x, \xi')^2$$

is also a $g_{\frac{1}{2},\frac{1}{2}}$-weight and the trace operator γ is a continuous and surjectif application from $H(M^s)$ to $H(m^{s-\frac{1}{2}})$ for all $s > 1/2$, where $m(x', \xi') = \widetilde{m}(0, x', \xi')$.

We have firstly the function M is a $\widetilde{g}$-weight and m is a g-weight.

The proof of Continuity of trace operators for no integer case is more complicate then integer case, we miss out this proof here, and attack directly to more interesting part: surjectivity of trace operators.

The idea of proof is to interpret the following classic formulai into Weyl-Hörmander's calculus: let $v \in H^{s-\frac{1}{2}}(\mathbf{R}^{d-1})$, set

$$u(x_1, x') = (2\pi)^{-(n-1)} C_s \mathcal{F}^{-1} \left(\frac{(1 + |\xi'|^2)^{s-\frac{1}{2}}}{(1 + |\xi|^2)^s} \widehat{v}(\xi') \right),$$

then $\|u\|_{H^s} \leq C \|v\|_{H^{s-\frac{1}{2}}}$ and $\gamma(u) = v$.

Take a function $\chi \in \mathcal{D}(\mathbf{R})$ with $\chi(t) = 1$ near to 0, we set for $v \in \mathcal{S}(\Sigma)$,

$$(R_\chi v)(x_1, x') = \int_{T^\star \Sigma} \mu_Y(x_1) (\phi_Y \#_{T^\star \Sigma} \theta_Y)^{w_\Sigma} v(x') dY, \tag{4}$$

with

$$\mu_Y(x_1) = C_s^{-1} m(Y)^{2s-1} \chi(x_1 \langle \eta \rangle^{\frac{1}{2}}) \int_{\mathbf{R}} e^{i x_1 \xi_1} M^{-2s}(0, y, \xi_1, \eta) d\xi_1.$$

Since $M^2(0, y, \xi_1, \eta) = \xi_1^2 + m^2(y, \eta)$, and $\chi(0) = 1$, then $\mu(0) = 1$, so that $\gamma \circ R_\chi =$Id. We have to prove that R_χ is a continuous linear application from $H(m^{s-\frac{1}{2}})$ to $H(M^s)$. The key lemma is the following.

Lemma 4 *For all integer N, there exists C such that for all function $f \in L^2(\Sigma)$, we have*

$$\begin{aligned} \|\psi^w_{\widetilde{Y}}(\mu_Y(\cdot) \phi_Y^{w_\Sigma} f)\|_{L^2(\mathbf{R}^d)} &\leq C \triangle(Y, \pi(\widetilde{Y}))^{-N} \frac{m^{2s-1}(Y)}{M^s(\widetilde{Y})} \\ &\times \langle \eta \rangle^{-\frac{1}{4}} (1 + \langle \eta \rangle \widetilde{y}_1^2)^{-1} \mathcal{J}_Y(\widetilde{\eta}_1) \|f\|_{L^2(\Sigma)} \end{aligned}$$

where $\mathcal{J}_Y$ is a function of real variable such that

$$\int_{\mathbf{R}} \mathcal{J}_Y^2(\tau) d\tau \leq C \frac{\langle \eta | \rangle}{m^{2s-1}(Y)}.$$

From this lemma, we can get immediately the continuity of R_χ, in fact, take $f = \theta_Y^w v$, we have

$$\begin{aligned} \|\psi^w_{\widetilde{Y}} R_\chi v\|_{L^2(\mathbf{R}^d)} \leq C M^{-s}(\widetilde{Y}) &\int_{T^\star \Sigma} \langle \eta \rangle^{-\frac{1}{4}} (1 + \langle \eta \rangle \widetilde{y}_1^2)^{-1} \mathcal{J}_Y(\widetilde{\eta}_1) \\ &\times \|\theta_Y^{w_\Sigma} v\|_{L^2(\Sigma)} \triangle(Y, \pi(\widetilde{Y}))^{-N} m^{2s-1}(Y) dY. \end{aligned}$$

Apply Cauchy-Schwarz inequality with measure $\triangle(Y,\pi(\widetilde{Y}))^{-N}m^{2s-1}(Y)dY$,

$$\begin{aligned}\|\psi^w_{\widetilde{Y}}R_\chi v\|^2_{L^2(\mathbf{R}^d)} &\leq CM^{-2s}(\widetilde{Y})\int_{T^\star\Sigma}\|\theta_Y^{w_\Sigma}v\|^2_{L^2(\Sigma)}\triangle(Y,\pi(\widetilde{Y}))^{-N}m^{2s-1}(Y)dY\\ &\times \int_{T^\star\Sigma}\langle\eta\rangle^{-\frac{1}{2}}(1+\langle\eta\rangle\widetilde{y}_1^2)^{-2}\mathcal{J}^2_Y(\widetilde{\eta}_1)\triangle(Y,\pi(\widetilde{Y}))^{-N}m^{2s-1}(Y)dY.\end{aligned}$$

By the definition of norm in $H(M^s)$,

$$\|R_\chi v\|^2_{H(M^s)} \leq \int_{Z\in T^\star\Sigma}F_1(Z)F_2(Z)dZ \quad \text{with}$$

$$F_1(Z) = \int_{T^\star\Sigma}\|\theta_Y^{w_\Sigma}v\|^2_{L^2(\Sigma)}\triangle(Y,Z)^{-N}m^{2s-1}(Y)dY,$$

$$F_2(Z) = \int_{T^\star\Sigma}\langle\eta|\rangle^{-\frac{1}{2}}\int_{\mathbf{R}^2}(1+\langle\eta\rangle t^2)^{-2}\mathcal{J}^2_Y(\tau)\triangle(Y,Z)^{-N}m^{2s-1}(Y)dtd\tau dY.$$

Since

$$\int_{\mathbf{R}^2}(1+\langle\eta\rangle t^2)^{-2}\mathcal{J}^2_Y(\tau)dtd\tau \leq \frac{C\langle\eta\rangle^{\frac{1}{2}}}{m^{2s-1}(Y)}.$$

The lemma 1 give

$$\begin{aligned}\|R_\chi v\|^2_{H(M^s)} &\leq C\int_{T^\star\Sigma\times T^\star\Sigma}m^{2s-1}(Y)\|\theta_Y^{w_\Sigma}v\|^2_{L^2(\Sigma)}\triangle(Y,Z)^{-N}dYdZ\\ &\leq \int_{T^\star\Sigma}m^{2s-1}(Y)\|\theta_Y^{w_\Sigma}v\|^2_{L^2(\Sigma)}dY\\ &\leq \|v\|^2_{H(m^{s-1/2})}.\end{aligned}$$

We have proved that the lemma 4 implies the theorem 1.

We prove now lemma 4, the essential idea is to use the confined properiety of unitaire partition and integration by part for a vecor of g-length ≤ 1. Set

$$F_{\widetilde{Y},Y} = \psi^w_{\widetilde{Y}}(\mu_Y(\,\cdot\,)\phi_Y^{w_\Sigma}f) = m^{2s-1}(Y)\widetilde{F}_{\widetilde{Y},Y},$$

with

$$\begin{aligned}\widetilde{F}_{\widetilde{Y},Y}(x_1,x') &= \int_{\mathbf{R}^3}e^{ix_1\tau-it(\tau-\tau')}\frac{\chi(t\langle\eta\rangle^{\frac{1}{2}})}{M^{2s}(0,y,\tau',\eta)}\\ &\times \left(\psi_{\widetilde{Y}}\left(\frac{x_1+t}{2},\cdot,\tau,\cdot\right)\#_{T^\star\Sigma}\phi_Y\right)^{w_\Sigma}f(x')dtd\tau d\tau'.\end{aligned}$$

By integrate by part for $\langle\widetilde{\eta}'\rangle\partial_\tau$, which is a vector of $\widetilde{g}_{\widetilde{Y}}$-length ≤ 1, then for all N,

$$\begin{aligned}\widetilde{F}_{\widetilde{Y},Y}(x_1,x') &= \int_{\mathbf{R}^3}e^{ix_1\tau-it(\tau-\tau')}\frac{\chi(t\langle\eta\rangle^{\frac{1}{2}})}{M^{2s}(0,y,\tau',\eta)}(1+\langle\widetilde{\eta}'\rangle(x_1-t)^2)^{-N}\\ &\times \left(\psi^{(N)}_{\widetilde{Y}}\left(\frac{x_1+t}{2},\cdot,\tau,\cdot\right)\#_{T^\star\Sigma}\phi_Y\right)^{w_\Sigma}f(x')dtd\tau d\tau'.\end{aligned}$$

where $\psi^{(N)}_{\widetilde{Y}} = (1-\langle\widetilde{\eta}'\rangle\partial_\tau^2)^N\psi_{\widetilde{Y}}$.

The vector

$$T_{\widetilde{Y},Y} = \left(\frac{1}{\langle\widetilde{\eta}'\rangle+\langle\eta\rangle}\right)^{\frac{1}{2}}\partial_t$$

is also a vector of $\widetilde{g}_{\widetilde{Y}}$-length ≤ 1. And

$$\begin{array}{rcl} |T^{j}_{\widetilde{Y},Y}(\langle\widetilde{\eta}'\rangle(x_1-t)^2)| & \leq & C(1+\langle\widetilde{\eta}'\rangle(x_1-t)^2)^{\frac{1}{2}} \\ |T^{j}_{\widetilde{Y},Y}(\chi(\langle\eta\rangle^{\frac{1}{2}}t))| & \leq & \|\chi^{(j)}\|_{L^\infty}. \end{array}$$

We get that

$$\begin{array}{rcl} \widetilde{F}_{\widetilde{Y},Y}(x_1,x') & = & \displaystyle\int_{\mathbf{R}^3} e^{ix_1\tau-it(\tau-\tau')}\frac{\widetilde{\chi}(t\langle\eta\rangle^{1/2})}{M^{2s}(0,y,\tau',\eta)}(1+\langle\widetilde{\eta}'\rangle(x_1-t)^2)^{-N} \\ & \times & \displaystyle\left(1+\frac{(\tau-\tau')^2}{\langle\widetilde{\eta}'\rangle+\langle\eta\rangle}\right)^{-N}(\mathcal{A}^{(N)}_{\widetilde{Y},Y}f)(x_1,x')dtd\tau d\tau' \end{array}$$

with

$$(\mathcal{A}^{(N)}_{\widetilde{Y},Y}f)(x_1,x')=\sum_{j+k\leq 2N}\mathcal{A}^{(N)}_{j}(t,\tau,\tau',\widetilde{Y},Y)\left(\psi^{N,k}_{\widetilde{Y},Y}\left(\frac{x_1+t}{2},\cdot,\tau,\cdot\right)\#_{T^\star\Sigma}\phi_Y\right)^{w_\Sigma}f(x')$$

where $\mathcal{A}^{(N)}_{j}$ is the bounded functions,

$$\psi^{N,k}_{\widetilde{Y},Y}=T^{k}_{\widetilde{Y},Y}(\mathrm{Id}-\langle\widetilde{\eta}'\rangle\partial_\tau^2)^N\psi_{\widetilde{Y}},$$

and $\widetilde{\chi}\in\mathcal{D}(\mathbf{R}),\widetilde{\chi}(t)=1$ for $t\in\text{Supp }\chi$. Since M is a $\widetilde{g}$-weight; there exists C and N_s such that

$$\begin{array}{rcl} M^{-s}(0,y,\tau',\eta) & \leq & CM^{-s}(\widetilde{Y})\left(1+\dfrac{1}{\langle\eta\rangle}|\tau'-\widetilde{\eta}_1|^2\right)^{N_s} \\ & \times & (1+\langle\eta\rangle\widetilde{y}_1^2)^{N_s}\triangle(Y,\pi(\widetilde{Y}))^{N_s}. \end{array}$$

Using now this estimations, $\widetilde{\chi}$ with compact support compact, and the biconfinement estimation (2); we have that for N big enough, there exists C such that

$$\begin{array}{l} \|\widetilde{F}_{\widetilde{Y},Y}(x_1,\cdot)\|_{L^2(\mathbf{R}^{d-1})}\leq C\triangle(Y,\pi(\widetilde{Y}))^{-N}M^{-s}(\widetilde{Y})\|f\|_{L^2(\mathbf{R}^{d-1})} \\ \times(1+\langle\eta\rangle x_1^2)^{-1}(1+\langle\eta\rangle\widetilde{y}_1^2)^{-1}\displaystyle\int_{\mathbf{R}^3}I_Y(t,\tau,\tau')M^{-s}(0,y,\tau',\eta)dtd\tau d\tau' \end{array}$$

with

$$I_Y(t,\tau,\tau')=(1+\langle\eta\rangle t^2)^{-1}\left(1+\frac{1}{\langle\eta\rangle}(\tau-\tau')^2\right)^{-1}\left(1+\frac{1}{\langle\eta\rangle}(\tau-\widetilde{\eta}_1)^2\right)^{-1}.$$

Using now the Cauchy-Schwarz inequality with measure $I_Y(t,\tau,\tau')dtd\tau d\tau'$, we get

$$\int_{\mathbf{R}^3}I_Y(t,\tau,\tau')M^{-s}(0,y,\tau',\eta)dtd\tau d\tau'\leq\mathcal{J}_Y^{\frac{1}{2}}(\widetilde{\eta}_1)$$

with

$$\begin{array}{rcl} \mathcal{J}_Y(\widetilde{\eta}_1) & = & \displaystyle\left(\int_{\mathbf{R}^2}\left(1+\frac{1}{\langle\eta\rangle}(\tau-\tau')^2\right)^{-1}\left(1+\frac{1}{\langle\eta\rangle}(\tau-\widetilde{\eta}_1)^2\right)^{-1}\right. \\ & & \displaystyle\left.\times M^{-2s}(0,y,\tau',\eta)d\tau d\tau'\right)^{\frac{1}{2}}. \end{array}$$

We have finally

$$\begin{aligned}\|\widetilde{F}_{\widetilde{Y},Y}(x_1,\cdot)\|_{L^2(\mathbf{R}^{n-1})} &\leq C\triangle(Y,\pi(\widetilde{Y}))^{-N'+N_1}M^{-s}(\widetilde{Y})\|f\|_{L^2(\mathbf{R}^{d-1})} \\ &\times (1+\langle\eta\rangle|x_1|^2)^{-1}(1+\langle\eta\rangle|\widetilde{y}_1|^2)^{-1}\mathcal{J}_Y(\widetilde{\eta}_1).\end{aligned}$$

And

$$\int_{\mathbf{R}}\mathcal{J}_Y(\tau)^2d\tau \leq C\frac{\langle\eta\rangle}{m^{2s-1}(Y)},$$

by integration in x_1, we have proved the lemma.

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