Sobolev embeddings in Weyl-Hörmander calculus

Jean-Yves CHEMIN and Chao-Jiang XU *

Introduction

The motivation for studying problems of Sobolev embeddings in Weyl-Hörmander calculus was the proof of Sobolev embeddings for Sobolev spaces associated to subelliptic systems of order two, see [4]. In this text, we focuse on L^p embeddings for abstract Sobolev spaces in the context of the Weyl-Hörmander calculus. The structure of this text will be the following:

in the first section, we briefly present Weyl-Hörmander calculus; then, we state the abstract Sobolev embedding;

in the second section, we introduce smoothing operators with respect to an Hörmander's metric g and a g-weight m,

in the third and last section, we prove the abstract embedding theorem.

1 Weyl-Hörmander calculus

In this section, we follow [5] and [3]. Weyl's quantization associates to $a \in \mathcal{S}(\mathbf{R}^n)$ the operator a^w defined by

$$a^w u(x) \stackrel{\text{def}}{=} (2\pi)^{-n} \int_{\mathbf{R}^{2n}} e^{i\langle x-z,\zeta\rangle} a\left(\frac{x+z}{2},\zeta\right) u(z)dzd\zeta.$$

Let us denote by $[X,Y]$ the standard symplectic form

$$[X,Y] = [(x,\xi),(y,\eta)] = \langle y,\xi\rangle - \langle x,\eta\rangle.$$

We have the following composition formula $a^w \circ b^w = (a\#b)^w$ with

$$(a\#b)(X) = \pi^{-2n} \int_{\mathbf{R}^{2n}\times\mathbf{R}^{2n}} e^{-2i[X-Y_1,X-Y_2]} a(Y_1)b(Y_2)dY_1dY_2.$$

Let us define the concept of Hörmander's metric.

* Geometrical optics and related topics, Ed. F. Colombini, N. Lerner, 1997, Brikhauser, p79–93.

Definition 1.1 *Let g measurable map from $\mathbf{R}^{2n}$ to the set of positive defined quadratic form on $\mathbf{R}^{2n}$. The metric g is an Hörmander's metric if and only if the following three conditions are satisfied.*

$$g_X(X-Y) \leq \frac{1}{C_0} \Rightarrow C_0^{-1} g_X \leq g_Y \leq C_0 g_X \tag{1}$$

$$g_X \leq g_X^\sigma \quad with \quad g_X^\sigma(T) \stackrel{\text{def}}{=} \sup_{W\neq 0} \frac{[T,W]}{g_X(W)}\ ; \tag{2}$$

An integer N_0 exists so that for any (X,Y) of $\mathbf{R}^{2n} \times \mathbf{R}^{2n}$,

$$C_0^{-1}(1+g_Y^\sigma(X-Y))^{-N_0} g_X \leq g_Y \leq C_0(1+g_Y^\sigma(X-Y))^{N_0} g_X. \tag{3}$$

Considering an Hörmander's metric g, let us fix a strictly positive real number r strictly smaller than C_0^{-1}, where the constant C_0 is this of assertions (1). In all that follows, let us denote by U_X the g_X-ball of center X and of radius r, i.e.

$$U_X \stackrel{\text{def}}{=} \{Y \in \mathbf{R}^{2n} \ / \ g_X(Y-X) < r\}.$$

Let us define the following function $\triangle$.

$$\begin{aligned} \triangle(X,Y) &\stackrel{\text{def}}{=} 1+\max\{g_X^\sigma(U_X-U_Y), g_Y^\sigma(U_X-U_Y)\} \ \text{ with} \\ g_X^\sigma(U_X-U_Y) &= \inf_{(X',Y')\in U_X\times U_Y} g_X^\sigma(X'-Y'). \end{aligned} \tag{4}$$

In all that follows, we drop the fact that this function depends on r. As proved in [3], we may substitute conditions (1) and (3) by

$$\frac{1}{C_0}\triangle(X,Y)^{-N_0} g_X \leq g_Y \leq C_0 \triangle(X,Y)^{N_0} g_X. \tag{5}$$

One of the key properties of function $\triangle$, obviously symmetric, is the following lemma proved in [3].

Lemma 1.2 *An integer N_1 exists so that*

$$\sup_{X\in\mathbf{R}^{2n}} \int_{Y\in\mathbf{R}^{2n}} \triangle(X,Y)^{-N_1} |g_Y|^{\frac{1}{2}} dY < \infty, \tag{6}$$

where $|g_Y|$ denotes the determinant of the quadratic form g_Y in any symplectic basis of $\mathbf{R}^{2n}$.

An Hörmander metric describe a localization procedure in the phase spaces. Let us notice that if a and b are smooth and compactly supported functions on $\mathbf{R}^{2n}$, there is no reason why $a\#b$ should be so. The acquired notion is the following.

Definition 1.3 *Let γ be a strictly positive defined quadratic form on $\mathbf{R}^{2n}$ such that $\gamma^\sigma \geq \gamma$ and Y a point of $\mathbf{R}^{2n}$. Let us define on $\mathcal{S}(\mathbf{R}^{2n})$ the following semi-norms*

$$\|a\|_{k,Conf(\gamma,Y)} \stackrel{\text{def}}{=} \sup_{\substack{X\in\mathbf{R}^{2n}\\ j\leq k,\gamma(T_j)\leq 1}} (1+\gamma^\sigma(X-B_\gamma(Y,r)))^{\frac{k}{2}}|\partial_{T_1}\cdots\partial_{T_j}a(X)|.$$

Let g be an Hömander's metric and $(a_Y)_{Y\in\mathbf{R}^{2n}}$ a family of functions of $\mathcal{S}(\mathbf{R}^{2n})$. This family is uniformely confined if and only if, for any integer k,

$$\|(a_Y)\|_{k,Conf(g)} \stackrel{\text{def}}{=} \sup_{Y\in\mathbf{R}^{2n}} \|a_Y\|_{k,Conf(g_Y,Y)} < \infty.$$

The key estimate, proved in [3] is the following.

Theorem 1.4 *Let g be an Hörmander's metric and a and b two functions of $\mathcal{S}(\mathbf{R}^{2n})$. For any couple of integers (k,N), an integer ℓ and a constant C exist such that, for any couple (Y,Z) of $\mathbf{R}^{2n}\times\mathbf{R}^{2n}$, we have*

$$\|a\#b\|_{k,Conf(g_Y,Y)} + \|a\#b\|_{k,Conf(g_Z,Z)} \leq C\triangle(Y,Z)^{-N}\|a\|_{\ell,Conf(g_Y,Y)}\|b\|_{\ell,Conf(g_Z,Z)}.$$

In all that follows, we shall assume, for sake of simplicity, that the metric g is strongly temperate (see [2] for a precise definition). The hypothesis implies in particular the following theorem.

Theorem 1.5 *Two uniformely confined families (φ_Y) and (ψ_Y) exists so that, for any $X\in\mathbf{R}^{2n}$,*

$$\int_{Y\in\mathbf{R}^{2n}} \varphi_Y(X)|g_Y|^{\frac{1}{2}}dY = \int_{Y\in\mathbf{R}^{2n}} (\psi_Y\#\varphi_Y)(X)|g_Y|^{\frac{1}{2}}dY = 1. \qquad (7)$$

As proved in [2], the above theorem 1.5 is always true with series of integrals instead of an integral. All the results that follows are true in this case. In the proofs, just substitute integral on $\mathbf{R}^{2n}$ by series of integrals on $\mathbf{R}^{2n}$.

Convention In all that follows, we denote by (φ_Y) and (ψ_Y) any two uniformely confined families satisfying (7).

Let us define the concepts of g-weight and of symbols associated to some g-weight.

Definition 1.6 *Let g be an Hörmander's metric, a measurable function m defined on $\mathbf{R}^{2n}$ with value in $\mathbf{R}_+^\star$ is a g-weight if and only if*

$$\exists\widetilde{C}\ /\ \left(\frac{m(X)}{m(Y)}\right)^{\pm 1} \leq \widetilde{C}\triangle(X,Y)^{\widetilde{N}}. \qquad (8)$$

Definition 1.7 *Let m be a g-weight. Let us denote by $S(m,g)$ the set of all smooth functions a so that, for any integer k,*

$$\|a\|_{k;S(m,g)} \stackrel{\text{def}}{=} \sup_{\substack{j\leq k, X\in\mathbf{R}^{2n}\\ g_X(T_j)\leq 1}} \frac{|\partial_{T_1}\cdots\partial_{T_j}a(X)|}{m(X)} < \infty$$

where $\partial_T a$ denotes the map $\langle da, T\rangle$.

As we have good localization procedure in the phase space, we introduce, following R. Beals's paper [1] (see also [6] and [2]), a "Littlewood-Paley" definition of Sobolev spaces.

Definition 1.8 *Let g an Hörmander's metric and m a g-weight. The space $H(m,g)$ is the set of tempered distributions u so that*

$$\|u\|_{H(m,g)} \stackrel{\text{def}}{=} \left(\int m(Y)^2 \|\varphi_Y^w u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY\right)^{\frac{1}{2}} < \infty.$$

As it is proved for instance in [2], the space $H(m,g)$ is the set of tempered distributions u on $\mathbf{R}^n$ such that for any $a\in S(m,g)$, $a^w u\in L^2$. In [2], it is also proved that the space $H(1,g)$ is L^2. Moreover the space $H(m,g)$ is "almost independant" of the metric g. And it is also proved that, for any g-weight m, a symbol $\mathcal{M}$ belonging to $S(m,g)$ and a constant C exists such that $u\in H(m,g) \Leftrightarrow \mathcal{M}^w u\in L^2$ and

$$C^{-1}\|u\|_{H(m,g)} \leq \|\mathcal{M}^w u\|_{L^2} \leq C\|u\|_{H(m,g)}.$$

From theorem 18.6.6 of [5], we deduce immediately the following theorem.

Theorem 1.9 *Let m and m_2 be two g-weights so that*

$$\lim_{X\to\infty} \frac{m_1(X)}{m_2(X)} = +\infty,$$

then the space $H(m_1,g)$ is compactly included in $H(m_2,g)$.

We are interested in Sobolev embeddings. So it is natural to do the following hypothesis on the metric g.

Definition 1.10 *Let g an Hörmander's metric, it is a splitted one if and only if we have, for any X of $\mathbf{R}^{2n}$,*

$$g_X(dx,d\xi)^2 = g_{1,X}(dx^2) + g_{2,X}(d\xi^2).$$

Let us notice that if g is splitted, then the metric g^σ defined by (2) satisfies

$$g_X^\sigma(dx,d\xi)^2 = g_{2,X}^{-1}(dx^2) + g_{1,X}^{-1}(d\xi^2).$$

Uncertainty principle $g\leq g^\sigma$ implies that

$$g_{1,X} \leq g_{2,X}^{-1} \quad \text{and obviously} \quad g_{2,X}\leq g_{1,X}^{-1}. \tag{9}$$

Convention In all that follows, we denote by g a splitted Hörmander's metric strongly temperate and denote $H(m,g)$ by $H(m)$.

In Weyl-Hörmander calculus, lemmas of Cotlar type are very important, see for instance [5] and [3]. We prove here a Cotlar type lemma, which take into account the localization in x-space.

Lemma 1.11 (localized Cotlar) *Let $(\theta_Y)_{Y\in\mathbf{R}^{2n}}$ be a uniformely confined family and $(u_Y)_{Y\in\mathbf{R}^{2n}}$ a family of functions of $L^2(\mathbf{R}^n)$ such that*

$$\int \|u_Y\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY < +\infty. \tag{10}$$

Then, for any N, a constant C and an integer k exist such that

$$\begin{aligned}\left\|\int_{\mathbf{R}^{2n}} \theta_Y^w u_Y |g_Y|^{\frac{1}{2}} dY\right\|_{L^2}^2 &\leq C\|(\theta_Y)\|_{k,Conf(g)}^2 \\ &\times \int_{Y\in\mathbf{R}^{2n}} \left\|\left(1+g_{2,Y}^{-1}(\cdot - U_Y)\right)^{-N} u_Y\right\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY.\end{aligned} \tag{11}$$

This lemma is proved in [4]. We give here only a sketch of the proof.

$$I = \int_{\mathbf{R}^{2n}\times\mathbf{R}^{2n}} (\theta_Y^w u_Y | \theta_Z^w u_Z)_{L^2}\, |g_Y|^{\frac{1}{2}} |g_Z|^{\frac{1}{2}} dY dZ.$$

Defining $\Theta_{Y,Z} \overset{\text{def}}{=} \overline{\theta}_Z \# \theta_Y$, we have

$$I = \int_{\mathbf{R}^{2n}\times\mathbf{R}^{2n}} I_{Y,Z} |g_Y|^{\frac{1}{2}} |g_Z|^{\frac{1}{2}} dY dZ \quad \text{with} \quad I_{Y,Z} = \left(\Theta_{Y,Z}^w u_Y | u_Z\right)_{L^2}.$$

Let us define the constant coefficients differential operator $L_{Y,Z}$ by

$$L_{Y,Z} f(\xi) = f(\xi) - \sum_{1\leq i,j\leq n} (g_{2,Y} + g_{2,Z})_{i,j}^{-1} \frac{\partial^2}{\partial\xi_i \partial\xi_j} f(\xi).$$

This operator $L_{Y,Z}$ is a finite sum of derivations of $g_{2,Y}+g_{2,Z}$-length smaller than 1. Using integrations by part with respect to $L_{Y,Z}$ and theorem 1.4, we get

$$\begin{aligned}|I_{Y,Z}| \;\leq\; & C\triangle(Y,Z)^{-N_1} \int \left(1 + g_Y^\sigma\left(\left(\frac{x+t}{2},\tau\right) - U_Y\right)\right)^{-\frac{N+n}{2}} \\ & \times (1+(g_{2,Y}^{-1}(x-t))^{-\frac{N+n}{2}} |u_Y(t)| \left(1 + g_Z^\sigma\left(\left(\frac{x+t}{2},\tau\right) - U_Z\right)\right)^{-\frac{N+n}{2}} \\ & \times (1+(g_{2,Z}^{-1}(x-t))^{-\frac{N+n}{2}} |u_Z(x)| dx dt d\tau.\end{aligned} \tag{12}$$

The metric g is splitted and satisfies (2), so standard computations on quadratic form implies that we have, for any $Y=(y,\eta)$ of $\mathbf{R}^{2n}$,

$$\begin{aligned}&(1+g_{2,Y}^{-1}(x-t))^{-\frac{N+n}{2}} \left(1 + g_Y^\sigma\left(\left(\frac{x+t}{2},\tau\right) - U_Y\right)\right)^{-\frac{N+n}{2}} \\ &\quad \leq C(1+g_{2,Y}^{-1}(t-U_Y))^{-\frac{N}{2}} (1+g_{2,Y}^{-1}(x-t))^{-\frac{n}{2}} (1+g_{2,Y}(\tau-\eta))^{-\frac{n}{2}}\end{aligned}$$

Using the inequality (12) and Schwarz inequality, we get

$$|I_{Y,Z}| \le C\triangle(Y,Z)^{-N_1}\int (1+g_{2,Y}^{-1}(x-t))^{-\frac{n}{2}}(1+g_{2,Y}(\tau-\eta))^{-\frac{n}{2}}\mathcal{U}_Y(t) \times (1+g_{2,Z}^{-1}(x-t))^{-\frac{n}{2}}(1+g_{2,Z}(\tau-\zeta))^{-\frac{n}{2}}\mathcal{U}_Z(x)dxdtd\tau.$$

where $\mathcal{U}_Y$ is defined by $\mathcal{U}_Y(t) \stackrel{\text{def}}{=} (1+g_{2,Y}^{-1}(t-U_Y))^{-\frac{N}{2}}|u_Y(t)|$. Schwarz inequality implies that

$$\begin{aligned} |I_{Y,Z}| &\le C\triangle(Y,Z)^{-N_1}J_Y^{\frac{1}{2}}J_Z^{\frac{1}{2}} \quad \text{with} \\ J_Y &\stackrel{\text{def}}{=} \int (1+g_{2,Y}^{-1}(x-t))^{-n}(1+g_{2,Y}(\tau-\eta))^{-n} \\ &\qquad \times (1+g_{2,Y}^{-1}(t-U_Y))^{-N}|u_Y(t)|^2 dxdtd\tau. \end{aligned}$$

With the change of variables $x' = g_{2,Y}^{-\frac{1}{2}}(x-t)$, $\tau' = g_{2,Y}^{\frac{1}{2}}(\tau-\eta)$ and $t'=t$, whose jacobian is 1, we find that

$$J_Y \le C\|(1+g_{2,Y}^{-1}(\cdot-U_Y))^{-N}u_Y\|_{L^2}^2.$$

So, we have proved that $|I_{Y,Z}|$ is smaller than

$$C\triangle(Y,Z)^{-N_1}\|(1+g_{2,Y}^{-1}(\cdot-U_Y))^{-N}u_Y\|_{L^2}\|(1+g_{2,Z}^{-1}(\cdot-U_Z))^{-N}u_Z\|_{L^2}.$$

By Schwarz inequality with measure $\triangle(Y,Z)^{-N_1}|g_Y|^{\frac{1}{2}}|g_Z|^{\frac{1}{2}}dYdZ$, we get

$$I^2 \le \int \|(1+g_{2,Y}^{-1}(\cdot-U_Y))^{-N}u_Y\|_{L^2}^2\triangle(Y,Z)^{-N_1}|g_Y|^{\frac{1}{2}}|g_Z|^{\frac{1}{2}}dYdZ.$$

Condition (6) on $\triangle$ implies the lemma.

Now, let us state the main results of the paper. First of all, let us recall theorem 4.7 of [2].

Theorem 1.12 *Let m a g-weight , let us denote by Ω_∞ the set of all x of $\mathbf{R}^n$ so that*

$$\Pi_\infty^2(x) \stackrel{\text{def}}{=} \int_{\mathbf{R}^n} m^{-2}(x,\xi)d\xi < \infty. \tag{13}$$

Then, for any x of Ω_∞, the linear form on $\mathcal{S}(\mathbf{R}^n)$ defined by

$$\begin{cases} \mathcal{S}(\mathbf{R}^n) & \to \quad \mathbf{C} \\ \quad u & \mapsto \quad u(x) \end{cases}$$

can be extended in a continuous linear form on $H(m)$ and we have

$$\forall u \in H(m,g)\,,\ \forall x \in \Omega_\infty\,,\ |u(x)| \le C\,\Pi_\infty(x)\|u\|_{H(m)}. \tag{14}$$

Sobolev embeddings in L^p where then obtained by interpolation, so it was impossible to catch critical cases. The aim of the paper is to prove the two following theorems. The first one is already proved in [4].

Theorem 1.13 *Let m be a g-weight greater than 1 and A_0 a real number strictly greater than 1. Let us denote by Ω_p the set of all x so that*

$$\begin{aligned}\Pi_p^2(x) &\stackrel{\text{def}}{=} \sup_{A\geq A_0} \Pi_p^2(A,x) \quad \textit{with}\\ \Pi_p^2(A,x) &\stackrel{\text{def}}{=} A^{-\frac{4}{p-2}} \int_{\{\xi\,/\ m(x,\xi)\leq A\}} m(x,\xi)^{-2} d\xi < \infty.\end{aligned}$$

If the Lebesgue measure of Ω_p is strictly positive, the map from $\mathcal{S}(\mathbf{R}^n)$ to L^∞_{loc} defined by $u \mapsto \dfrac{u}{\Pi_p}$ can be extended in a linear continuopus map from $H(m)$ to $L^p(\Omega_p, d\mu_p)$, where $d\mu_p$ denote the measure $\Pi_p^2(x)dx$.

Theorem 1.14 *Consider m a g-weight as in theorem 1.13 above. Assume moreover that*

$$\lim_{A\to\infty} \frac{\Pi_p(A,x)}{\Pi_p(x)} = 0\,, \textit{ uniformely in } x \textit{ and that} \qquad \lim_{(x,\xi)\to\infty} m(x,\xi) = +\infty,$$

then $H(m)$ is compactly embedded in $L^p(\Omega_p, d\mu_p)$.

Let us notice that when $m(x,\xi) = (1+|\xi|^2)^s$, we recover usual Sobolev embeddings. For corollaries involving subelliptic systems, see [4].

2 Smoothing operators in Weyl-Hörmander calculus

Before introducing the concept of smoothing operators, let us present a proof of usual Sobolev embeddings which is

$$\|f\|_{L^p} \leq C \left(\int_{\mathbf{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \quad \text{with} \quad p = \frac{2d}{d-2s}\cdot$$

That will justify the definition of smoothing operators given in a while. This proof is based on classical real interpolation ideas. Let us write

$$\|f\|_{L^p}^p = p \int_0^\infty \lambda^{p-1} \mu(|f| > \lambda) d\lambda.$$

We use the following decomposition in a low and high frequency part. We write $f = f_{1,A} + f_{2,A}$ with $f_{1,A} = \mathcal{F}^{-1}(\mathbf{1}_{B(0,A)}\widehat{f})$ and $f_{2,A} = \mathcal{F}^{-1}(\mathbf{1}_{B^c(0,A)}\widehat{f})$. It is obvious that

$$\|f_{1,A}\|_{L^\infty} \leq \|\widehat{f_{1,A}}\|_{L^1} \leq \frac{C}{d-2s} A^{\frac{d}{2}-s} |f|_{\dot{H}^s}. \tag{15}$$

From the above inequality (15), we have

$$A = A_\lambda \stackrel{\text{def}}{=} \left(\frac{\lambda(d-2s)}{4C|f|_{\dot H^s}}\right)^{\frac{p}{d}} \Rightarrow \mu\left(|f_{1,A}| > \frac{\lambda}{2}\right) = 0.$$

As $(|f| > \lambda) \subset (2|f_{1,A}| > \lambda) \cup (2|f_{2,A}| > \lambda)$, we infer that

$$\|f\|_{L^p}^p \leq p\int_0^\infty \lambda^{p-1}\mu(2|f_{2,A_\lambda}| > \lambda)d\lambda.$$

Using Bienaymé-Tchebichev inequality, we get

$$\|f\|_{L^p}^p \leq 4p\int_0^\infty \lambda^{p-3}\|f_{2,A_\lambda}\|_{L^2}^2 d\lambda. \tag{16}$$

Using Fourier-Plancherel theorem, we have from inequality (16) that

$$\|f\|_{L^p}^p \leq 4p(2\pi)^d \int_{\mathbf{R}_+\times\mathbf{R}^d} \lambda^{p-3}\mathbf{1}_{\{(\lambda,\xi)\,/\,\,|\xi|\geq A_\lambda\}}(\lambda,\xi)|\widehat{f}(\xi)|^2 d\xi d\lambda.$$

But by definition de A_λ, we have

$$|\xi| \geq A_\lambda \Leftrightarrow \lambda \leq C_\xi \stackrel{\text{def}}{=} \frac{4C|f|_{\dot H^s}}{d-2s}|\xi|^{\frac{d}{p}}.$$

Using Fubini's theorem , we have

$$\|f\|_{L^p}^p \leq 4\frac{p(2\pi)^d}{p-2}\left(\frac{4C}{d-2s}\right)^{p-2}|f|_{\dot H^s}^{p-2}\int_{\mathbf{R}^d}|\xi|^{\frac{d(p-2)}{p}}|\widehat{f}(\xi)|^2 d\xi.$$

As $2s = \dfrac{d(p-2)}{p}$, the usual Sobolev embedding is proved.

The problem here is to substitute $m(x,\xi)$ to $|\xi|$ in the proof written above. We are going to define a smoothing operator. Let us choose a cut-off function $\chi \stackrel{\text{def}}{=} \mathbf{1}_{[0,1]}$. For any g-weight m, any strictly positive real number A and any $Y = (y,\eta)$ in $\mathbf{R}^{2n}$, we define

$$\chi_{Y,m,A}(x) \stackrel{\text{def}}{=} \chi\left(\frac{m(x,\eta)}{A}\right).$$

Let us define the operator $S_{m,A}$ by

$$S_{m,A}u \stackrel{\text{def}}{=} \int_{\mathbf{R}^{2n}} \psi_Y^w(\chi_{Y,m,A}\varphi_Y^w u)|g_Y|^{\frac{1}{2}}dY. \tag{17}$$

This operator is the analogous of the frequency cut-off usued in the preceeding proof. It is a smoothing operator in the spaces $H(m^\sigma m_1)$ in the following sense.

Theorem 2.1 *Let m and m_1 two g-weights and $\sigma > 0$. A constant C exists such that, for any real number $A > 1$, we have*

$$\|S_{m,A}\|_{\mathcal{L}(H(m_1),H(m^\sigma m_1))} \leq CA^\sigma.$$

This theorem is proved in [4].

The following L^∞-estimate, which is the equivalent of inequalty (15) will be very important for us. An analogous statement is proved in [4].

Theorem 2.2 *Let m be a g-weight and (δ, σ) a couple of real numbers so that δ is positive. Let us denote by $\Omega_{\delta,\sigma}$ the set of all x of $\mathbf{R}^n$ such that, for some strictly positive real number A_0, we have*

$$\begin{aligned} \Pi^2(x) &\stackrel{\text{def}}{=} \sup_{A>A_0} \Pi^2(A,x) < +\infty \quad \textit{with} \qquad (18)\\ \Pi^2(A,x) &\stackrel{\text{def}}{=} A^{-2\delta} \int_{m(x,\eta)\leq A} m(x,\eta)^{-2} d\eta. \end{aligned}$$

Then a constant C exists so that, for any $A \geq A_0$ and for any x of $\Omega_{\delta,\sigma}$, the linear form defined on $\mathcal{S}(\mathbf{R}^n)$ by

$$\left\{ \begin{array}{ccc} \mathcal{S}(\mathbf{R}^n) & \to & \mathbf{C} \\ u & \mapsto & S_{m,A}u(x) \end{array} \right.$$

can be extended to a continuous linear form on $H(m)$ so that

$$\forall u \in H(m)\,, \forall x \in \Omega_{\delta,\sigma}\,,\ |S_{m,A}(u)(x)| \leq CA^\delta \|u\|_{H(m)} \sup_{B\geq A} \Pi(B,x). \qquad (19)$$

The proof of this theorem relies on the following pointwise estimate, proved in [4] and implicitely contained in the proof of lemma 4.8 of [2].

Lemma 2.3 *For any couple of integers (N, N'), a constant C and an integer k exist so that for any function $\theta \in \mathcal{S}(\mathbf{R}^{2n})$ and for any splitted quadratic form*

$$\gamma(dx^2, d\xi^2) = \gamma_1(dx^2) + \gamma_2(d\xi^2) \quad \textit{with} \quad \gamma \leq \gamma^\sigma,$$

for any $Y \in \mathbf{R}^{2n}$ and for any $x \in \mathbf{R}^n$, we have,

$$\begin{aligned} &|(1+\gamma_2^{-1}(x-U_{1,Y}))^N \theta^w u(x)| \\ &\qquad\qquad \leq C\|\theta\|_{k,Conf(\gamma,Y)} |\gamma_2|^{-1/2} ((1+\gamma_2^{-1}(\cdot))^{-N'} * |u|)(x). \end{aligned}$$

Let us go back to the proof of theorem 2.2. From lemma 2.3, for any couple of integers (N, N'), a integer k and a constant C exist such that

$$\begin{aligned} |S_{m,A}u(x)| \leq C &\int_{(Y,z)\in\mathbf{R}^{2n}\times\mathbf{R}^n} (1+g_{2,Y}^{-1}(x-U_Y))^{-N} |g_{2,Y}|^{-\frac{1}{2}} \\ &\times (1+g_{2,Y}^{-1}(x-z))^{-N'} m(Y)\chi_{Y,m,A}(z) m(Y)^{-1} |(\varphi_Y^w u)(z)| |g_Y|^{\frac{1}{2}} dY dz. \end{aligned}$$

As the weight m is temperate, applying (8), we infer the existence of an integer N_2 so that

$$m(Y)^{-1} \leq Cm(z,\eta)^{-1}(1+g_{2,Y}^{-1}(z-U_{1,Y}))^{N_2}.$$

As we know that

$$g_{2,Y}^{-1}(z-U_{1,Y}) \leq 2g_{2,Y}^{-1}(x-z) + 2g_{2,Y}^{-1}(x-U_{1,Y}),$$

we deduce from this that

$$S_{m,A}u(x)^2 \leq C\int_{\mathcal{I}_m(A)} (1+g_{2,Y}^{-1}(x-U_Y))^{N_2-N}(1+g_{2,Y}^{-1}(x-z))^{N_2-N'} \times m(z,\eta)^{-1}|g_{2,Y}|^{-\frac{1}{2}} m(Y)|(\varphi_Y^w u)(z)||g_Y|^{\frac{1}{2}}dY\,dz \quad \text{with}$$

$$\mathcal{I}_m(A) \stackrel{\text{def}}{=} \{(Y,z)\in \mathbf{R}^{2n}\times\mathbf{R}^n \;/\; m(z,\eta)\leq A\}.$$

Schwarz inequality for the measure $|g_Y|^{\frac{1}{2}}dY\,dz$ implies that

$$\begin{aligned} S_{m,A}u(x)^2 &\leq C\|u\|^2_{H(m)}I(x) \quad \text{with} \\ I(x) &\stackrel{\text{def}}{=} \int_{\mathcal{I}_m(A)} m(z,\eta)^{-2}(1+g_{2,Y}^{-1}(x-U_Y))^{2N_2-2N} \\ &\qquad \times (1+g_{2,Y}^{-1}(x-z))^{2N_2-2N'}|g_{1,Y}|^{\frac{1}{2}}|g_{2,Y}|^{-\frac{1}{2}}dY\,dz. \end{aligned}$$

The metric g is temperate, so, stating $X=(x,\eta)$, we have

$$|g_{1,Y}|^{\frac{1}{2}}|g_{2,Y}|^{-\frac{1}{2}} \leq |g_{1,X}|^{\frac{1}{2}}|g_{2,X}|^{-\frac{1}{2}}(1+g_{2,Y}^{-1}(x-U_Y))^{N_0 n} \quad \text{and}$$

$$(1+g_{2,Y}^{-1}(x-z))^{2N_2-2N'} \leq (1+g_{2,X}^{-1}(x-z))^{2N_2-2N'} \times (1+g_{2,Y}^{-1}(x-U_Y))^{2N_0(N'-N_2)}.$$

Using uncertainty principle and the fact that g is temperate, we have

$$(1+g_{2,Y}^{-1}(x-U_Y))^{-n} \leq C(1+g_{1,X}(x-y))^{-n}(1+g_{2,Y}^{-1}(x-U_Y))^{N_0 n}.$$

Then we claim that the quantity

$$(1+g_{2,Y}^{-1}(x-U_Y))^{2N_2-2N}(1+g_{2,Y}^{-1}(x-z))^{2N_2-2N'}|g_{1,Y}|^{\frac{1}{2}}|g_{2,Y}|^{-\frac{1}{2}}$$

is smaller than

$$C(1+g_{1,X}(x-y))^{-n}(1+g_{2,X}^{-1}(x-z))^{2N_2-2N'}|g_{1,X}|^{\frac{1}{2}}|g_{2,X}|^{-\frac{1}{2}} \times (1+g_{2,Y}^{-1}(x-U_Y))^{2N_0(N'-N_2)+n(N_0+1)+2(N_2-N)}.$$

So choosing $N=(1-N_0)N_2+N'+n\left[\dfrac{N_0+1}{2}\right]$, we find that

$$I(x) \leq C\int_{\mathcal{I}_m(A)} m(z,\eta)^{-2}(1+g_{2,X}^{-1}(x-z))^{2N_2-2N'} \times (1+g_{1,X}(x-y))^{-n}|g_{1,X}|^{\frac{1}{2}}|g_{2,X}|^{-\frac{1}{2}}dY\,dz.$$

As m is a g-weight, we have, by definition of $\mathcal{I}_m(A)$, that

$$I(x) \leq \int_{\mathcal{J}_m(A)} m(x,\eta)^{-2}(1+g_{2,X}^{-1}(x-z))^{4N_2-2N'} \times (1+g_{1,X}(x-y))^{-n}|g_{1,X}|^{\frac{1}{2}}|g_{2,X}|^{-\frac{1}{2}}dY\,dz \quad \text{with}$$

$$\mathcal{J}_m(A) \stackrel{\text{def}}{=} \{(y,\eta,z)\in \mathbf{R}^{2n}\times\mathbf{R}^n \ / \ m(x,\eta)\leq CA(1+g_{2,X}^{-1}(x-z))^{N_0}\}.$$

With the change of variables $\eta'=\eta$, $y'=g_{1,X}^{\frac{1}{2}}(x-y)$ and $z'=g_{2,X}^{-\frac{1}{2}}(x-z)$, whose jacobian is $|g_{1,X}|^{\frac{1}{2}}|g_{2,X}|^{-\frac{1}{2}}$, we get

$$I(x)\leq C\int_{\mathcal{K}_m(A)} m(x,\eta')^{-2}(1+|z'|^2))^{2N_2-2N'}(1+|y'|^2)^{-n}dy'd\eta'dz'$$

with $\mathcal{K}_m(A)=\{(y',\eta',z')\,/\ m(x,\eta')\leq CA(1+|z'|^2)^{N_0}\}$. As it is obvious that $A\leq CA(1+|z'|^2)^{N_0}$, we have, by definition of functions $\Pi(A,\cdot)$,

$$\int_{\{\eta\,/\ m(x,\eta)\leq A(1+|z|^2)^{N_0}\}} m(x,\eta)^{-2}d\eta \leq C\sup_{B\geq A}\Pi^2(B,x)A^{2\delta}(1+|z|^2)^{2N_0\delta}.$$

So we get

$$I(x)\leq CA^{2\delta}\sup_{B\geq A}\Pi^2(B,x)\int(1+|z|^2))^{4N_2+2N_0\delta-2N'}(1+|y|^2)^{-n}dydz.$$

Choosing for instance $N'=[N_2\delta]+2N_2+n+1$, we conclude the proof of the theorem.

3 Proof of the embedding theorems in L^p

We begin the proof exactly as in the classical case by writing

$$\begin{aligned}\left\|\frac{u}{\Pi_p}\right\|^p_{L^p(\Omega_p,d\mu_p)} &\leq p\int_{\lambda\leq\lambda_0}\lambda^{p-1}\mu_p\left(\frac{|u|}{\Pi_p}>\lambda\right)d\lambda\\ &+ p\int_{\lambda\geq\lambda_0}\lambda^{p-1}\mu_p\left(\frac{|S_{m,A}u|}{\Pi_p}>\frac{\lambda}{2}\right)d\lambda\\ &+ p\int_{\lambda\geq\lambda_0}\lambda^{p-1}\mu_p\left(\frac{|u-S_{m,A}u|}{\Pi_p}>\frac{\lambda}{2}\right)d\lambda.\end{aligned}$$

If C is the constant given by inequality (19) of theorem 2.2, let us define

$$A_\lambda=\left(\frac{\lambda}{4C\|u\|_{H(m)}}\right)^{(p-2)/2} \quad \text{and} \quad \lambda_0=4C\|u\|_{H(m)}A_0^{2/(p-2)},$$

Theorem 2.2 applied with $\delta = \dfrac{2}{p-2}$ and $A = A_\lambda$ gives

$$\left(\frac{|S_{m,A}u|}{\Pi_p} > \frac{\lambda}{2}\right) = \emptyset.$$

So we get

$$\left\|\frac{u}{\Pi_p}\right\|^p_{L^p(\Omega_p,d\mu_p)} \leq p\int_{\lambda\leq\lambda_0}\lambda^{p-1}\mu_p\left(\frac{|u|}{\Pi_p}>\lambda\right)d\lambda + p\int_{\lambda\geq\lambda_0}\lambda^{p-1}\mu_p\left(\frac{|u-S_{m,A_\lambda}u|}{\Pi_p}>\frac{\lambda}{2}\right)d\lambda.$$

We infer that

$$\left\|\frac{u}{\Pi_p}\right\|^p_{L^p(\Omega_p,d\mu_p)} \leq \lambda_0^{p-2}p\int_{\lambda\leq\lambda_0}\lambda\mu_p\left(\frac{|u|}{\Pi_p}>\lambda\right)d\lambda + p\int_{\lambda\geq\lambda_0}4\lambda^{p-3}\left\|\frac{u-S_{m,A_\lambda}u}{\Pi_p}\right\|^2_{L^2(\Omega_p,d\mu_p)}d\lambda.$$

Let us notice that, for any function v of $L^2(\mathbf{R}^n,dx)$, we have

$$\left\|\frac{v}{\Pi_p}\right\|^2_{L^2(\Omega_p,d\mu_p)} = \|v\|^2_{L^2(\Omega_p,dx)};$$

so we have

$$\lambda_0^{p-2}p\int_{\lambda\leq\lambda_0}\lambda\mu_p\left(\frac{|u|}{\Pi_p}>\lambda\right)d\lambda \leq \lambda_0^{p-2}\|u\|^2_{L^2(\Omega_p,dx)} \leq (4C)^{p-2}A_0^2\|u\|^p_{H(m)}$$

and then

$$\left\|\frac{u}{\Pi_p}\right\|^p_{L^p(\Omega_p,d\mu_p)} \leq (4C)^{p-2}A_0^2\|u\|^p_{H(m)} + p\int_{\lambda\geq\lambda_0}4\lambda^{p-3}\|u-S_{m,A_\lambda}u\|^2_{L^2(\mathbf{R}^n,dx)}d\lambda.$$

Now let us estimate $\|u-S_{m,A_\lambda}u\|^2_{L^2}$. We apply the localized Cotlar lemma 1.11 with $\theta_Y = \psi_Y$ and $u_Y(x) = (1-\chi_{Y,m,A_\lambda})(x)\varphi_Y^w u(x)$; so we get

$$\|u-S_{m,A_\lambda}u\|^2_{L^2} \leq C\int(1+g_{2,Y}^{-1}(x-U_{1,Y}))^{-N}(1-\chi_{Y,m,A_\lambda}(x))^2 \times|\varphi_Y^w u(x)|^2|g_Y|^{\frac{1}{2}}dY\,dx.$$

So we infer that

$$p\int_{\lambda\geq\lambda_0}\lambda^{p-3}\|u-S_{m,A_\lambda}u\|_{L^2}^2d\lambda\leq C\int(1+g_{2,Y}^{-1}(x-U_{1,Y}))^{-N}$$
$$\times(1-\chi_{Y,m,A_\lambda})(x))^2|\varphi_Y^w u(x)|^2\lambda^{p-3}|g_Y|^{\frac{1}{2}}dYdxd\lambda.$$

By definition of χ_{Y,m,A_λ} and A_λ, we have

$$\lambda\leq 4C\|u\|_{H(m)}m(x,\eta)^{2/(p-2)}=\lambda(x,\eta).$$

So we get

$$\begin{aligned} I_p &\stackrel{\text{def}}{=} p\int_{\lambda\geq\lambda_0}\lambda^{p-3}\|u-S_{m,A_\lambda}u\|_{L^2}^2d\lambda \\ &\leq \frac{p}{p-2}\int(1+g_{2,Y}^{-1}(x-U_{1,Y}))^{-N_0}\lambda(x,\eta)^{p-2}dYdx \\ &\leq C_N^p\|u\|_{H(m)}^{p-2}\int(1+g_{2,Y}^{-1}(x-U_{1,Y}))^{-N}m^2(x,\eta)|\varphi_Y^w u(x)|^2dYdx. \end{aligned}$$

As m is a g-weight, we have $m^2(x,\eta)\leq Cm^2(y,\eta)(1+g_{2,Y}^{-1}(x-U_{1,Y}))^{N_0}$; so we get

$$p\int_{\lambda\geq\lambda_0}\lambda^{p-3}\|u-S_{m,A_\lambda}u\|_{L^2}^2d\lambda\leq C\|u\|_{H(m)}^p.$$

This proves theorem 1.13.

Now let us prove the theorem 1.14. The main step consists in the proof of the following interpolation lemma.

Lemma 3.1 *Let us assume the hypotheses of theorem 1.14. For any strictly positive real number ε, a constant C_ε exists so that such that, for any u in $H(m)$, we have*

$$\left\|\frac{u}{\Pi_p}\right\|_{L^p(\Omega_p;d\mu_p)}\leq C_\varepsilon\|u\|_{L^2(\Omega_p;dx)}+\varepsilon\|u\|_{H(m)}.$$

Let us start from inequality (20) and assume that $\varepsilon<1$. A strictly positive real number A_ε exists so that

$$\forall x\in\Omega_p\,,\ \sup_{B\geq A_\varepsilon}\Pi_p(B,x)\leq\varepsilon\Pi_p(x).$$

Consider the constant C of inequality (19) of theorem 2.2 and let us define

$$A_{\lambda,\varepsilon}=\left(\frac{\lambda}{4C\|u\|_{H(m)}\varepsilon}\right)^{(p-2)/2}\quad\text{and}\quad\lambda_{0,\varepsilon}=4C\varepsilon\|u\|_{H(m)}A_\varepsilon^{\frac{2}{p-2}}.$$

The theorem 2.2 implies that $\|S_{m,A_{\lambda,\varepsilon}}\|_{L^\infty}\leq\lambda/2$. So, we have

$$\left\|\frac{u}{\Pi_p}\right\|_{L^p(\Omega_p;d\mu_p)}^p\leq\lambda_{0,\varepsilon}^{p-2}p\|u\|_{L^2(\Omega_p,d\mu_p)}^2+p\int_{\lambda\geq\lambda_0}\lambda^{p-3}\|u-S_{m,A_{\lambda,\varepsilon}}u\|_{L^2}^2d\lambda.$$

Applying the localized Cotlar lemma 1.11, and using the fact that, by definition of $\chi_{Y,m,A}$, we have

$$\left\|\frac{u}{\Pi_p}\right\|^p_{L^p(\Omega_p;d\mu_p)} \leq \lambda_{0,\varepsilon}^{p-2} p\|u\|^2_{L^2(K,d\mu_p)}$$
$$+ C_{p,N}\int_{\lambda\leq\lambda_\varepsilon(x,\eta)}(1+g_{2,Y}^{-1}(x-U_{1,Y}))^{-N}|\varphi_Y^w u(x)|^2\lambda^{p-3}|g_Y|^{\frac{1}{2}}dY\,dxd\lambda$$

with $\lambda_\varepsilon(x,\xi) \stackrel{\text{def}}{=} 4C\varepsilon\|u\|_{H(m)}m(x,\eta)^{\frac{2}{p-2}}$. Along the sames lines as in the proof of theorem 1.13 and using the definition of $\lambda_{0,\varepsilon}$ we proof that

$$\left\|\frac{u}{\Pi_p}\right\|^p_{L^p(\Omega_p;d\mu_p)} \leq C_p A_\varepsilon^2\|u\|^{p-2}_{H(m)}\|u\|^2_{L^2(K,d\mu_p)} + C_p\varepsilon^{p-2}\|u\|^p_{H(m)}.$$

This implies the lemma. Using theorem 1.9, the theorem comes from the above lemma 3.1 with standard functionnal analysis arguments.

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Jean-Yves Chemin
Analyse Numérique, Tour 55-65, 5ème étage
Université Pierre and Marie CURIE, 4 Place Jussieu
75230 Paris Cedex 05, France
and
Chao-Jiang Xu
Institut de Mathématiques, Université de Wuhan
Wuhan 430072, R. P. China