Calc. Var. 5, 323–343 (1997)

Higher interior regularity for quasilinear subelliptic systems

Chao-Jiang Xu[1,⋆], **Claude Zuily**[2]

[1] Wuhan University, Institute of Mathematics, 430072 Wuhan, P.R. of China
[2] Université de Paris Sud, Département de Mathématiques, F-91405 Orsay Cedex, France
e-mail: claude.zuily@math.u-psud.fr

Received August 8, 1995 / In revised form April 9, 1996 / Accepted February 26, 1996

1. Introduction

This work deals with interior regularity for solutions of quasilinear systems of the form

$$(1.1)\qquad \sum_{i,j=1}^{m} X_j^*\left(a^{ij}(x,u(x))X_i u^\alpha(x)\right) = f^\alpha(x,u(x),Xu(x)), \quad 1 \le \alpha \le N\,,$$

where $X = (X_1,\ldots,X_m)$ is a set of C^∞ vector fields satisfying Hörmander's condition in a neighborhood of the closure of an open set Ω in $\mathbb{R}^n$, $n \ge 2$, and the matrix $(a^{ij}(x,y))$ is symmetric and positive definite. The right hand side in (1.1) will be assumed to be at most of quadratic growth in Xu, which means that

$$(1.2)\qquad ||f(x,y,p)|| \le a\,|p|^2 + b \ \text{ for } (x,y,p) \text{ in } \Omega\times\mathbb{R}^N\times\mathbb{R}^{mN}\,.$$

We shall consider weak solutions of (1.1) *i.e.* solutions belonging to the space $M^1(\Omega) = \left\{u \in L^2(\Omega,\mathbb{R}^N) : X_i u \in L^2(\Omega,\mathbb{R}^N),\, i = 1,\ldots,m\right\}$. A particular case of our main result will be the following

Theorem 1.1. *Assume that the data a^{ij} and f^α are C^∞ functions of their arguments and that f^α satisfies (1.2). Then every weak solution $u \in M^1(\Omega)$ of (1.1) which is continuous in Ω is C^∞ in Ω.*

Non linear equations and systems involving vector fields have received much attention during the recent years. For equations coming from variational problems Xu [X1] has proved the existence of a Hölder continuous weak solution. More recently systems which are particular case of (1.1) appear in connection with existence of subelliptic harmonic maps in a recent work by Jost and Xu [JX] and

⋆ The work of first author is partially supported by NSF of China.

again the existence of a Hölder continuous weak solution is proved. For equations, G. Lu, [L] has recently announced the Hölder continuity of weak solutions. Our Theorem 1.1 applies to all these cases and provides higher regularity of weak solutions.

Our work has been, of course, greatly influenced by the elliptic case (*i.e.* $m = n$, $X_i = \frac{\partial}{\partial x_i}$), where fairly complete results have been obtained through the works of Campanato [C], Giaquinta-Giusti [G] etc... There are two main difficulties in our case: the lack of commutation and the lack of homogeneity of the vector fields. The first one does not allow us to differentiate the equation. Instead, after lifting the vector fields as in Rothschild-Stein [RS], we prove a lemma which allows us to handle the first derivative as the solution itself. The same lemma allows us to prove precise Sobolev inequalities on balls which, in the elliptic case, are proved by scaling from inequalities on the unit ball. As soon as this has been done, the proof goes along the same lines as in Giaquinta [G], using the Poincaré inequality proved by Jerison [J], the maximal regularity results of Rothschild-Stein [RS] and some a priori estimates proved in Xu [X2].

This work has partially been done while the first author was visiting the University of Paris XI-Orsay. He would like to thank this institution for support and hospitality.

2. Geometry of vector fields and functions spaces

In this section we collect some classical facts and we prove some preliminary results.

2.1. Let $(X_1, \ldots, X_m)$ be a set of real and C^∞ vector fields, in an open set $\tilde{\Omega}$ of $\mathbb{R}^n$, $n \geq 2$, satisfying Hörmander's condition up to the order r. By the Rothschild-Stein lifting theorem (see [RS]), the vector fields $(X_1, \ldots, X_m)$ can be lifted by adding extra variables to free vector fields $(\tilde{X}_1, \ldots, \tilde{X}_m)$ on $\tilde{\Omega} \times T \subset R^n \times R^{d-n}$, where T is the unit ball in $\mathbb{R}^{d-n}$ and

$$\tilde{X}_j = X_j + \sum_{k=1}^{d-n} \lambda_{kj}(x,t)\partial_{t_k} .$$

Since for $u = u(x)$, we have $X_j u(x) = \tilde{X}_j u(x)$, then if $u(x)$ is a solution of equation (1.1), it is also a solution of following equation

$$\sum_{ij=1}^{m} \tilde{X}_j^*(a^{ij}(x, u(x))\tilde{X}_i u^\alpha(x)) = \tilde{f}^\alpha(x, t, u(x), \tilde{X}u(x)), \;\; 1 \leq \alpha \leq N .$$

So we may assume without loss of generality that these vector fields $(X_1, \ldots, X_m)$ are free up to the order r. As usual, we associate to this set a continuous subelliptic local metric $\rho(x, y)$ which satisfies on each compact set K in $\tilde{\Omega}$

$$C^{-1}|x-y| \leq \rho(x,y) \leq C\,|x-y|^{\frac{1}{r}} . \tag{2.1}$$

For small positive δ one can then define the ball $B(x,\delta)$, of center x and radius δ, with respect to this metric. Then (see [NSW], [X3])

$$|B(x,\delta)| = |\lambda(x)|\,\delta^Q \tag{2.2}$$

where $|B|$ denotes the Lebesgue measure, Q is the homogeneous dimension and $0 < C_1 \leq |\lambda(x)| \leq C_2$ on any compact K in Ω.

2.2. The functions spaces (see [X2])

Let Ω be an open set in $\tilde{\Omega}$. For $k \in \mathbb{N}$ and $\alpha \in]0,1]$ we define

$$S^{0,\alpha}(\Omega) = \left\{ f \in L^\infty(\Omega) : [f]^X_{\alpha,\Omega} = \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{\rho(x,y)^\alpha} < +\infty \right\} \tag{2.3}$$

$$S^{k,\alpha}(\Omega) = \left\{ f \in S^{0,\alpha}(\Omega) : X_I f \in S^{0,\alpha}(\Omega),\ |I| \leq k \right\} \tag{2.4}$$

where, as usual, $X_I = X_{i_1} \ldots X_{i_p}$ if $I = (i_1, \ldots, i_p)$.

Then $S^{k,\alpha}(\Omega)$ is a Banach space when it is endowed with the obvious norm deduced from m its definition. Moreover by Hörmander's condition and (2.1) we have

$$S^{kr,\alpha}(\Omega) \subset C^{k,\frac{\alpha}{r}}(\Omega) \ \text{ for all } \ k \in \mathbb{N}. \tag{2.5}$$

We can then define the spaces $S^{k,\alpha}_{\mathrm{loc}}(\Omega)$ as usual.

We shall also consider, for any integer k, the Sobolev space

$$M^k(\Omega) = \left\{ f \in L^2(\Omega) : X_I f \in L^2(\Omega),\ |I| \leq k \right\}. \tag{2.6}$$

Endowed with the obvious scalar product this is a Hilbert space. The closure of $C_0^\infty(\Omega)$ in $M^k(\Omega)$ will be denoted by $M_0^k(\Omega)$. It is proved in [X3] that for $k > \frac{Q}{2}$, elements in $M^k(\Omega)$ are continuous functions in Ω.

Let us now recall the Poincaré-type inequalities.

For any open set Ω with $\overline{\Omega} \subset \tilde{\Omega}$ one can find positive constants R_0, C such that for any x_0 in Ω and R in $]0, R_0]$ we have

$$||u||_{L^2(B(x_0,R))} \leq C\,R \sum_{j=1}^m ||X_j u||_{L^2(B(x_0,R))}, \quad \forall u \in M_0^1(B(x_0,R)) \tag{2.7}$$

$$||u - u_{x_0,R}||_{L^2(B(x_0,R))} \leq C\,R \sum_{j=1}^m ||X_j u||_{L^2(B(x_0,R))}, \quad \forall u \in M^1(B(x_0,R)) \tag{2.8}$$

where $u_{x_0,R} = \frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} u\,dx$ is the average of u on $B(x_0,R)$.

The inequality (2.7) is easy but (2.8) is hard and has been proved by Jerison [J].

As in the Euclidian case the space $S^{0,\alpha}$ can be characterized in terms of integral conditions.

Proposition 2.1. *Let u be in $L^2(\Omega)$, then the following conditions are equivalent.*

a) $u \in S^{0,\alpha}_{\mathrm{loc}}(\Omega)$.
b) One can find positive constants R_0 and C such that for any R in $]0, R_0]$ and any x_0 in Ω such that $B(x_0, 2R)$ is contained in Ω one has

$$\int_{B(x_0,R)} |u(x) - u_{x_0,R}|^2 \, dx \le C \, |B(x_0, R)| \, R^{2\alpha} . \tag{2.9}$$

Since the metric ρ is continuous the proof is very close to that in the Euclidian case (see [C], [G]).

2.3. The following results are crucial for the sequel. Let P be a second order differential operator of the form

$$P = \sum_{i,j=1}^{m} X_j^*(a_{ij} X_i) \tag{2.10}$$

were a_{ij} are functions and denote by A the matrix $(a_{ij})_{1\le i,j\le m}$.

Lemma 2.2. *For any x_0 in Ω one can find coordinates in a neighborhood V of x_0 and a matrix $T(x) \in GL(m, \mathbb{R})$ which is C^∞ in V such that if we set $(Y_1, \ldots, Y_m) = T(x)(X_1, \ldots, X_m)$ we have*

i) $Y_j = \dfrac{\partial}{\partial x_j} + \displaystyle\sum_{k=m+1}^{n} g_{jk} \dfrac{\partial}{\partial x_k}, \; 1 \le j \le m,$

ii) the set $(Y_1, \ldots, Y_m)$ satisfies Hörmander's condition of order r and is free up to the order r in V,

iii) $P = \displaystyle\sum_{i,j=1}^{m} Y_j^*(b_{ij} Y_i)$ *where $(b_{ij}) = {}^tC(x)AC(x)$ and $C(x)$ is an invertible matrix with C^∞ entries in V.*

Corollary 2.3. *Let us set $x = (x', x'') \in \mathbb{R}^n$ with $x' \in \mathbb{R}^m$ and for $k \in \mathbb{N}^*$ and $(C_J)_{|J|\le k}$ in $\mathbb{R}^d$, $h(x') = \sum_{|J|\le k} \frac{1}{J!} C_J (x' - x_0')^J$. Then for any I with $|I| = k$ we have $Y_I h = C_I$.*

Proof of Lemma 2.2.
i) Since the vector fields $(X_1(x_0), \ldots, X_m(x_0))$ are linearly independent and commute one can find coordinates near x_0 in which $X_j(x_0) = \frac{\partial}{\partial x_j}$, $1 \le j \le m$. In these coordinates $X_j = \sum_{k=1}^{n} c_{jk}(x) \frac{\partial}{\partial x_k}$ with $c_{jk}(0) = 0$ if $j \ne k$ and $c_{jj}(0) = 1$, $1 \le j \le m$. Let us set $Z_j = \sum_{k=1}^{m} c_{jk}(x) \frac{\partial}{\partial x_k}$, $R_j = \sum_{k=m+1}^{n} c_{jk}(x) \frac{\partial}{\partial x_k}$, $Z = (Z_1, \ldots, Z_m)$, $R = (R_1, \ldots, R_m)$, $\partial = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$, $C = (c_{jk})_{1\le j,k\le m}$. Then $X = Z + R$ and

$Z = C\partial$. Since $C(0) = \mathrm{Id}$ one can find a neighborhood V of x_0 in which C is invertible with C^∞ entries. Let us set $Y = C^{-1}X$. Then $Y = C^{-1}(Z+R) = \partial + C^{-1}R$ which proves i).

ii) is obvious since $X = CY$ where C is smooth and invertible.

iii) We have $P = \sum_{i,j=1}^{m} X_j^*(a_{ij}X_i) = \sum_{i,j=1}^{m}\sum_{k,\ell=1}^{m} Y_k^* c_{jk}\, a_{ij}\, c_{i\ell}\, Y_\ell = \sum_{k,\ell=1}^{m} Y_k^* b_{k\ell} Y_\ell$ with $B = {}^t C A C$. In particular B is positive definite if A is so.

Proof of Corollary 2.3.

It is obvious if we use i) of Lemma 2.2.

With the aid of the above results one can prove the following

Proposition 2.4. *Let $(Y_j)_{1\le j\le m}$ be the vector fields defined in Lemma 2.2 and $k > \frac{Q}{2}$. Then one can find positive constants R_0, C independent of x_0 such that for any $R \le R_0$ and u in $M^k(\Omega)$ we have*

$$\sup_{B\left(x_0,\frac{R}{4}\right)} |u| \le C \sum_{|J|\le k} R^{|J|-\frac{Q}{2}}\, ||Y_J u||_{L^2(B(x_0,R))}. \tag{2.11}$$

Let us remark that in the Euclidian case or in the case of left invariant vector fields on nilpotent Lie groups, this inequality can be deduced by scaling from the Sobolev embedding in a fixed domain. The problem here is then the lack of homogeneity.

Proof of Proposition 2.4. For $1 \le \ell \le k$ let us set

$$p_{\ell,x_0,R}(x' - x_0') = \sum_{|J|\le \ell} \frac{1}{J!}\,(Y_J u)_{x_0,R}(x'-x_0')^J \tag{2.12}$$

where $(v)_{x_0,R}$ denotes the average on the ball $B(x_0,R)$ and $x' \in \mathbb{R}^m$.
First claim : for any $1 \le \ell \le k$ and for small R we have

$$\int_{B(x_0,R)} \left|u(x) - p_{\ell-1,x_0,R}(x'-x_0')\right|^2 dx \le C\, R^{2\ell} \sum_{|I|=\ell} ||Y_I u||_{L^2(B(x_0,R))}. \tag{2.13}$$

According to Corollary 2.3, this claim can be proved by a straightforward induction from the Poincaré inequality (2.8).

Second claim : we have

$$\left|u_{x_0,\frac{1}{2}R} - u_{x_0,R}\right| \le C\, R^{k-\frac{Q}{2}} \sum_{|I|=k} ||Y_I u||_{L^2(B(x_0,R))}. \tag{2.14}$$

Here k and x_0 are fixed so we shall write for short p_R, u_R, B_R instead of $p_{k-1,x_0,R}$, $u_{x_0,R}$, $B(x_0,R)$. Then we set $p_R^0 = p_R - u_R$. We have

$$\left|B_{\frac{1}{2}R}\right| \cdot \left|u_{\frac{1}{2}R} - u_R\right|^2 = \int_{B_{\frac{1}{2}R}} \left|u_{\frac{1}{2}R} - u_R\right|^2 dy$$
$$= \int_{B_{\frac{1}{2}R}} \left|p_{\frac{1}{2}R} - p^0_{\frac{1}{2}R} - u(y) + u(y) - (p_R - p^0_R)\right|^2 dy\,.$$

Therefore

$$\left|u_{\frac{1}{2}R} - u_R\right|^2 \le C\,R^{-Q}\Big\{ \underbrace{\int_{B_{\frac{1}{2}R}} \left|u(y) - p_{\frac{1}{2}R}\right|^2 dy}_{(1)} + \underbrace{\int_{B_R} \left|u(y) - p_R\right|^2 dy}_{(2)} + \underbrace{\int_{B_R} \left|p^0_R - p^0_{\frac{1}{2}R}\right|^2 dy}_{(3)} \Big\}.$$

It follows from (2.13) that

$$(1) + (2) \le C\,R^{2k} \sum_{|I|=k} ||Y_I u||^2_{L^2(B(x_0,R))}\,. \tag{2.15}$$

According to (2.12), since on B_R one has $|x' - x'_0| \le R$, we have

$$(3) \le C \sum_{1\le |J|\le k-1} R^{Q+2|J|} \left|(Y_J u)_R - (Y_J u)_{\frac{1}{2}R}\right|^2. \tag{2.16}$$

We deduce from (2.15) and (2.16) that for any $k \ge 1$ we have

$$\begin{aligned} &\left|u_{\frac{1}{2}R} - u_R\right|^2 \\ &\le C\Big\{R^{2k-Q} \sum_{|I|=k} ||Y_I u||^2_{L^2(B_R)} + \sum_{1\le|J|\le k-1} R^{2|J|} \left|(Y_J u)_R - (Y_J u)_{\frac{1}{2}R}\right|^2\Big\}. \end{aligned} \tag{2.17}$$

We now prove by induction on ℓ with $1 \le \ell \le k-1$ that for every K such that $|K| = \ell$ we have for $u \in M^k(B_R)$,

$$\left|(Y_K u)_{\frac{1}{2}R} - (Y_K u)_R\right|^2 \le C\,R^{-Q+2(k-|K|)} \sum_{|I|=k} ||Y_I u||_{L^2(B_R)}\,. \tag{2.18$_\ell$}$$

This is true for $\ell = k-1$. Indeed if $|K| = k-1$

$$\left|B_{\frac{1}{2}R}\right|^2 \left|(Y_K u)_{\frac{1}{2}R} - (Y_K u)_R\right|^2 = \left|\int_{B_{\frac{1}{2}R}} \Big(Y_K u(y) - (Y_K u)_R\Big)\,dy\right|^2$$
$$\le \left|B_{\frac{1}{2}R}\right| \int_{B_R} \left|Y_K u(y) - (Y_K u)_R\right|^2 dy$$

so by the Poincaré inequality (2.8) we get

$$\left|(Y_K u)_{\frac{1}{2}R} - (Y_K u)_R\right|^2 \leq C\, R^{-Q+2} \sum_{|I|=k} ||Y_I u||_{L^2(B_R)}\,.$$

Now assume (2.18) true for k, $(k-1), \ldots, \ell$ and let $|K| = \ell - 1 \geq 1$. Let us apply (2.17) with $Y_K u$ instead of u and $k - \ell + 1$ instead of k. We get

$$\left|(Y_K u)_{\frac{1}{2}R} - (Y_K u)_R\right|^2 \leq C \Bigg\{ R^{2(k-\ell+1)-Q} \sum_{|I|=k} ||Y_I u||^2_{L^2(B_R)} + \sum_{1\leq |J| \leq k-\ell} R^{2|J|} \left|(Y_J\, Y_K u)_{\frac{1}{2}R} - (Y_J\, Y_K u)_R\right|^2 \Bigg\}.$$

Since $|J| + |K| \geq \ell$ the second term in the right hand side can be estimated by $(2.18)_\ell$ so we get $(2.18)_{\ell-1}$.

Inserting (2.18) in (2.17) gives (2.14).

Now using (2.14) with $R_j = 2^{-j} R$, $j = 0, \ldots, q$ and taking the sum we obtain

$$\left|u_{x_0,R} - u_{x_0,R_q}\right| \leq C \sum_{j=0}^{q-1} 2^{-j\left(k-\frac{Q}{2}\right)} R^{k-\frac{Q}{2}} \sum_{|I|=k} ||Y_I u||_{L^2(B(x_0,R))}\,.$$

Now since $k > \frac{Q}{2}$, the function u is continuous, then for any $\varepsilon > 0$, there is $q_0 > 0$ such that if $q \geq q_0$, we have $|u(x) - u(x_0)| \leq \varepsilon$ for all $x \in B(x_0, R_q) \subset \{y \in \mathbb{R}^n; |y - x_0| \leq 2^{-q_0} R_0\}$, and

$$|u_{x_0,R_q} - u(x_0)| = \left| |B(x_0, R_q)|^{-1} \int_{B(x_0,R_q)} (u(x) - u(x_0))\,dx \right| \leq \varepsilon\,.$$

So $\lim_{q\to+\infty} u_{x_0,R_q} = u(x_0)$ and the sum in the right hand side converges. Therefore

$$\left|u(x_0) - u_{x_0,R}\right| \leq C\, R^{k-\frac{Q}{2}} \sum_{|I|=k} ||Y_I u||_{L^2(B(x_0,R))}\,.$$

Since $|u_{x_0,R}| \leq C\, R^{-\frac{Q}{2}}\, ||u||_{L^2(B(x_0,R))}$ we get

$$|u(x_0)| \leq C \sum_{|J|\leq k} R^{|J|-\frac{Q}{2}}\, ||Y_J u||_{L^2(B(x_0,R))}\,. \tag{2.18}$$

An obvious covering argument gives (2.11). ♢

We recall now that we can construct cut off functions which are adapted to our balls.

Lemma 2.5. (see [X2]) *For small R, c_1, c_2 with $c_1 < c_2$, there exists a function ζ in $C_0^\infty(B(x_0, c_2 R))$ such that $\zeta = 1$ on $B(x_0, c_1 R)$, $0 \leq \zeta \leq 1$ and*

$$|X_I\, \zeta| \leq c_I\, R^{-|I|}\,, \tag{2.19}$$

where c_I are independent of R.

Finally we quote from [G] the following useful lemma.

Lemma 2.6. ([G] lemma 2.1) *Let $\phi(R)$ be a non negative, non decreasing function on $0 < R \leq R_1$. Assume*

$$\phi(R) \leq A\left\{ \left(\left(\frac{R}{R_0}\right)^{\alpha} + \varepsilon \right) \phi(R_0) + B\, R_0^{\beta} \right\} \tag{2.20}$$

for all R, R_0, $0 < R \leq R_0 \leq R_1$, with A, B, ε, α, β being non negative constants $A \geq 1$, $\beta < \alpha$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$ such that if $\varepsilon \leq \varepsilon_0$

$$\phi(R) \leq C\left\{ \left(\frac{R}{R_0}\right)^{\beta} \phi(R_0) + B\, R^{\beta} \right\} \tag{2.21}$$

for all $0 < R \leq R_0 \leq R_1$ with $C = C(A, \alpha, \beta)$.

3. Regularity for solutions of linear equations

3.1. Linear homogeneous equations with constant coefficients

In the sequel we shall use the Einstein summation convention. We shall consider in this section equations of the following type.

$$Lu = a^{ij} X_i X_j u = 0 \tag{3.1}$$

where (a^{ij}) is a constant coefficient matrix which is assumed to be real, symmetric and satisfies

$$\lambda |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2, \text{ for all } \xi \text{ in } \mathbb{R}^m, \tag{3.2}$$

where λ and Λ are positive constants.

A weak solution of (3.1) in Ω will be a function $u \in M^1(\Omega)$ such that

$$\int_{\Omega} a^{ij} X_j u \cdot X_i^* \varphi \, dx = 0, \text{ for all } \varphi \text{ in } M_0^1(\Omega). \tag{3.3}$$

For these solutions we have a Caccioppoli-type inequality.

Proposition 3.1. *Let $u \in M^1(\Omega)$ be a weak solution of (3.1). Then $u \in C^\infty(\Omega)$ and for all x_0 in Ω, all R in $]0, 1[$ such that $B(x_0, R) \subset \Omega$ and all $k \in \mathbb{N}$ we have*

$$||u||_{M^k\left(\left(x_0, \frac{R}{2^k}\right)\right)} \leq C\, R^{-k}\, ||u||_{L^2(B(x_0, R))} \tag{3.4}$$

where the constant C is independent of x_0, R and u.

Proof. The function u is C^∞ since L is hypoelliptic. We shall prove (3.4) by induction on k. We shall write B_R instead of $B(x_0, R)$ and $|Xu| = \left(\sum_{j=1}^{m} |X_j u|^2\right)^{1/2}$.

Let $\zeta \in C_0^\infty(B_R)$, $0 \le \zeta \le 1$, $\zeta = 1$ on $B_{\frac{R}{2}}$ and $|X_j\zeta| \le C\,R^{-1}$, $j = 1,\dots,m$. Let us take in (3.3) $\varphi = u\,\zeta^2 \in M_0^1(\Omega)$. We get

$$\int_{B_R} a^{ij}\,X_i u\cdot X_j u\cdot\zeta^2\,dx = \int_{B_R} a^{ij}\,X_j u\cdot c_i\cdot u\cdot\zeta^2\,dx - 2\int_{B_R} a^{ij}\,X_j u\cdot u\cdot\zeta\cdot X_i\zeta\,dx\,.$$

Using (3.2) we deduce

$$\lambda\int_{B_R}|Xu|^2\zeta^2\,dx \le C\,R^{-1}\int_{B_R}|Xu|\cdot|u|\cdot\zeta\,dx$$
$$\le \varepsilon\int_{B_R}|Xu|^2\zeta^2\,dx + \frac{C_\varepsilon}{R^2}\int_{B_R}|u|^2\,dx\,.$$

Taking ε small enough and since $\zeta = 1$ on $B_{\frac{R}{2}}$ we get (3.4) for $k = 1$. Assume now (3.4) true up to the order $k-1$ with $k \ge 2$. Let $\zeta \in C_0^\infty\big(B_{\frac{R}{2^{k-1}}}\big)$ be such $\zeta = 1$ on $B_{\frac{R}{2^k}}$, $0 \le \zeta \le 1$ and $|X_I\zeta| \le C_k\,R^{-|I|}$. Using the maximum regularity for the operator L proved by Rothschild-Stein [RS] we get

$$\sum_{|I|=k} ||X_I u||_{L^2\left(B_{\frac{R}{2^k}}\right)} \le ||\zeta u||_{M^k(\Omega)} \le C\left\{||L\zeta u||_{M^{k-2}(\Omega)} + ||\zeta u||_{L^2(\Omega)}\right\}.$$

Since $Lu = 0$ we have $L\zeta u = [L,\zeta]u$ so, using the Leibniz formula, the estimates on ζ and the induction we get (3.4) for k.

We begin now to prove estimates for the weak solutions of (3.1). In the sequel x_0 is a fixed point in Ω and B_R stands for $B(x_0, R)$.

Theorem 3.2. *There exist positive constants C_0, R_0 such that for every x_0 in Ω, for any $R \le R_0$ and any weak solution $u \in M^1(\Omega)$ of (3.1) we have*

$$\int_{B_R}|u|^2\,dx \le C_0\left(\frac{R}{R_0}\right)^Q\int_{B_{R_0}}|u|^2\,dx \tag{3.5}$$

$$\int_{B_R}|Xu|^2\,dx \le C_0\left(\frac{R}{R_0}\right)^Q\int_{B_{R_0}}|Xu|^2\,dx \tag{3.6}$$

$$\int_{B_R}|u - u_{x_0,R}|^2\,dx \le C_0\left(\frac{R}{R_0}\right)^{Q+2}\int_{B_{R_0}}|u - u_{x_0,R_0}|^2\,dx \tag{3.7}$$

$$\int_{B_R}|Xu - (Xu)_{x_0,R}|^2\,dx \le C_0\left(\frac{R}{R_0}\right)^{Q+2}\int_{B_{R_0}}|Xu - (Xu)_{x_0,R_0}|^2\,dx \tag{3.8}$$

where $(v)_{x_0,R}$ denotes the average of v on $B_R = B(x_0, R)$.

Proof. Without loss of generality, according to Lemma 2.2, one may assume that the vector fields X_j satisfy the conclusion of corollary 2.3.

If $\lambda_0 \in]0,1[$ is fixed, (3.5) to (3.8) are true (with a constant C_0 depending on λ_0) if $\lambda_0 < \frac{R}{R_0} \leq 1$. This is obvious for (3.5) and (3.6). For (3.7) and (3.8) this follows from the fact that $\int_{B_R} |u - u_{x_0,R}|^2\,dx = \inf_c \int_{B_R} |u-c|^2\,dx$.

i) Let k be a fixed integer, $k > \frac{Q}{2}$. Let us take $R \leq \frac{R_0}{2^{k+2}}$ where R_0 is such that $B(x_0,R_0) \subset \Omega$ and (2.11) is true for $\frac{R_0}{2^k}$. Then by (2.11)

$$\int_{B_R} |u|^2\,dx \leq |B_R| \sup_{B_{\frac{R_0}{2^{k+2}}}} |u|^2 \leq C\,|B_R| \sum_{|J|\leq k} R_0^{2|J|-Q}\,||X_J u||^2_{L^2\left(B_{\frac{R_0}{2^k}}\right)}.$$

Now by (3.4) we have

$$||X_J u||^2_{L^2\left(B_{\frac{R_0}{2^k}}\right)} \leq ||X_J u||^2_{L^2\left(B_{\frac{R_0}{2^{|J|}}}\right)} \leq C\,R_0^{-2|J|}\,||u||^2_{L^2(B_{R_0})}.$$

Therefore (3.5) follows from the fact that $|B_R| \leq C\,R^Q$.

ii) We take $R \leq \frac{R_0}{2^{k+3}}$ and R_0 as above. Then by (2.11) and (3.4)

$$\begin{aligned}\int_{B_R} |X_i u|^2\,dx &\leq C\,R^Q \sup_{B_{\frac{R_0}{2^{k+3}}}} |X_i u|^2 \leq C_1 R^2 \sum_{|J|\leq k} R_0^{2|J|-Q}\,||X_J X_i u||^2_{L^2\left(B_{\frac{R_0}{2^{k+1}}}\right)} \\ &\leq C\,R^Q \sum_{|J|\leq k} R_0^{2|J|-Q} R_0^{-2(|J|+1)}\,||u||_{L^2(B_{R_0})}.\end{aligned}$$

Since $u - u_{x_0,R_0}$ is still a weak solution of (3.1) we get

$$\int_{B_R} |X_i u|^2\,dx \leq C\left(\frac{R}{R_0}\right)^Q R_0^{-2} \int_{B_{R_0}} |u - u_{x_0,R_0}|^2\,dx.$$

Then (3.6) follows from the Poincaré inequality (2.8) applied to the right hand side and (3.7) from the same inequality applied to the left hand side.

iii) (3.8) will follow from

$$\int_{B_R} |X^2 u|^2\,dx \leq C\left(\frac{R}{R_0}\right)^Q R_0^{-2} \int_{B_{R_0}} |Xu|^2\,dx \tag{3.9}$$

where $|X^2 u|^2 = \sum_{i,j} |X_i X_j u|^2$. Indeed let us take in Corollary 2.3, $k = 1$ and $c_j = (X_j u)_{x_0,R_0}$ then $h(x) = c_0 + \sum_{j=1}^m c_j(x_j' - x_{0j}')$ satisfies $X^2 h = 0$ and $X_j h = (X_j u)_{x_0,R_0}$. Then $u - h$ is still a solution of (3.1). Therefore we get from (3.9)

$$\int_{B_R} |X^2 u|^2\,dx \leq C\left(\frac{R}{R_0}\right)^Q R_0^{-2} \int_{B_{R_0}} |Xu - (Xu)_{x_0,R_0}|^2\,dx. \tag{3.10}$$

Now the Poincaré inequality (2.8) implies

$$\int_{B_R} |Xu - (Xu)_{x_0,R}|^2\, dx \le C\, R^2 \int_{B_R} |X^2 u|^2\, dx\,. \tag{3.11}$$

Therefore (3.8) follows from (3.10) and (3.11). Let us prove (3.9). We take $R \le \frac{R_0}{2^{k+2}}$ and R_0 small. Then from (2.11) and (3.4) we get

$$\begin{aligned}\int_{B_R} |X^2 u|^2\, dx &\le C\, R^Q \sup_{B_{\frac{R_0}{2^{k+4}}}} |X^2 u|^2 \\ &\le C_1 R^Q \sum_{|J|\le k} R_0^{2|J|-Q}\, ||X_J X^2 u||^2_{L^2\left(B_{\frac{R_0}{2^{k+2}}}\right)} \\ &\le C_2 R^Q \sum_{|J|\le k} R_0^{2|J|-Q} R_0^{-2(|J|+2)}\, ||u||^2_{L^2(B_{R_0})} \\ &\le C_3 \left(\frac{R}{R_0}\right)^Q R_0^{-4}\, ||u||^2_{L^2(B_{R_0})}\,.\end{aligned}$$

Now, since $u - u_{x_0,R_0}$ is still a weak solution of (3.1) we get

$$\int_{B_R} |X^2 u|^2\, dx \le C_3 \left(\frac{R}{R_0}\right)^Q R_0^{-4} \int_{B_{R_0}} |u - u_{x_0,R_0}|^2\, dx\,,$$

so (3.9) follows from the Poincaré inequality (2.8) applied to the right hand side.

3.2. Linear equations with Hölder coefficients

We consider now an equation of the form

$$X_j^*\left(a^{ij}(x) X_i u\right) = X_j^* f^j + g\,. \tag{3.12}$$

The matrix (a^{ij}) is assumed to be real, symmetric with entries belonging to some $S^{0,\mu}(\Omega)$, where $\mu \in]0,1[$, and satisfies

$$\lambda |\xi|^2 \le a^{ij}(x)\xi_i \xi_j \le \Lambda |\xi|^2\,, \quad \forall x \in \Omega\,,\ \forall \xi \in \mathbb{R}^n \tag{3.13}$$

$$\left[a^{ij}\right]^X_{\Omega,\mu} \le H\,, \quad \forall i, j\,. \tag{3.14}$$

We shall assume moreover

$$f = (f^1, \dots, f^m) \in S^{0,\mu}(\Omega, \mathbb{R}^m) \text{ and } g \in L^\infty(\Omega)\,. \tag{3.15}$$

The purpose of this section is to prove the

Theorem 3.3. *Let $u \in M^1(\Omega)$ be a weak solution of (3.12). Under conditions (3.13), (3.14) and (3.15) we have $u \in S^{1,\mu}_{\rm loc}(\Omega)$ and for all x_0 in Ω there exists $R_0 > 0$ such that for $0 < R \le R_0$ we have*

$$\operatorname*{osc}_{B(x_0,R)} Xu \le C\, R^{\mu}\left\{ ||Xu||_{L^2(\Omega)} + [f]^X_{\Omega,\mu} + ||g||_{L^\infty(\Omega)} \right\} \tag{3.16}$$

where the constant C depends on $n, m, \lambda, \dots, H$ and $d(x_0, \partial\Omega)$ but is independent of R and u.

Proof. In what follows, x_0 will be a fixed point in Ω and $B_R = B(x_0, R)$. According to Proposition 2.1, our conclusion will follow (by (2.8)) from

$$\int_{B_R} |Xu|^2\, dx \le C_\varepsilon\, B\, R^{Q-\varepsilon}, \quad \text{for every small } \varepsilon > 0. \tag{3.17}$$

$$\int_{B_R} |Xu - (Xu)_{x_0,R}|^2\, dx \le C\, B^2\, R^{Q+2\mu}, \tag{3.18}$$

where $B = ||Xu||_{L^2(\Omega)} + [f]^X_{\Omega,\mu} + ||g||_{L^\infty(\Omega)}$ and C as in (3.16).

Proof of (3.17). We rewrite equation (3.12) as follows

$$X_j^*(a^{ij}(x_0)X_i u) = X_j^* f^j + g - X_j^*\big((a^{ij}(x) - a^{ij}(x_0))X_i u\big).$$

Let R_0 be such that Theorem 3.2 can be applied. Let $v \in M^1(B_{R_0})$ be a solution of the problem

$$\begin{cases} X_j\big(a^{ij}(x_0)X_i v\big) = 0 \ \text{ in } \ B_{R_0} \\ u - v \in M_0^1(B_{R_0}). \end{cases} \tag{3.19}$$

Then $u - v$ is a weak solution of the following equation on B_{R_0}.

$$X_j^*\big(a^{ij}(x_0)X_i(u-v)\big) = X_j^* f^j + g - X_j^*\big((a^{ij}(x) - a^{ij}(x_0))X_i u\big) - c_j\, a^{ij}(x_0)X_i v$$

since $X_j^* = -X_j + c_j$. Now, since $u - v \in M_0^1(B_{R_0})$ we get

$$\begin{aligned} \underbrace{\int_{B_{R_0}} a^{ij}(x_0)X_i(u-v)X_j(u-v)\,dx}_{(1)} &= \underbrace{\int_{B_{R_0}} f^j X_j(u-v)\,dx}_{(2)} \\ &+ \underbrace{\int_{B_{R_0}} g(u-v)\,dx}_{(3)} - \underbrace{\int_{B_{R_0}} \big(a^{ij}(x) - a^{ij}(x_0)\big) X_i u\, X_j(u-v)\,dx}_{(4)} \\ &- \underbrace{\int_{B_{R_0}} c_j\, a^{ij}(x_0) X_i v\,(u-v)\,dx}_{(5)}. \end{aligned} \tag{3.20}$$

It follows from (3.13) that

$$(1) \ge \lambda \int_{B_{R_0}} |X(u-v)|^2\, dx. \tag{3.21}$$

Now, since $u - v \in M_0^1(B_{R_0})$ we have

$$(2) = \int_{B_{R_0}} (f^j(x) - f^j(x_0)) X_j(u - v)\, dx\,,$$

so

$$|(2)| \leq \frac{\lambda}{10} \int_{B_{R_0}} |X(u - v)|^2\, dx + C\, R_0^{Q+2\mu} \left([f]^X_{\Omega,\mu}\right)^2. \tag{3.22}$$

Furthermore by the Poincaré inequality

$$\begin{aligned} |(3)| &\leq \left(\int_{B_{R_0}} |g|^2\, dx\right)^{1/2} \left(\int_{B_{R_0}} |u - v|^2\, dx\right)^{1/2} \\ &\leq C\, R_0^{\frac{Q}{2}+1}\, ||g||_{L^\infty(\Omega)} \left(\int_{B_{R_0}} |X(u - v)|^2\, dx\right)^{1/2}. \end{aligned}$$

Therefore

$$|(3)| \leq \frac{\lambda}{10} \int_{B_{R_0}} |X(u - v)|^2\, dx + C\, R_0^{Q+2}\, ||g||^2_{L^\infty(\Omega)}\,. \tag{3.23}$$

Now by (3.14),

$$|(4)| \leq \frac{\lambda}{10} \int_{B_{R_0}} |X(u - v)|^2\, dx + C\, H^2\, R_0^{2\mu} \int_{B_{R_0}} |Xu|^2\, dx\,. \tag{3.24}$$

Finally,

$$\begin{aligned} |(5)| &\leq C_0 \left(\int_{B_{R_0}} |Xv|^2\, dx\right)^{1/2} \left(\int_{B_{R_0}} |u - v|^2\, dx\right) \\ &\leq C_1\, R_0 \left(\int_{B_{R_0}} |Xv|^2\, dx\right)^{1/2} \left(\int_{B_{R_0}} |X(u - v)|^2\, dx\right)^{1/2} \end{aligned}$$

$$|(5)| \leq \frac{\lambda}{10} \int_{B_{R_0}} |X(u - v)|^2\, dx + C_2\, R_0^2 \int_{B_{R_0}} |Xv|^2\, dx\,.$$

Therefore

$$\begin{aligned} |(5)| \leq \frac{\lambda}{10} \int_{B_{R_0}} |X(u - v)|^2\, dx &+ 2\, C_2\, R_0^2 \int_{B_{R_0}} |X(u - v)|^2\, dx \\ &+ 2\, C_2\, R_0^2 \int_{B_{R_0}} |Xu|^2\, dx \end{aligned} \tag{3.25}$$

where C_2 depends only on H in (3.14) and on the coefficients of X_j on Ω. Taking $2\, C_2\, R_0^2 \leq \frac{\lambda}{10}$ and using (3.20) to (3.25) we get, since $\mu < 1$,

(3.26)
$$\int_{B_{R_0}} |X(u-v)|^2\,dx \le C\left\{R_0^{Q+2\mu}\left[\left([f]^X_{\Omega,\mu}\right)^2 + ||g||^2_{L^\infty(\Omega)}\right] + R_0^{2\mu}\int_{B_{R_0}} |Xu|^2\,dx\right\}.$$

Now for $R \le R_0$ we write

$$\int_{B_R} |Xu|^2\,dx \le 2\int_{B_R} |Xv|^2\,dx + 2\int_{B_R} |X(u-v)|^2\,dx\,. \tag{3.27}$$

Since v is a weak solution of (3.19) it follows from (3.6) in Theorem 3.2,

$$\begin{aligned}\int_{B_R} |Xv|^2\,dx &\le C\left(\frac{R}{R_0}\right)^Q \int_{B_{R_0}} |Xv|^2\,dx \\ &\le 2C\left(\frac{R}{R_0}\right)^Q \left\{\int_{B_{R_0}} |Xu|^2\,dx + \int_{B_{R_0}} |X(u-v)|^2\,dx\right\}.\end{aligned} \tag{3.28}$$

We deduce from (3.27), (3.28) and (3.26) that

$$\begin{aligned}&\int_{B_R} |Xu|^2\,dx \le \\ &\le C\left\{\left[\left(\frac{R}{R_0}\right)^Q + R_0^{2\mu}\right]\int_{B_{R_0}} |Xu|^2\,dx + R_0^{Q+2\mu}\left(||g||^2_{L^\infty(\Omega)} + \left([f]^X_{\Omega,\mu}\right)^2\right)\right\}.\end{aligned} \tag{3.29}$$

It follows from Lemma 2.5 that for every $\varepsilon > 0$,
(3.30)
$$\int_{B_R} |Xu|^2\,dx \le C'\left\{\left(\frac{R}{R_0}\right)^{Q-\varepsilon}\int_{B_{R_0}} |Xu|^2\,dx + R_0^{Q-\varepsilon}\left(||g||^2_{L^\infty(\Omega)} + \left([f]^X_{\Omega,\mu}\right)^2\right)\right\}.$$

This proves (3.17).

Proof of (3.18). Let us take $0 < R_1 \le R_0$ and v solution of (3.19) on B_{R_1}. Then, for any $0 < R \le R_1$

$$\begin{aligned}\int_{B_R} |Xu-(Xu)_{x_0,R}|^2\,dx &\le C\Bigg\{\underbrace{\int_{B_R} |X(u-v)|^2\,dx}_{(1)} \\ &+ \underbrace{\int_{B_R} |Xv-(Xv)_{x_0,R}|^2\,dx}_{(2)} + \underbrace{\int_{B_R} |(Xv)_{x_0,R}-(Xu)_{x_0,R}|^2\,dx}_{(3)}\Bigg\}.\end{aligned} \tag{3.31}$$

It is easily seen that

$$(3) \le (1)\,. \tag{3.32}$$

Now, by (3.8)

$$(2) \le C\left(\frac{R}{R_1}\right)^{Q+2}\int_{B_{R_0}} \left|Xv - (Xv)_{x_0,R_1}\right|^2 dx\,,$$

so

$$(2) \le C\left(\frac{R}{R_1}\right)^{Q+2}\left\{\int_{B_{R_1}} |X(v-u)|^2\,dx + \int_{B_{R_1}} \left|Xu - (Xu)_{x_0,R_1}\right|^2 dx + \int_{B_{R_1}} \left|(Xu)_{x_0,R_1} - (Xv)_{x_0,R_1}\right|^2 dx\right\}$$

$$(2) \le C\left(\frac{R}{R_1}\right)^{Q+2}\int_{B_{R_1}} \left|Xu - (Xu)_{x_0,R_1}\right|^2 dx + 2C\int_{B_{R_1}} |X(u-v)|^2\,dx\,. \tag{3.33}$$

It follows from (3.31), (3.32) and (3.26)

$$\int_{B_R} \left|Xu - (Xu)_{x_0,R}\right|^2 dx \le \left\{\left(\frac{R}{R_1}\right)^{Q+2}\int_{B_{R_1}} \left|Xu - (Xu)_{x_0,R_1}\right|^2 dx + R_1^{Q+2\mu}\left([f]^X_{\Omega,\mu} + ||g||_{L^\infty(\Omega)}\right)^2 + R_1^{2\mu}\int_{B_{R_1}} |Xu|^2\,dx\right\}.$$

If we use (3.17) with $R = R_1$ we deduce

$$\int_{B_R} \left|Xu - (Xu)_{x_0,R}\right|^2 dx \le C\left\{\left(\frac{R}{R_1}\right)^{Q+2}\int_{B_{R_1}} \left|Xu - (Xu)_{x_0,R_1}\right|^2 dx + B^2 R_1^{Q+2\mu}\right\}$$

where $B = [f]^X_{\Omega,\mu} + ||g||_{L^\infty(\Omega)} + ||Xu||_{L^2(\Omega)}$. Here C depends on the distance of x_0 to $\partial\Omega$. It follows from Lemma 2.6 that

$$\int_{B_R} \left|Xu - (Xu)_{x_0,R}\right|^2 dx \le C\left(\frac{R}{R_1}\right)^{Q+2\mu}\int_{B_{R_1}} \left|Xu - (Xu)_{x_0,R_1}\right|^2 dx + B\,R^{Q+2\mu}\,. \tag{3.34}$$

Therefore

$$\int_{B_R} \left|Xu - (Xu)_{x_0,R}\right|^2 dx \le C' R^{Q+2\mu}\left([f]^X_{\Omega,\mu} + ||g||_{L^\infty(\Omega)} + ||Xu||_{L^2(\Omega)}\right)^2$$

which proves (3.28) and completes the proof of Theorem 3.3. ♢

We consider now higher regularity results. Let us consider equations on the form

$$Pu = X_j^*\left(a^{ij}(x)X_i u\right) + b^j X_j u + cu = f\,. \tag{3.35}$$

We assume as before uniform ellipticity of (a^{ij}) *i.e.* $\forall\,\Omega' \subset\subset \Omega$,

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\,\xi_j \le \Lambda|\xi|^2, \quad \forall x \in \Omega',\ \ \forall \xi \in \mathbb{R}^n\,. \tag{3.36}$$

The purpose is now to prove the

Theorem 3.4. *Let $k \in \mathbb{N}$ and $\mu \in]0,1[$. Assume $a^{ij} \in S^{k+1,\mu}_{\mathrm{loc}}(\Omega)$, $b^j, c, f \in S^{k,\mu}_{\mathrm{loc}}(\Omega)$.*

Then if $u \in M^1(\Omega) \cap S^{1,\mu}_{\mathrm{loc}}(\Omega)$ is a weak solution of (3.35) then $u \in S^{k+2,\mu}_{\mathrm{loc}}(\Omega)$.

Proof. Let Ω' be an open subset such that $\overline{\Omega}' \subset \Omega$ and let $\psi \in C_0^\infty(\Omega)$, $0 \le \psi \le 1$, $\psi = 1$ on Ω'. We can find Ω_1, Ω_2 open with $\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ such that $\psi(x) \ge \frac{1}{4}$ on Ω_2, $\psi(x) \le \frac{3}{4}$ on $\Omega \setminus \Omega_1$. Let us define

$$Y_j = \psi X_j\,,\quad 1 \le j \le m\,,\quad Y_{m+\ell} = (1-\psi)\partial_{x_\ell}\,,\quad 1 \le \ell \le n\,. \tag{3.37}$$

This new set of vector fields $Y = (Y_1, \dots, Y_{m+n})$ satisfy also Hörmander's condition on $\overline{\Omega}$ but it is elliptic on $\Omega \setminus \Omega_1$.

If we denote by $S^{k,\mu}_Y(\Omega)$ the Hölder space associated with Y we have $S^{k,\mu}_Y(\Omega') = S^{k,\mu}_X(\Omega')$ and $S^{k,\mu}_Y(\overline{\Omega} \setminus \Omega_1) = C^{k,\mu}(\overline{\Omega} \setminus \Omega_1)$ which is the usual Hölder space. We set

$$\tilde{P} = X_j^*\left(\psi^2 a^{ij} X_i\right) - \sum_{i=1}^n \partial_{x_i}(1-\psi)^2\partial_{x_i} \tag{3.38}$$

$$\tilde{H} = \sum_{j=1}^m X_j^* \psi^2 X_j - \sum_{i=1}^n \partial_{x_i}(1-\psi)^2\partial_{x_i}\,. \tag{3.39}$$

Then we can state

Theorem 3.5. *For any $\mu \in]0,1[$, $k \in \mathbb{N}$, f in $S^{k,\mu}_Y(\Omega)$ and φ in $C^{k+2,\mu}(\partial\Omega)$ the Dirichlet problem*

$$\tilde{H}u = f\,,\quad u|_{\partial\Omega} = \varphi \tag{3.40}$$

has an unique solution $u \in S^{k+2}_Y(\Omega)$ and we have the estimate

$$||u||_{S^{k+2,\mu}_Y(\Omega)} \le C\left\{||f||_{S^{k,\mu}_Y(\Omega)} + ||\varphi||_{S^{k+2,\mu}_Y(\partial\Omega)}\right\}, \tag{3.41}$$

with C independent of u, f, φ.

Proof. Since $\varphi \in C^{k+2,\mu}(\partial\Omega)$ there exists $\tilde{\varphi} \in C^{k+2,\mu}(\overline{\Omega}) \subset S^{k+2,\mu}_Y(\overline{\Omega})$ with $\tilde{\varphi}|_{\partial\Omega} = \varphi$ so we can take $\varphi = 0$ in (3.40). Let $\zeta \in C_0^\infty(\Omega)$. If v is a weak solution of the Dirichlet problem $\tilde{H}v = \zeta f$, $v|_{\partial\Omega} = 0$, the strong maximum principle of Bony [B] implies that supp $v \subset$ supp $\zeta \cap \overline{\Omega}$ and

$$v(x) = \int_\Omega G(x,y)(\zeta f)(y)\,dy \tag{3.42}$$

where G is the Green function of $\tilde{H}$ on Ω. Let us take a partition of unity $(\zeta_\ell)_{\ell=0\dots N}$ with $\zeta_0 \in C_0^\infty(\Omega)$ and $\zeta_\ell \in C_0^\infty(\overline{\Omega} \setminus \operatorname{supp}\psi)$, $\ell = 1, \dots, N$. Then Theorem 4.1 in [X2] ensures that $v_0(x) = \int_\Omega G(x,y)(\zeta_0 f)(y)\,dy \in S^{k+2,\mu}_Y(\Omega)$, supp $v_0 \subset\subset \Omega$. For $1 \le \ell \le N$ let us set, $v_\ell(x) = \int_{\text{supp } \zeta_\ell \cap \overline{\Omega}} G(x,y)(\zeta_\ell f)(y)\,dy$. Then

supp $v_\ell \subset \overline{\Omega} \setminus \text{supp } \psi$. Therefore we have $\tilde{H}\, v_\ell = \Delta v_\ell = \zeta_\ell f$ on $\Omega \cap \text{supp } \zeta_\ell$, $v_\ell = 0$ on the boundary of $\overline{\Omega} \cap \text{supp } \zeta_\ell$. It follows that $v_\ell \in C^{k+2,\mu}(\overline{\Omega} \setminus \text{supp } \psi)$. Then

$u = \sum_{\ell=0}^{N} v_\ell \in S_Y^{k+2,\mu}(\overline{\Omega})$ is our solution. The uniqueness is given by the maximum principle and the a priori estimate by that on v_ℓ.

Theorem 3.6. *Let $k \in \mathbb{N}$ and $\mu \in]0,1[$. Assume that*

$a^{ij} \in S_{\text{loc}}^{k+1,\mu}(\Omega)$, $b_j, c, f \in S_{\text{loc}}^{k,\mu}(\Omega)$, $\varphi \in S_Y^{k+2,\mu}(\partial\Omega)$. *Then the Dirichlet problem*

$$\tilde{P} u = f \ \text{ on } \Omega, \quad u|_{\partial\Omega} = \varphi \tag{3.43}$$

has an unique solution $u \in S_Y^{k+2,\mu}(\Omega)$.

Proof. The uniqueness is still given by the maximum principle. The existence will be proved by the continuity method using Theorem 3.5. The main step will be the proof of an a priori estimate for smooth solution of (3.43) *i.e.*

$$||v||_{S_Y^{k+2,\mu}(\Omega)} \le C\left\{||\tilde{P}v||_{S_Y^{k,\mu}(\Omega)} + ||v||_{S_Y^{k+2,\mu}(\partial\Omega)}\right\} \tag{3.44}$$

with C independent on v.

Since $\left(a^{ij}(x)\right)$ is positive definite on supp ψ, for any x_0 in supp ψ, the operator

$$\tilde{P}_0 = \sum_{i,j=1}^{m} X_j^*\left(\psi^2 a^{ij}(x_0) X_i\right) - \sum_{i=1}^{n} \partial_{x_i}(1-\psi)^2 \partial_{x_i}$$

can be rewrited in the form (3.39) with vector fields $\tilde{X}_j$ such that $\tilde{X} = B(x_0)X$ where $B(x_0)$ is a non singular $m \times m$ matrix. So(3.41) gives the estimate (3.44) for $\tilde{P}_0$. Then (3.44) for $\tilde{P}$ follows from an easy perturbation argument using a partition of unity.

Proof of Theorem 3.4. Let $\zeta \in C_0^\infty(\Omega')$. Then $\zeta u \in S^{1,\mu}(\Omega')$. From (3.35) we get $L\zeta u = F$, $L = X_j^*\left(a^{ij}(x)X_i\right)$ and $F(x) = \zeta(f - b^j X_j u - cu) + uL\zeta + a^{ij}\left(X_j^* \zeta X_i u + X_j^* u X_i \zeta\right)$.

Since $u \in S_{\text{loc}}^{1,\mu}(\Omega)$ it follows that $F \in S^{0,\mu}(\Omega')$ and supp $F \subset \Omega'$. Then $\zeta u \in S_Y^{1,\mu}(\Omega)$, $F \in S_Y^{0,\mu}(\Omega)$ and $\tilde{P}(\zeta u) = L(\zeta u) = F$, $\zeta u = 0$ on $\partial\Omega$. Then the regularity and uniqueness results given by the Theorem 3.6 imply that $\zeta u \in S^{2,\mu}(\Omega')$. Since this is true for any Ω' and ζ it follows that $u \in S_{\text{loc}}^{2,\mu}(\Omega)$. For higher k we use an induction. This completes the proof of Theorem 3.4.

Remark 3.7. If the equations are written for $u \in S_{\text{loc}}^{2,\mu}(\Omega)$ in the form

$$a^{ij} X_i X_j u + b^j X_j u + cu = f$$

it has been proved in [X2] that for any $k \in \mathbb{N}$ we have $u \in S^{k+2,\mu}_{\mathrm{loc}}(\Omega)$ if a^{ij}, b^j, c, f are in $S^{k,\mu}_{\mathrm{loc}}(\Omega)$.

4. Quasilinear subelliptic systems

The content of this section is very close to [G] which deals with the elliptic case (see also Sect. 2.4 in [Sc]).

We consider a quasilinear subelliptic system of the form

$$X_j^*\left(a^{ij}(x,u)X_i u^\alpha\right) = f^\alpha(x,u,Xu), \quad \alpha = 1,\dots,N \tag{4.1}$$

were $X = (X_1, \dots X_m)$ is a set of vector fields satisfying Hörmander's condition. We shall assume that we have a solution u such that

$$u \in M^1(\Omega, \mathbb{R}^N) \cap C^0(\Omega, \mathbb{R}^N). \tag{H.1}$$

We assume moreover that

(H.2)

$$\begin{cases} \text{the coefficients } a^{ij} \text{ are locally Hölder continuous on } \Omega\times\mathbb{R}^N \text{ with} \\ \text{exponent } \mu \in]0,1[\text{ and for any } \Omega' \subset\subset \Omega \text{ there exist some positive } \lambda, \Lambda \\ \text{such that } \lambda|\xi|^2 \le a^{ij}(x,z)\xi_i\xi_j \le \Lambda|\xi|^2, \ \forall (x,z,\xi)\in\Omega'\times\mathbb{R}^N\times\mathbb{R}^m, \end{cases}$$

(H.3)

$$\begin{cases} \text{the functions } x \mapsto f^\alpha(x,u(x),Xu(x)) \text{ are measurable on } \Omega \text{ and satisfy} \\ \text{for some positive } a,b : |f(x,u(x),Xu(x))| \le a|Xu(x)|^2 + b, \ \text{a.e.} \end{cases}$$

We shall fix positive constants M and H such that

$$||u||_{M^1(\Omega)} \le M, \quad [a^{ij}]_{\Omega\times\mathbb{R}^N,\mu} \le H. \tag{4.2}$$

Remark. From (H.1) u is uniformly continuous on each compact set in Ω. Therefore for each x_0 in Ω in a small neighborhood V of x_0, u has a modulus of continuity that is a function $\omega : [0,a[\to [0,+\infty[$, continuous, non decreasing, with $\omega(0) = 0$ and $|u(x) - u(x')| \le \omega(|x - x'|)$ for all x, x' in V.

The first result is the following

Theorem 4.1. *Let $u \in M^1(\Omega)$ be a solution of (4.1). Under conditions (H.1), (H.2), (H.3) there exists $\alpha \in]0,1[$ such that $u \in S^{1,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and for $x_0 \in \Omega$ there is a radius R_0 and a constant C, which depends only on $n, N, m, \lambda, \Lambda, \mu, H, a$, b, M and* $\mathrm{dist}(x_0, \partial\Omega)$ *such that* $\operatorname*{osc}_{B(x_0,R)} Xu \le C R^\alpha$.

As soon as we have this result we can apply Theorem 3.4 to get higher regularity.

Theorem 4.2. *Let $u \in M^1(\Omega)$ be a weak solution of (4.1). Assume (H.1), (H.2), (H.3) and that a^{ij}, f^α are C^∞ functions. Then $u \in C^\infty(\Omega)$.*

Proof. It follows closely [G]. (See also [Sc], Theorem 2.4.3). We give the details only for sake of completeness. The constants $n, N, m, \lambda, \Lambda, H, a, b, M$ will be called the data. Let $x_0 \in \Omega$ be fixed.

Step 1: for all $\varepsilon \in]0,1[$ there exists a radius $R_1 > 0$ and a positive constant $C(R_1)$ depending on the data and on $d(x_0, \partial\Omega)$ such that for all R in $]0, R_1]$.

$$\int_{B_R} |Xu|^2\, dx \le C(R_1) R^{Q-\varepsilon} \tag{4.3}$$

Proof. We write our system as

$$X_j^*\left(a^{ij}(x_0, u(x_0)) X_i u^\alpha\right) = f^\alpha + X_j^*\left(\left(a^{ij}(x_0, u(x_0)) - a^{ij}(x, u(x))\right) X_i u^\alpha\right).$$

Without loss of generality we may assume the vector fields $X_1 \dots X_m$ free, satisfying Lemma 2.2 and Corollary 2.3.

Arguing as in the proof of (3.17) in Theorem 3.3 with v given on B_{R_1} for $0 < R_1 \le R_0$ we get

$$\begin{aligned}\lambda \int_{B_{R_1}} \left|X(u^\alpha - v^\alpha)\right|^2 dx \le C \Bigg\{ & \left|f^\alpha(x, u(x), Xu(x))\right| |u^\alpha - v^\alpha|\, dx + \\ & + \int_{B_{R_1}} \left|a^{ij}\left(x_0, u(x_0)\right) - a^{ij}(x, u(x))\right| |X_i u^\alpha|\, |X_j(u^\alpha - v^\alpha)|\, dx + \\ & + \int_{B_{R_1}} \left|a^{ij}(x_0, u(x_0))\right| |X_i u^\alpha|\, |u^\alpha - v^\alpha|\, dx \Bigg\}.\end{aligned}$$

Using (H.2), (H.3) and the Poincaré inequality (2.7) we get

$$\begin{aligned}&\int_{B_{R_1}} |X(u-v)|^2\, dx \\ &\le \left\{ \left[\sup_{B_{R_1}} |u - v| + R_1^{2\mu} + \omega(R_1)^{2\mu} \right] \int_{B_{R_1}} |Xu|^2\, dx + \int_{B_{R_1}} |u - v|\, dx \right\}.\end{aligned} \tag{4.4}$$

By the maximum principle applied to the solution v of (3.19) on B_{R_1}, we have

$$\begin{aligned}\sup_{\bar B_{R_1}} |u - v| = v(x_1) - u(x_1) &\le \sup_{B_{R_1}} v - u(x_1) \le \sup_{\partial B_{R_1}} u - u(x_1) \\ &\le \sup_{\bar B_{R_1}} u - \inf_{\bar B_{R_1}} u \le \omega(R_1).\end{aligned}$$

(Same reasoning if $\sup |u - v| = u(x_1) - v(x_1)$).

Applying once more Poincaré inequality to handle $\int_{B_{R_1}} |u - v|\, dx$ we get

$$(4.5)\quad \int_{B_{R_1}} |X(u-v)|^2\,dx \le C\left\{\left(\omega(R_1)+R_1^{2\mu}+\omega(R_1)^{2\mu}\right)\int_{B_{R_1}} |Xu|^2\,dx + R_1^{Q+2}\right\}.$$

Writing for $R \le R_1$

$$\int_{B_R} |Xu|^2\,dx \le 2\left\{\int_{B_R} |Xv|^2\,dx + \int_{B_R} |X(u-v)|^2\,dx\right\}$$

and using (3.6) we get easily

$$\int_{B_R} |Xu|^2\,dx \le C\left\{\left[\left(\frac{R}{R_1}\right)^Q + \omega(R_1) + R_1^{2\mu} + \omega(R_1)^{2\mu}\right]\int_{B_{R_1}} |Xu|^2\,dx + R_1^{Q+2}\right\}.$$

We use now Lemma 2.6. It follows that if R_1 is small enough (depending only on the data) we have for $R \le R_1$ and all small $\varepsilon > 0$,

$$\int_{B_R} |Xu|^2\,dx \le C\left\{\left(\frac{R}{R_1}\right)^{Q-\varepsilon}\int_{B_{R_1}} |Xu|^2\,dx + R^{Q-\varepsilon}\right\}.$$

This gives (4.3) since $\int_{B_{R_1}} |Xu|^2\,dx \le ||u||^2_{M^1(\Omega)} \le M^2$.

It follows from Poincaré inequality (2.8) and from Proposition 2.1 that

$$(4.6)\qquad u \in S^{0,\sigma}_{\mathrm{loc}}(\Omega) \text{ for every } 0 < \sigma < 1.$$

Step 2: We come back to the proof in step 1 using (4.6) instead of (H.1). Instead of (4.5) we get, with $\nu = \min(1, 2\mu)$

$$(4.7)\qquad \int_{B_{R_1}} |X(u-v)|^2\,dx \le C\left\{R_1^{\nu\sigma}\int_{B_{R_1}} |Xu|^2\,dx + R_1^{Q+2}\right\}$$

since $R_1^{\sigma} + R_1^{2\mu} + R_1^{2\mu\sigma} \le C\,R_1^{\nu\sigma}$ where $\nu = \min(1, 2\mu)$.

We fix now R_1 as in step 1 and 2. We take $0 < R_0 < R_1$, R_0 small enough depending on the data and we take w solution of (3.19) on B_{R_0}. With this w we can do the same computations as in step 1 and step 2 to get the analogue of (4.7) with R_0 instead of R_1 *i.e.*

$$(4.8)\qquad \int_{B_{R_0}} |X(u-w)|^2\,dx \le C\left\{R_0^{\nu\sigma}\int_{B_{R_0}} |Xu|^2\,dx + R_0^{Q+2}\right\}.$$

Now since $R_0 < R_1$ we can use (4.3). Then (4.8) implies for $\varepsilon > 0$,

$$(4.9)\qquad \int_{B_{R_0}} |X(u-w)|^2\,dx \le C(R_1)R_0^{Q-\varepsilon+\nu\sigma}.$$

Step 3: There exist $\alpha > 0$ and $R_0 > 0$, $C > 0$ depending only on the data and on $d(x_0, \partial\Omega)$ such that for all $R \le R_0$,

$$(4.10)\qquad \int_{B_R} \left|Xu - (Xu)_{x_0,R}\right|^2 dx \le C\,R^{Q+2\alpha}.$$

Proof. We write, with w as above,

$$\int_{B_R} \left| Xu - (Xu)_{x_0,R} \right|^2 dx \le C \left\{ \int_{B_R} |X(u-w)|^2 dx + \int_{B_R} \left| Xw - (Xw)_{x_0,R} \right|^2 dx \right\}.$$

So, using (3.8) and (4.9) we get for $R \le R_0$,

$$\int_{B_R} \left| Xu - (Xu)_{x_0,R} \right|^2 dx \le C \left\{ \left(\frac{R}{R_0} \right)^{Q+2} \int_{B_{R_0}} \left| Xu - (Xu)_{x_0,R_0} \right|^2 dx + R_0^{Q-\varepsilon+\nu\sigma} \right\}.$$

Since $0 < \nu\sigma - \varepsilon < 1$ we can use again Lemma 2.6 to get (4.10) with $2\alpha = \nu\sigma - \varepsilon$.

The conclusion of Theorem 4.1 follows from (4.10) and Proposition 2.1.

References

[B] Bony J.-M.: Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier **19** (1969) 227–304

[C] Campanato S.: Equazioni ellittiche del secondo ordine e spazi $\mathscr{L}^{2,\lambda}$, Ann. Mat. Pura e Appl. **69** (1965) 321–380

[G] Giaquinta M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems, Annals of Math. Studies, **105**, Princeton Univ. Press, New Jersey, 1983

[GT] Gilbarg D., Trudinger N.S.: Elliptic partial differential equations of second order, Springer Verlag, Heidelberg, New York (1977)

[H] Hörmander L.: Hypoelliptic second order differential equations, Acta Math., **119** (1967) 141–171

[J] Jerison D.: The Poincaré inequality for vector fields satisfying Hörmander's condition, Duke Math. J., **53** (1986) 503–523

[JSC] Jerison D., Sánchez-Calle A.: Subelliptic second order differential operators, Lecture Notes in Math., **1277** Springer Verlag

[JX] Jost J., Xu C.J.: Subelliptic harmonic maps, Preprint

[L] Lu G.: Embedding theorem on Campanato-Morrey spaces for vector fields and applications, C.R.A.S. **320** (1995) 429–434

[NSW] Nagel A., Stein E.M. and Wainger S.: Balls and metrics defined by vector fields I, basic properties, Acta Math., **155** (1985) 103–147

[RS] Rothschild L., Stein E.M.: Hypoelliptic operators and nilpotent Lie Groups, Acta Math. **137** (1977) 247–320

[SC] Sánchez-Calle A.: Fundamental solutions and geometry of the sum of squares of vector fields, Invent. Math., **78** (1984) 143–160

[Sc] Schulz F.: Regularity theory for quasi linear elliptic systems, Lectures Notes in Math. **1445**, Springer Verlag

[X1] Xu C.J.: Subelliptic variational problems, Bull. Soc. Math. France, **118** (1990) 147–169

[X2] Xu C.J.: Regularity for quasilinear second order subelliptic equations, Comm. Pure Appl. Math., (1992) 77–96

[X3] Xu C.J.: Semilinear subelliptic equations and Sobolev inequality for vector fields satisfying Hörmander's condition, Chinese Ann. Math., **15A** (1994) 65–72. English version : Chinese J. Contemp. Math. **15** (1994) 34–40

[X4] Xu C.J.: Application équilibre d'un espace homogène dans une variété riemannienne, Séminaire EDP, Ecole Polytechnique, 1994–1995