# Cauchy-Dirichlet Problems For A Class Of Quasilinear Degenerate Parabolic Equations *

By XU Chao-Jiang and NIE Lei
*Institute of Mathematics, Wuhan University*
*430072, Wuhan, P.R.China*

**ABSTRACT**

In this work, we study the quasilinear Cauchy-Dirichlet problem, $\partial_t u + \sum_{i,j=1}^m X_i^* A_{ij}(x,t,u)X_j u - \sum_{j=1}^m B_j(x,t,u)X_j u + c(x,t,u) = 0$, in $]0,T[\times\Omega; u|_{t=0} = \varphi$, on $\Omega$; $u = \psi$, on $[0,T[\times\partial\Omega.$, where $X = \{X_1,\dots,X_m\}$ is a system of real smooth vector fields which is defined on a open domain $M$ of $\mathbb{R}^n$, and satisfies the Hörmander's condition, $\Omega \subset M$. Assume that $\partial\Omega$ is non characteristic for the system $X_1,\dots,X_m$. Under some hypothesis for the boundary of domain and the elliptic structure condition for the non-linear coefficients $A_{ij}$, $B_j$, $C$, $i,j = 1,\dots,m$, we have proved that the existence and regularity of solution for above Cauchy-Dirichlet problems.

**Key Words** degenerate parabolic equation, vector fields, "non-isotropic" Hölder's space, Cauchy-Dirichlet problems.

**Classification** 35I, 35H.

## § 1 Introduction

In this work, we study the following quasilinear Cauchy-Dirichlet problem:

$$Lu \equiv \partial_t u + \sum_{i,j=1}^m X_i^* A_{ij}(x,t,u)X_j u$$

$$+\sum_{j=1}^m B_j(x,t,u)X_j u + C(x,t,u) = 0, \text{ in } ]0,T[\times\Omega \tag{1}$$

$$u|_{t=0} = \varphi, \qquad \text{on } \partial\Omega \tag{2}$$

$$u = \psi \qquad \text{on } [0,T[\times\partial\Omega. \tag{3}$$

*J. Wuahn Univ. 17(1996) 167–180

where $X = \{X_1, \ldots, X_m\}$ is a system of real smooth vector fields defined in an open domain $M \subset \mathbb{R}^n, n \geq 2$, $\Omega$ is a bounded open subdomain of $M$ with $\partial\Omega$ smooth. $X_j^* = -X_j + c_j$ is the formula adjoint of $X_j$. We assume that the system of vector fields $X = \{X_1, \ldots, X_m\}$ satisfies the following Hörmander's condition (H):

> $X_1, \ldots, X_m$ together with their commutators $X_\alpha = [X_{\alpha_1}, \ldots [X_{\alpha_{s-1}}, X_{\alpha_s}] \ldots]$ up to some fixed length $r$ span the tangent space at each point of $M$.

We will study the problem (1)–(3) similar to the case of second order parabolic equations. The role of Laplaceian $-\triangle_x$ is substituted by the Hörmander's operators $H = \sum_{j=1}^m X_j^* X_j + c$, where $c \geq c_0 > 0$. Actually by using the geometry and the function spaces associated with the system of vector fields $X$, the operators $H$ seems to satisfy nearly all properties of Laplacian $-\triangle_x$ (see [1, 4, 6, 11]).

We assume that the system of vector fields $X$ and the boundary $\partial\Omega$ satisfies the following additional conditions (S. E. $\partial\Omega$)

> $\partial\Omega$ is non characteristic for the system $X$. And for all $1 \leq j \leq r$, we have $\mathcal{X}_j' = \mathcal{X}_j \bigcap T_x(\partial\Omega)$ for all $x \in \partial\Omega$, and the dimension of $\mathcal{X}_j$ is constant in a neighborhood of $\overline{\Omega}$.

Where $\mathcal{X}_1$ is the linear space spanned by the vector fields $X_1, \ldots, X_m$ with smooth real coefficients in $C^\infty(M), \mathcal{X}_j = [\mathcal{X}_1, \mathcal{X}_{j-1}]$. And for $x \in \partial\Omega, \mathcal{X}_1' = \mathcal{X}_1 \bigcap T_x(\partial\Omega)$, $\mathcal{X}_j' = [\mathcal{X}_1', \mathcal{X}_{j-1}']$. $S^{k,\alpha}(\overline{\Omega})$ is the "non-isotropic" Hörlder space associated with the system of vector fields $X$ (see [8] and Section 2).

Then, the Hörmander's condition implies that $\mathcal{X}_r(x) = T_x M$ for all $x \in M$. And the condition (S. E. $\partial\Omega$) implies that the bases of $\mathcal{X}_1'$ (vector fields defined on $\partial\Omega$) satisfies the Hörmander's condition as well on the manifold $\partial\Omega$ at order $r$.

For the nonlinear coefficients, we assume that there exists constants $\lambda, \Lambda > 0$ and $g(x,t) \in C^0([0,T[\times\overline{\Omega})$ such that:

$$A_{i,j},\, B_j,\, C \in C^\infty([0,T[\times\overline{\Omega}\times\mathbb{R}),\ i,j = 1, \ldots, m; \tag{4}$$

$$|A_{ij}(x,t,z)|, |B_j(x,t,z)|, |C(x,t,z)| \leq g(x,t); \tag{5}$$

$$\lambda|\xi|^2 \leq \sum_{i,j}^m A_{ij}(x,t,z,)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall (x,t,z,\xi) \in [0,T[\times\overline{\Omega} \subset \mathbb{R}^{n+1}. \tag{6}$$

We prove in this work the following theorem:

**Theorem 1** *Assume that the system of vector fields $X_1, \ldots, X_m$ and $\partial\Omega$ satisfies the Hörmander's condition and (S. E. $\partial\Omega$). Let the nonlinear coefficients of equation (1) satisfy the conditions (4)-(6). then there exists a solution of problems (1)–(3) in the class $C^\infty([0,T[\times\overline{\Omega})$, if $\varphi, \psi$ are in the class of $C^\infty$.*

We will give the essential notations and well know results of Hörmander's operators in Section 2. We study linear problems in Section 3, and nonlinear problems in Section 4.

## § 2 Preliminary Lemmas And Notations

We define now the sub-unit metric on $M$ associated with $X$ as in [4] and [8].

**Definition 1** *Let $C(\delta)$ be a class of absolutely continuous mappings $\phi : [0,1] \to M$ which almost everywhere satisfy the differential equation*

$$\phi'(t) = \sum_{|J| \le r} a_J(t) X_J(\phi(t)) \tag{7}$$

*with $|a_J(t)| < \delta^{|J|}$, then we define*

$$\rho(x,y) = inf\{\delta > 0 \mid \exists \phi \in C(\delta) \text{ with } \phi(0) = x,\ \phi(1) = y\}. \tag{8}$$

Then, $\rho$ is a local metric on $M$. We introduce now a class of "non-isotropic" Hölder continuous functions. For $1 > \alpha > 0$, we define the function spaces $S^0([0,T[\times\overline{\Omega}) = C^0([0,T[\times\overline{\Omega})$, and $S^\alpha([0,T[\times\overline{\Omega})$ by

$$\Big\{ f \in S^0([0,T[\times\overline{\Omega});\ \sup_{x,y\in\overline{\Omega};t\in[0,T[} \frac{|f(x,t)-f(y,t)|}{\rho(x,y)^\alpha} < +\infty$$
$$\sup_{x\in\overline{\Omega};t,\tau\in[0,T[} \frac{|f(x,t)-f(x,\tau)|}{|t-\tau|^{\alpha/2}} < +\infty \Big\}. \tag{9}$$

And for $k, l \in \mathbb{N}$, $1 > \alpha \ge 0$, we define $S^{k,l}_\alpha([0,T[\times\overline{\Omega})$ by

$$\{u \in S^\alpha([0,T[\times\overline{\Omega})\ ;\ \partial_t^h X^J u \in S^\alpha([0,T[\times\overline{\Omega}),$$
$$\forall h \le k, 2h + |J| \le l\}. \tag{10}$$

The definition of norm of $S^{k,l}_\alpha([0,T[\times\overline{\Omega})$ is classic, and it is also a Banach space as in [8] (see also [5, 12]).

Using the hypotheses (S. E. $\partial\Omega$) on $\partial\Omega$, we can also define the functions spaces $S^{k,l}_\alpha([0,T[\times\partial\Omega)$ by use the bases of $\mathcal{X}'_1$ as in (9) (10) (see [5, 12]). We have also the following interpolation inequality and compactness results.

**Lemma 1** *For any $k_1 \le k_2$, $l_1 \le l_2, \alpha \le \beta$, $k_1 + l_1 + \alpha < k_2 + l_2 + \beta$, and $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that for any $u \in S^{k_2,l_2}_\beta([0,T[\times\overline{\Omega})$, we have*

$$\|u\|_{S^{k_1,l_1}_\alpha([0,T[\times\overline{\Omega})} \le \varepsilon \|u\|_{S^{k_2,l_2}_\beta([0,T[\times\overline{\Omega})} + C_\varepsilon \|u\|_{L^\infty(]0,T[\times\Omega)}. \tag{11}$$

*Then the embedding from $S^{k_2,l_2}_\beta([0,T[\times\overline{\Omega})$ to $S^{k_1,l_1}_\alpha([0,T[\times\overline{\Omega})$ is a compact map.*

## § 3 Existence and estimation of solutions for linear problems

We consider in this section the following linear problems:

$$\partial_t u + Hu = f(x,t), \quad \text{in } ]0,T[\times\Omega, \tag{12}$$
$$u|_{t=0} = \varphi \quad \text{on } \Omega, \tag{13}$$
$$u = \psi \quad \text{on } [0,T]\times\partial\Omega. \tag{14}$$

We will use the method of Hilbert spaces to study existence of weak solution for above problems. We define now

$$\begin{aligned} M^1(]0,T[\times\Omega) &= \{u \in L^2(]0,T[\times\Omega);\ \partial_t u \in L^2(]0,T[\times\Omega), \\ &\qquad X_j u \in L^2(]0,T[\times\Omega),\ 1 \le j \le m\}; \\ m^1(\Omega) &= \{u \in L^2(\Omega);\ X_j u \in L^2(\Omega),\ 1 \le j \le m\}. \end{aligned}$$

The norm of $M^1(]0,T[\times\Omega), m^1(\Omega)$ is classic, and it is Hilbert spaces. $m_0^1(\Omega)$ denote the closure of $\mathcal{D}(\Omega)$ in $m^1(\Omega)$, and we have $D(H) = m_0^1(\Omega)$.

For $u, v \in m_0^1(\Omega)$, we define

$$\begin{aligned} a(u,v) &= (Hu,v)_{m^1(\Omega)\times m^1(\Omega)} \\ &= \int_\Omega \sum_{j=1}^m X_j u \overline{X_j v} dx + + \int_\Omega c u \bar{v} dx \end{aligned}$$

Then we have

$$a(u,u) \ \ge\ c_0 \|u\|^2_{m^1(\Omega)}; \tag{15}$$
$$|a(u,v)| \ \le\ C\|u\|_{m^1(\Omega)} \|v\|_{m^1(\Omega)}. \tag{16}$$

Using the abstract existence theorem (see [3]), we obtain the existence of weak solution for degenerate parabolic problems :

**Theorem 2** *If $f \in L^2(]0,T[;\ m_0^{1\prime}(\Omega))$, $\varphi \in L^2(\Omega)$. Then there exists an unique solution $u \in W(]0,T[;\ m_0^1(\Omega)) = \{v \in L^2(]0,T[;\ m_0^1(\Omega));\ \partial_t v \in L^2(]0,T[;\ m_0^{1\prime}(\Omega))\}$ of following Cauchy problems:*

$$\partial_t u + Hu(t) = f(t), \tag{17}$$
$$u(0) = u_0. \tag{18}$$

*where $m_0^{1\prime}(\Omega)$ is the adjoint Hilbert spaces of $m_0^1(\Omega)$.*

Since $L^2(]0,T[\times\Omega) \subset L^2(]0,T[;\ m_0^{1\prime}(\Omega))$, then for $f \in L^2(]0,T[\times\Omega), \varphi \in L^2(\Omega), \tilde{\psi} \in M^1(]0,T[\times\Omega)$, we have proved the existence of weak solution for problems (12)–14) with $\psi = \tilde{\psi}|_{[0,T]\times\partial\Omega}$. Here the existence of trace for function of $M^1(]0,T[\times\Omega)$ is given in [2, 5, 12]. We have also $W(]0,T[;\ m_0^1(\Omega)) \subset$

$M^1(]0,T[\times\Omega)$. Let $u = v - \tilde{\psi}$, then $v$ will be a solution of problems (17)–(18) with $\tilde{f}(x,t) = f(x,t) + \partial_t\tilde{\psi}(x,t) + H\tilde{\psi}(x,t) \in L^2(]0,T[;\ m_0'(\Omega))$.

For the regularity of this weak solution, if $f \in C^\infty([0,T[\times\overline{\Omega}), \varphi \in C^\infty(\overline{\Omega}), \psi \in C^\infty([0,T[\times\partial\Omega)$, then $u$ is also in $C^\infty([0,T[\times\overline{\Omega})$. But we have to use a more precise regularity results in the associated Hölder spaces $S^{k,l}_\alpha([0,T[\times\overline{\Omega})$ to study nonlinear problems. So we will use the following estimate of heat kernel. Since the operators $\partial_t + H$ is hypoelliptic, and the boundary $[0,T[\times\partial\Omega$ is noncharacteristic, there exists the heat kernel $K_t(x,y)$ of Cauchy-Dirichlet problems (12)-(14).

**Lemma 2** *For any $N \in \mathbb{N}$ and $t > 0$ small, we have the following estimate of heat kernel:*

$$\begin{aligned} |\partial_t^k X^J K_t(x,y)| &\leq C_{N,|J|,k} t^{-k-\frac{|J|}{2}} |B(x;t^{1/2})|^{-1} \\ &\quad \times \left(1 + \frac{\rho(x,y)^2}{t}\right)^{-N}. \end{aligned} \tag{19}$$

*where the differentiations of $X^J$ are taken in $x$ or $y$.*

This is just Theorem II of [6]. Using the hypothesis (S. E. $\partial\Omega$). We can prolong the functions of $S^{k,l}_\alpha([0,T[\times\overline{\Omega})$ and heat kernel $K_t$ to the exterior of $[0,T[\times\Omega$, so we have also the estimate (19) near the boundary of domain. The following is maximum principle for degenerate parabolic operators (see [12]).

**Lemma 3** *If $f \in C^0([0,T[\times\overline{\Omega})$, $\varphi \in C^0(\overline{\Omega})$, $\psi \in C^0([0,T[\times\partial\Omega)$, and $u \in S^{1,2}_\alpha(]0,T[\times\Omega)$ is a solution of problems (12)-(14). Then we have the following a priori estimate for all $t \in [0,T[$,*

$$\begin{aligned} \|u\|_{L^\infty(]0,T[\times\Omega)} &\leq C\{\|\varphi\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty([0,T[\times\partial\Omega)} \\ &+ \|f\|_{L^\infty(]0,T[\times\Omega)}\}. \end{aligned} \tag{20}$$

Using the estimate of heat kernel and maximum principle, we have proved the following precise regularity results (see [12]).

**Theorem 3** *For $k \in \mathbb{N}$, $0 \leq \alpha \leq 1$, let $f \in S^{k,2k}_\alpha([0,T[\times\overline{\Omega}), \varphi \in S^{2k+2,\alpha}(\overline{\Omega})$, $\psi \in S^{k+1,2k+2}_\alpha([0,T[\times\partial\Omega)$, there exists an unique solution of problems (12)-(14) in the function space $S^{k+1,2k+2}_\alpha([0,T[\times\overline{\Omega})$. And the solution verifies following estimate:*

$$\begin{aligned} \|u\|_{S^{k+1,2k+2}_\alpha([0,T[\times\overline{\Omega})} &\leq C\{\|\varphi\|_{S^{2k+2,\alpha}(\overline{\Omega})} + \|\psi\|_{S^{k+1,2k+2}_\alpha([0,T[\times\partial\Omega)} \\ &+ \|f\|_{S^{k,2k}_\alpha([0,T[\times\overline{\Omega})}\}, \end{aligned} \tag{21}$$

*where the constant $C$ is independent of $u$.*

Using the freezing coefficients method, we can also study the following linear problems

$$\partial_t u + \sum_{i,j=1}^m X_i^* a_{ij}(x,t) X_j u + \sum_{j=1}^m b_j(x,t) X_j u$$

$$
\begin{aligned}
&+c(x,t)u = f(x,t), \quad \text{in } ]0,T[\times\Omega && (22)\\
u &= \varphi, \quad \text{on } \{t=0\}\times\Omega && (23)\\
u &= \psi, \quad \text{on } [0,T[\times\partial\Omega. && (24)
\end{aligned}
$$

Assume that, for some $k \in \mathbb{N}, \alpha > 0$, the coefficients $a_{ij}, b_j, c$ satisfies the conditions:

$$
\lambda|\xi|^2 \le \sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2,
$$

$$
\forall (x,t,\xi) \in [0,T[\times\overline{\Omega}\times\mathbb{R}^n; \tag{25}
$$

$$
\sum_{i,j=1}^{m} \|a_{ij}\|_{S_\alpha^{k,2k}} + \sum_{j=1}^{m} \|b_j\|_{S_\alpha^{k,2k}} + \|c\|_{S_\alpha^{k,2k}} \le \Lambda. \tag{26}
$$

We have the following results.

**Corollary 1** *Assume that the coefficients $a_{ij}, b_j, c$ satisfies the conditions (25)-(26), and $f \in S_\alpha^{k,2k}([0,T[\times\overline{\Omega}), \varphi \in S^{2k+2,\alpha}(\overline{\Omega}), \psi \in S_\alpha^{k+1,2k+2}([0,T[\times\partial\Omega)$. Then the solutions of problems (22)-(24) is in the class $S_\alpha^{k+1,2k+2}([0,T[\times\overline{\Omega})$, and we have also the estimate*

$$
\begin{aligned}
\|u\|_{S_\alpha^{k+1,2k+2}([0,T[\times\overline{\Omega})} &\le C\{\|\varphi\|_{S^{2k+2,\alpha}(\overline{\Omega})} + \|\psi\|_{S_\alpha^{k+1,2k+2}([0,T[\times\partial\Omega)}\\
&+ \|f\|_{S_\alpha^{k,2k}([0,T[\times\overline{\Omega})}\}, && (27)
\end{aligned}
$$

*where $C$ depends only on $\lambda, \Lambda, r$.*

The proof of this Corollary is given in [12]. We will use those results to study the existence of smooth solution for quasilinear problems (1)-(3) in Section 4.

## § 4 Existence of solution for quasilinear problems

We consider now the quasilinear problems (1)-(3). Using the results of Corollary 1 and Leray-Schauder's fixed point theorem, we have the following existence theorem.

**Theorem 4** *Assume that $\varphi \in S^{2,\alpha}(\overline{\Omega})$, $\psi \in S_\alpha^{1,2}([0,T[\times\partial\Omega)$. Under the hypothesis of Theorem 1, if there exist constant $B > 0, \beta > 0$ such that for all solution $u \in S_\alpha^{1,2}([0,T[\times\overline{\Omega})$ of following problems ($0 \le \sigma \le 1$):*

$$
\partial_t u + \sum_{i,j=1}^{m} X_i^* A_{ij}(x,t,u)X_j u + \sigma \sum_{j=1}^{m} B_j(x,t,u)X_j u
$$

$$
\begin{aligned}
&+\sigma C(x,t,u) = 0, \quad \textit{in} ]0,T[\times\Omega && (28)\\
&u|_{t=0} = \sigma\varphi, \quad \textit{on } \partial\Omega && (29)\\
&u = \sigma\psi \quad \textit{on } [0,T[\times\partial\Omega, && (30)
\end{aligned}
$$

*satisfies the a priori estimate:*

$$\|u\|_{S^\beta([0,T[\times\overline{\Omega})} \le B. \tag{31}$$

*Then there exist a solution* $u \in S^{1,2}_\beta([0,T[\times\overline{\Omega})$ *for quasilinear problems (1)-(3).*

In fact, we define the operator $T:\ S^\beta([0,T[\times\overline{\Omega}) \to S^\beta([0,T[\times\overline{\Omega})$ by $u = Tv$ is the solution of following linearization problems:

$$\begin{array}{ll} \partial_t u + \sum_{i,j=1}^m X_i^* A_{ij}(x,t,v)X_j u + \sum_{j=1}^m B_j(x,t,v)X_j u & \\ +C(x,t,v) = 0, & \text{in } ]0,T[\times\Omega \\ u|_{t=0} = \varphi, & \text{on } \partial\Omega, \\ u = \psi & \text{on } [0,T[\times\partial\Omega. \end{array}$$

Since $a_j(x,t) = A_j(x,t,v(x,t)), b_j(x,t) = B_j(x,t,v(x,t)), c(x,t) = C(x,t,v(x,t)) \in S^\beta([0,T[\times\overline{\Omega})$, using the results of Corollary 1, we have $T:\ S^\beta([0,T[\times\overline{\Omega}) \to S^{1,2}_\beta([0,T[\times\overline{\Omega})$. Then Lemma 1 give that $T:\ S^\beta([0,T[\times\overline{\Omega}) \to S^\beta([0,T[\times\overline{\Omega})$ is a compact operator, so that $T$ has a fixed point in $S^\beta([0,T[\times\overline{\Omega})$. Corollary 1 give the results of Theorem 4.

Now, the proof of Theorem 1 is to construct the a priori estimate (31). We sckicth this process in two step.

**(1) Estimate of** $\max|u|$.

Denote by $\Gamma = \{\{t=0\}\times\Omega\}\bigcup\{[0,T[\times\partial\Omega\}$, then we have

**Lemma 4** *Assume that the nonlinear coefficients* $A_{ij}, B_j, C,\ i,j = 1,\ldots,m$ *satisfies the hypothesis (4)-(6), and* $u \in S^{1,2}_\alpha([0,T[\times\overline{\Omega})$ *verifies equation (1). Then there exists constant* $C = C(\sup_{[0,T[\times\overline{\Omega}}|g|, \lambda,\ \Lambda,\ |\Omega|, \sup_\Gamma |u|)$*, such that*

$$\sup_{]0,T[\times\Omega} |u| \le C. \tag{32}$$

The proof of this Lemma is as following. We have the following Sobolev inequality, for $u \in M^1(]0,T[\times\Omega)$, there exists $q \in ]2, \frac{2n}{n-2/r}[$ such that

$$\|u\|_{L^{q,l}(]0,T[\times\Omega)} \le C\|u\|_{M^1(]0,T[\times\Omega)}, \tag{33}$$

where $\|u\|_{L^{q,l}(]0,T[\times\Omega)} = \left(\int_0^T \left(\int_\Omega |u(x,t)|^q dxdt\right)^{l/q}\right)^{1/q}$, and $l$ verifies $1/q = \frac{2-(2/r)}{nl} + \frac{1-(2/l)}{2}$. Using Nash-Moser processor and inequality (33), we can obtain estimate (32) as in [7] (see also [5]).

**(2) Estimate of** $\|u\|_{S^\beta}$**.**

The estimate of $\|u\|_{S^\beta}$ is given by Harnack's inequality for quasilinear degenerate parabolic equation (1), there is also two step for this processor, one is in the interior of domain and on a ball of $B(x,\varepsilon)$, another is near the boundary and on a semi-ball $B(x,\varepsilon)^+$. The method is standard for degenerate elliptic and parabolic equation under the hypothesis of Theorem 1, we reference the proof to the detail paper [5].

**End of proof of Theorem 1**

Now we have proved that for $\varphi \in S^{2,\alpha}(\overline{\Omega})$, $\psi \in S^{1,2}_\alpha([0,T[\times\partial\Omega)$, there exists a solution $u \in S^{1,2}_\beta(]0,T[\times\Omega)$ for quasilinear problems (1)-(3). Using the linear regularity results of Theorem 3 with freezing method, we can prove that if $\varphi \in S^{2k+2,\alpha}(\overline{\Omega})$, $\psi \in S^{k+1,2k+2}_\alpha([0,T[\times\partial\Omega)$, then $u \in S^{k+1,2k+2}_\beta(]0,T[\times\Omega)$ for any $k \in \mathbb{N}$. Since $S^{k+1,2k+2}_\beta(]0,T[\times\Omega) \subset C^{(2k+2)/r}([0,T[\times\overline{\Omega})$, we have proved Theorem 1.

## References


[1] Bony, J. M., Principe du maximum, inégalité de Harnack et uninité du problème de Cauchy pour les opérateurs elliptiques dégénérées , *Ann. Inst. Fourier* 19 (1969), 227–304.

[2] Derrdj, M., Un problème aux limites pour une classe d'opérateurs du second ordre hypoelliptiques, *Ann. Inst. Fourier* 21 (1971), 99–148.

[3] Ladyzenskaja O. A. & Solonnikov V. A. & Uralceva N. N., Linear and quasilinear equation of parabolic type, *Translation of Math. Mono. 23, AMS 1968.*

[4] Nagel, A., Stein, E. M., and Wainger, S., Balls and metrics defined by vector fields I, basic properties, *Acta Math.,* 155 (1985), 103–147.

[5] Nie, L., Existence of solutions for a class of quasilinear degenerate parabolic equation, *Master thesis of Wuhan University, April 1993, and paper to appear.*

[6] Sanchez-Calle, A., Fundamental solutions and geometry of the sum of squares of vector fields, *Invent. Math.,* 78 (1984), 143–160.

[7] Xu, C. J., Subelliptic variational problems, *Bull. Soc. Math. France,* 118 (1990), 147–159.

[8] Xu C. J., Regularity for quasilinear second order subelliptic equations, *Comm. Pure Appl. Math.,* (1992), 77–96.

[9] Xu, C. J., Hypoellipticity of nonlliear second order partial differential equations, *Jour. P. D. E.* , 1 (1988), 85–95.

[10] Xu, C. J., Problème de Dirichlet pour les équations associées à un système de champs de vecteurs, *Séminaire E. D. P., Ecole Polytechnique, 1990-1991, n. 17.*

[11] Xu, C. J., Problème de Dirichlet pour les opérateurs de Hörmander, *Prepublication de l'Ecole Polytechnique, Palaiseau, 1991.*

[12] Xu, C. J. & Nie, L., Existence and regularity of solutions for a class of degenerate parabolic equations, *preprint and paper to appear.*