AN EXISTENCE RESULT FOR A CLASS OF SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS *

XU Chao-Jiang
Institute of Mathematics, Wuhan University, Wuhan 430072, China

ABSTRACT

In this work, we study the semilinear degenerate Dirichlet problem, $\sum_{j=1}^m X_j^* X_j u + cu + f(x, u, Xu) = 0$, in Ω; $u = \varphi$, on $\partial\Omega$, where $X = \{X_1, \ldots, X_m\}$ is a system of real smooth vector fields which satisfies the Hörmander's condition. Assume that $X_1, \ldots, X_m$ satisfies some supplementary conditions on the boundary $\partial\Omega$, $f \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m), \partial_z f(x, z, \xi) \geq 0, \operatorname{sign} z f(x, z, 0) \geq \mu > -\infty, c(x) \geq c_0 > 0$. With some growth hypothesis of $f(x, z, \xi)$ in the variables ξ, we have proved the existence and the uniqueness of solution $u \in C^\infty(\overline{\Omega})$ of above semilinear Dirichlet problem, if $\varphi \in C^\infty(\partial\Omega)$.

Key Words Semilinear degenerate elliptic equation, vector fields, "non-isotropic" Hölder's space, Dirichlet problems.

Classification 35I, 35H.

1 Introduction

In this work, we study the following semilinear Dirichlet problem:

$$
(1) \qquad \begin{cases} Lu \equiv \sum_{j=1}^m X_j^* X_j u + cu + f(x, u, Xu) = 0, & \text{in} \quad \Omega \\ u = \varphi, & \text{on} \quad \partial\Omega \end{cases}
$$

where $X = \{X_1, \ldots, X_m\}$ is a system of real smooth vector fields defined in an open domain $M \subset \mathbb{R}^n, n \geq 2$, Ω is a bounded open subdomain of M with $\partial\Omega$ smooth, $c(x) \geq c_0 > 0$. $X_j^* = -X_j + c_j$ is the adjoint of X_j. We assume that the system of vector fields $X = \{X_1, \ldots, X_m\}$ satisfies the following Hörmander's condition:

> $X_1, \ldots, X_m$ together with their commutators $X_\alpha = [X_{\alpha_1}, \ldots [X_{\alpha_{s-1}}, X_{\alpha_s}] \ldots]$ up to some fixed length r span the tangent space at each point of M.

*Adv. in Math. (China) 25 (1996) pp. 492–499

We will study the problem (1) and similar to the case of second order elliptic equations. The role of Laplaceian $-\triangle_x$ is substituted by the Hörmander's operators $H = \sum_{j=1}^m X_j^* X_j + c$. Actually by using the geometry and the function spaces associated with the system of vector fields X, the operators H seems to satisfy nearly all properties of Laplacian $-\triangle_x$ (see [1], [2], [3], [4], [5]). For example, we have proved in [10] the following linear Dirichlet problem:

$$\begin{cases} Hu = f, & \text{in} \quad \Omega, \\ u = \varphi, & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

has a solution $u \in S^{k+2,\alpha}(\overline{\Omega})$, if $f \in S^{k,\alpha}(\overline{\Omega})$, $\varphi \in S^{k+2,\alpha}(\partial\Omega)$, and $\partial\Omega$ satisfies following additional conditions (S. E. $\partial\Omega$)

> $\partial\Omega$ is non characteristic for the system X. And for all $1 \leq j \leq r$, we have $\mathcal{X}'_j = \mathcal{X}_j \bigcap T_x(\partial\Omega)$ for all $x \in \partial\Omega$, and the dimension of $\mathcal{X}_j$ is constant in a neighborhood of $\overline{\Omega}$.

Where $\mathcal{X}_1$ is the linear space spanned by the vector fields $X_1, \ldots, X_m$ with smooth real coefficients in $C^\infty(M)$, $\mathcal{X}_j = [\mathcal{X}_1,\, \mathcal{X}_{j-1}]$. And for $x \in \partial\Omega$, $\mathcal{X}'_1 = \mathcal{X}_1 \bigcap T_x(\partial\Omega)$, $\mathcal{X}'_j = [\mathcal{X}'_1,\, \mathcal{X}'_{j-1}]$. $S^{k,\alpha}(\overline{\Omega})$ is the "non-isotropic" Hörlder space associated with the system of vector fields X (see [7] and Section 2).

Then, the Hörmander's condition implies that $\mathcal{X}_r(x) = T_x M$ for all $x \in M$. And the condition (S. E. $\partial\Omega$) implies that the bases of $\mathcal{X}'_1$ (vector fields defined on $\partial\Omega$) satisfies the Hörmander's condition as well on the manifold $\partial\Omega$ at order r.

Using the results of [8] (Theorem 2), we prove in this work the following theorem:

Theorem 1 *Assume that the system of vector fields $X = \{X_1, \ldots, X_m\}$ and $\partial\Omega$ satisfies the Hörmander's condition and (S. E. $\partial\Omega$). Let $f \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m)$, $\partial_z f(x, z, \xi) \geq 0$, $\operatorname{sign} z f(x, z, 0) \geq \mu > -\infty$, $\varphi \in S^{k+2,\alpha}(\partial\Omega)$, $k \in \mathbb{N}$, $1 > \alpha > 0$, and if there is $\theta \in]1,\, 2[$, such that $|f|_{\alpha,K,M} \leq C(K^\theta + 1)$ for all $K > 0$ and $0 < M \leq M_0$. Then there exists a solution $u \in S^{k+2,\beta}(\overline{\Omega})$ of Dirichlet problem (1) for some $\beta > 0$.*

Where we denote by

$$M_0 = \sup_{\partial\Omega} |\varphi| + c_0^{-1}\mu, \tag{3}$$

$$\Omega_{K,M} = \overline{\Omega} \times \{|z| \leq M\} \times \{|\xi| \leq K\}, \tag{4}$$

$$|f|_{\alpha,K,M} = \sup_{\Omega_{K,M}} |f(x, z, \xi)| + \sup_{\Omega_{K,M} \times \Omega_{K,M}} \frac{|f(x, z, \xi) - f(x', z', \xi')|}{\rho(x, x')^\alpha + |z - z'|^\alpha + K^{-\alpha}|\xi - \xi'|^\alpha}. \tag{5}$$

Since the equation (1) is degenerate elliptic and subelliptic, we call equation (1) semilinear subelliptic. Using the properties of Hörmander's operator H , we have proved the interior regularities for quasilinear second order subelliptic equation of form $\sum_{ij=1}^m A_{ij}(x,\, u,\, Xu) X_i X_j u + B(x, u, Xu) = 0$, and the existence of weak solution for variational problems (see [6],[7],[8],[9],[10]).

2 Preliminary Lemmas And Notations

We define now the sub-unit metric on M associated with X as in [4] and [7].

Definition 1 *Let $C(\delta)$ be a class of absolutely continuous mappings $\phi : [0,1] \to M$ which almost everywhere satisfy the differential equation*

$$\phi'(t) = \sum_{|J| \le r} a_J(t) X_J(\phi(t)) \tag{6}$$

with $|a_J(t)| < \delta^{|J|}$, then we define

$$\rho(x,y) = inf\{\delta > 0 \mid \exists \phi \in C(\delta) \ with \ \phi(0) = x, \ \phi(1) = y\}. \tag{7}$$

Then, ρ is a local metric on M, and for any small compact subset $K \subset M$, there exists a constant $C > 0$ such that

$$C^{-1}|x-y| \le \rho(x,y) \le C|x-y|^{1/r}$$

for any $x, y \in K$.

We introduce now a class of "non-isotropic" Hölder continuous functions. For $1 > \alpha > 0$, we define ($S^0(\overline{\Omega}) = C^0(\overline{\Omega})$)

$$S^\alpha(\overline{\Omega}) = \left\{ f \in S^0(\overline{\Omega});\ [f]^X_{\alpha,\overline{\Omega}} = \sup_{x,y\in\overline{\Omega}} \frac{|f(x)-f(y)|}{\rho(x,y)^\alpha} < +\infty \right\} \tag{8}$$

and for $k \in \mathbb{N}$, $1 > \alpha \ge 0$, we define

$$S^{k,\alpha}(\overline{\Omega}) = \{u \in S^\alpha(\overline{\Omega})\ ;\ X^J u \in S^\alpha(\overline{\Omega}),\ \forall |J| \le k\} \tag{9}$$

Set:

$$[u]^X_{k,0,\overline{\Omega}} = \sup_{|J|=k} \sup_{x\in\overline{\Omega}} |X^J u(x)|$$

and

$$[u]^X_{k,\alpha,\overline{\Omega}} = \sup_{|J|=k} [X^J u(x)]^X_{k,\alpha,\overline{\Omega}}.$$

The norms on $S^{k,\alpha}(\overline{\Omega})$ are given by

$$\|u\|_{S^{k,\alpha}(\overline{\Omega})} = \sum_{j=0}^{k} [u]^X_{j,0,\overline{\Omega}} + [u]^X_{k,\alpha,\overline{\Omega}}. \tag{10}$$

Then the norms of $S^{k,\alpha}(\overline{\Omega})$ is also convex, and $S^{k,\alpha}(\overline{\Omega})$ is a Banach space (see [7]). Hörmander's condition implies that $S^{k,\alpha}(\overline{\Omega}) \subset C^{k/r}(\overline{\Omega})$ for all $k \in \mathbb{N}$.

Using the hypotheses (S. E. $\partial\Omega$) on $\partial\Omega$, we can also define the functions spaces $S^{k,\alpha}(\partial\Omega)$ by the bases of $\mathcal{X}'_1$ as in (9) (see [8], [10]).

As for the classical Hölder space, we also have the interpolation inequalities in the space $S^{k,\alpha}(\Omega)$. For $j+\beta < k+\alpha$, $j,k \in \mathbb{N}$, $0 \le \alpha,\beta \le 1$, $u \in S^{k,\alpha}(\Omega)$, and any $\varepsilon > 0$, we have

$$\|u\|_{S^{j,\beta}(\Omega)} \le \varepsilon \|u\|_{S^{k,\alpha}(\Omega)} + C(\varepsilon, j, k, \Omega, r)\|u\|_{L^\infty(\Omega)}. \tag{11}$$

In [8] and [10], we have proved a abstract existence results (see Theorem 2 of [8]). We rewrite in the form of this paper

Theorem 2 *Assume that the hypothesis (S. E. $\partial\Omega$) is satisfied, and $\varphi \in S^{2,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$. If for some fixed $0 < \beta < 1$, there exists a constant B such that for all solutions $u \in S^{2,\alpha}(\overline{\Omega})$ of following Dirichlet problems ($0 \le \sigma \le 1$):*

$$\begin{cases} \sum_{j=1}^{m} X_j * X_j u + \sigma(cu + f(x,u,Xu)) = 0, & in \quad \Omega, \\ u = \sigma\varphi, & on \quad \partial\Omega. \end{cases} \tag{12}$$

we have a priori estimates

$$\|u\|_{S^{1,\beta}(\overline{\Omega})} \le B. \tag{13}$$

Then the Dirichlet problem (1) has a solution in the class $S^{2,\beta}(\overline{\Omega})$. Furthermore if $\varphi \in S^{k+2,\alpha}(\partial\Omega)$ with $k \in \mathbb{N}$, then $u \in S^{k+2,\beta}(\overline{\Omega})$.

Now we have transformed the nonlinear degenerate Dirichlet problems (1) to the the problem of construction of apriori estimates (13).

3 Schauder Estimates For The Hörmander Operators

We study in this section the following linear Dirichlet problem:

$$Hu = f, \quad \text{in } \Omega; \qquad u = \varphi, \quad \text{on } \partial\Omega. \tag{14}$$

with $c(x) \ge c_0 > 0$. By [1], there exists Green's kernel $G(x,y)$ for the operators H. From [5] and [7] we have

Lemma 1 *For $n \ge 2$, $K \subset\subset \Omega$, and $(x,y) \in K \times K$, we have*

$$|X^J G(x,y)| \le C_J \rho(x,y)^{2-|J|} |B(x,\rho(x,y))|^{-1}, \tag{15}$$

where differential are taken in x or y.

We shall use the inequality (15) to prove the Schauder estimate of Hörmander operators in the "non-isotropic" Hölder spaces $S^{k,\alpha}$. Firstly, we have the maximum principle

Lemma 2 *If $u \in S^2(\overline{\Omega})$ is a solution of Dirichlet problem (14), $c(x) \ge c_0 > 0$. Then we have*

$$\|u\|_{L^\infty(\Omega)} \le c_0^{-1} \|f\|_{L^\infty(\Omega)}. \tag{16}$$

If $u \in S^2(\overline{\Omega}), u \le 0$ on $\partial\Omega$ verifies $Hu \le 0$ in Ω. Then $u \le 0$ in Ω

This is just the results of J.-M. Bony [1]. We have also

Lemma 3 *Let $u \in S^{2,\alpha}(\overline{\Omega}), u|_{\partial\Omega} = 0$, $\alpha > 0$, then there exists a constant C such that*

$$\|u\|_{S^{2,\alpha}(\overline{\Omega})} \le C \|Hu\|_{S^{\alpha}(\overline{\Omega})}. \tag{17}$$

The proof of this Lemma is in [10], so we have obtain Schauder type estimate in "non-isotropic" function spaces for degenerate elliptic operators. As in the elliptic case, we well use this Lemma to study nonlinear problems (1).

4 A Priori Estimate For Semilinear Equations

Using the maximum principle, we have the following comparison principle.

Lemma 4 *Let u, $v \in S^2(\overline{\Omega})$, $Lu \leq Lv$ in Ω, $u \leq v$ on $\partial\Omega$. Under the assumption of Theorem 1, we have $u \leq v$ in Ω.*

Proof: Set $w = u - v$, then $w \leq 0$ on $\partial\Omega$ and

$$H(w) + (f(x,u,Xu) - f(x,v,Xv) \leq 0$$

Remark that

$$\begin{aligned} & f(x,u(x),Xu(x)) - f(x,v(x),Xv(x)) \\ = \; & f(x,u(x),Xu(x)) - f(x,u(x),Xv(x)) \\ + \; & f(x,u(x),Xv(x)) - f(x,v(x),Xv(x)) \\ = \; & \sum_{j=1}^{m} \partial_{\xi_j} f(x,u(x),X\widetilde{u}(x))X_j w(x) + \partial_z f(x,\hat{u}(x),Xv(x))w(x). \end{aligned}$$

and $\partial_z f(x,\hat{u}(x),Xv(x)) \geq 0$. We have

$$\sum_{j=1}^{m}(X_j^* X_j w + b_j X_j w) + \widetilde{c}w \leq 0; \quad w|_{\partial\Omega} leq 0.$$

with $\widetilde{c}(x) = c(x) - \partial_u f(x,\widetilde{u}(x)) \geq c_0 > 0$. and $b_j, \widetilde{c} \in S^{1,\alpha}(\overline{\Omega})$. Then Lemma 2 implies that $w \leq 0$ in Ω.

Using this Lemma, we get a priori estimates of $\|u\|_{L^\infty}$.

Theorem 3 *Under the assumptions of Theorem 1, if $u \in S^2(\overline{\Omega})$, $Lu = 0$ in Ω, then*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + c_0^{-1}|\mu|. \tag{18}$$

Proof: Set

$$v(x) = \sup_{\partial\Omega} u^+ + c_0^{-1}|\mu|,$$

Since $u \leq v$ on $\partial\Omega$, $v \geq 0$ in Ω, then

$$f(x,v,Xv) = f(x,v,0) \geq \mu.$$

using the comparison principle, we have

$$Lv = c(x)v(x) + f(x,v,0) \geq c(x)c_0|\mu| + \mu \geq 0 = Lu, \text{ in } \Omega,$$

by Lemma 2, we have proved $u \leq \sup_{\partial\Omega} u^+ + c_0^{-1}|\mu|$ in Ω. In the other hand, set

$$v_1(x) = \inf_{\partial\Omega} u^- - c_0^{-1}|\mu|,$$

Since $u \geq v_1$ on $\partial\Omega,\ v_1 \leq 0$ in Ω, then

$$f(x, v_1, Xv_1) = f(x, v_1, 0) \leq -\mu.$$

using the comparison principle, we have

$$Lv_1 = c(x)v(x)_1 + f(x, v_1, 0) \leq -c(x)c_0|\mu| - \mu \leq 0 = Lu, \text{ in } \Omega,$$

by Lemma 2, we have proved $u \geq \inf_{\partial\Omega} u^- - c_0^{-1}|\mu|$ in Ω. Which prove the Theorem 3.

Using the notations of (3)–(5), we can prove now

Theorem 4 *Let $u \in S^{2,\alpha}(\overline{\Omega})$, $1 > \alpha > 0$ be a solution of Dirichlet problem (12). Under the assumption of Theorem 1, we have*

$$\|u\|_{S^{1,\beta}(\overline{\Omega})} \leq B < +\infty, \tag{19}$$

with $\beta = \min\{\frac{\alpha}{2\theta-\alpha},\ \frac{\theta-\alpha}{\alpha}\}$, $B = B(n, m, r, \alpha, \theta, c_0, \mu)$.

Proof: Set $K = \max\{1,\ [Xu]_{0,\overline{\Omega}}\}$, $K_\nu = \|Xu\|_{S^\nu(\overline{\Omega})}$ for $\nu \in]0,\ 1[$, $\bar{f}(x) = f(x, u(x), Xu(x))$. Since $\|u\|_{L^\infty} \leq M_0$, we have

$$\begin{aligned}
\|\bar{f}\|_{S^{\alpha\nu}(\overline{\Omega})} &= \|\bar{f}\|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega} \frac{|\bar{f}(x) - \bar{f}(y)|}{\rho(x,y)^{\alpha\nu}} \\
&\leq C(K^\theta + 1)(1 + K^{\alpha\nu} + K^{-\alpha}K_\nu^\alpha) \\
&\leq C_1(K^{\theta+\alpha\nu} + K^{\theta-\alpha}K_\nu^\alpha)
\end{aligned}$$

Using the Schauder's estimate (17) of linear Dirichlet problems

$$Hu = -\bar{f}, \text{ in } \Omega, \qquad u = \varphi, \text{ on } \partial\Omega.$$

We have

$$\begin{aligned}
\|u\|_{S^{2,\alpha\nu}(\overline{\Omega})} &\leq C\{\|u\|_{L^\infty(\overline{\Omega})} + \|\varphi\|_{S^{2,\alpha\nu}(\partial\Omega)} + \|\bar{f}\|_{S^{\alpha\nu}(\overline{\Omega})}\} \\
&\leq C\{M_1 + \|\bar{f}\|_{S^{\alpha\nu}(\overline{\Omega})}\} \\
&\leq C_2\{K^{\theta+\alpha\nu} + K^{\theta-\alpha}K_\nu^\alpha\}.
\end{aligned}$$

We need now the following precise interpolation inequality:

$$\|u\|_{S^2(\overline{\Omega})} \leq \varepsilon\|u\|_{S^{2,\alpha}(\overline{\Omega})} + C_\alpha\varepsilon^{-2/\alpha}\|u\|_{L^\infty(\Omega)}; \tag{20}$$

$$\|u\|_{S^{1,\beta}(\overline{\Omega})} \leq \varepsilon\|u\|_{S^2(\overline{\Omega})} + C_\beta\varepsilon^{-(1+\beta)/(1-\beta)}\|u\|_{L^\infty(\Omega)}; \tag{21}$$

$$\|u\|_{S^1(\overline{\Omega})} \leq \varepsilon\|u\|_{S^2(\overline{\Omega})} + C\varepsilon^{-1}\|u\|_{L^\infty(\Omega)}, \tag{22}$$

for any $\alpha, \beta, \varepsilon \in]0,\ 1[$.

Taking $\varepsilon = \frac{1}{C_2}$ in (20), we have

$$\|u\|_{S^2(\overline{\Omega})} \leq K^{\theta+\alpha\nu} + K^{\theta-\alpha}K_\nu^\alpha + C(\nu)M_0,$$

and take $\varepsilon = K^{\alpha-\theta}/2$ in (21), then

$$\|u\|_{S^{1,\nu}(\overline{\Omega})} \leq \frac{1}{2}K^{\alpha+\alpha\nu} + \frac{1}{2}K_\nu^\alpha + C(\nu, M_0)K^{(\theta-\alpha)(1+\nu)/(1-\nu)}.$$

Take $\beta = \nu$, using $K_\beta \geq 1$, we obtain

$$\|u\|_{S^{1,\beta}(\overline{\Omega})} \leq C_3K^\theta. \tag{23}$$

Which implies that

$$\|\bar{f}\|_{S^{\alpha\beta}(\overline{\Omega})} \leq CK^{\theta+\alpha\theta-\alpha} \leq CK^{2\theta}.$$

Hence, for all $\gamma, \varepsilon \in]0,\, 1[$, we have

$$\begin{aligned}
\|u\|_{S^1(\overline{\Omega})} &\leq \varepsilon\|u\|_{S^{2,\alpha\beta\gamma}(\overline{\Omega})} + C\varepsilon^{-1}M_0 \\
&\leq \varepsilon(M_1 + \|\bar{f}\|_{S^{\alpha\beta\gamma}(\bar{})}) + C\varepsilon^{-1}M_0 \\
&\leq \varepsilon\|\bar{f}\|_{S^{\alpha\beta\gamma}(\bar{})} + C(M_0)\varepsilon^{-1}.
\end{aligned}$$

Now convexity of norms in $S^\alpha(\overline{\Omega})$ give that

$$\begin{aligned}
\|\bar{f}\|_{S^{\alpha\beta\gamma}(\overline{\Omega})} &\leq 4\left(\|\bar{f}\|_{S^{\alpha\beta}(\overline{\Omega})}\right)^\gamma\left(\|\bar{f}\|_{L^\infty(\Omega)}\right) \\
&\leq CK^{2\theta\gamma}K^{\theta(1-\gamma)} = CK^{\theta(1+\gamma)}.
\end{aligned}$$

Take $\gamma = \frac{2-\theta}{2\theta}$, we have

$$K \leq \|u\|_{S^1(\overline{\Omega})} \leq C(\theta, M_0)\varepsilon^{-1} + \varepsilon C_4K^{1+\frac{\theta}{2}}.$$

Since $\frac{\theta}{2} < 1$, let $\varepsilon = \min\{\frac{1}{2},\ (2C_4K^{\theta/2})^{-1}\}$, we have

$$K \leq C_5,$$

where C_5 is independent on u. So we have proved Theorem 3 by use (23).

End of proof of Theorem 1

The uniqueness of solution of the Dirichlet problem (1) is immediate from the comparison principle Lemma 4. The existence of solution give by abstract Theorem 2 and a priori estimates Theorem 3.

References

[1] Bony, J. M., Principe du maximum, inégalité de Harnack et uninité du problème de Cauchy pour les opérateurs elliptiques dégénérées , *Ann. Inst. Fourier* 19 (1969), 227–304.

[2] Derrdj, M., Un problème aux limites pour une classe d'opérateurs du second ordre hypoelliptiques, *Ann. Inst. Fourier* 21 (1971), 99–148.

[3] Hörmander, L., Hypoelliptic second order differential equations, *Acta Math.,* 119 (1967), 141–171.

[4] Nagel, A., Stein, E. M., and Wainger, S., Balls and metrics defined by vector fields I, basic properties, *Acta Math.,* 155 (1985), 103–147.

[5] Sanchez-Calle, A., Fundamental solutions and geometry of the sum of squares of vector fields, *Invent. Math.,* 78 (1984), 143–160.

[6] Xu, C. J., Subelliptic variational problems, *Bull. Soc. Math. France,* 118 (1990), 147–159.

[7] Xu C. J., Regularity for quasilinear second order subelliptic equations, *Comm. Pure Appl. Math.,* (1992), 77–96.

[8] Xu, C. J., Problème de Dirichlet pour les équations associées à un système de champs de vecteurs, *Séminaire E. D. P. de l'Ecole Polytechnique, 1990-1991, n. 17.*

[9] Xu C. J., Existence of bounded solutions for quasilinear subelliptic problems, *To appear in Journ.P.D.E.*

[10] Xu, C. J., Dirichlet problems for the quasilinear second order subelliptic equations, *To appear in Acta Math. Sinica.*