Nonlinear Microlocale Analysis [1]

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[1] General theory of partial differential equations and microlocal analysis (Trieste, 1995), 155–182, Pitman Res. Notes Math. Ser., 349, Longman, Harlow, 1996

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In this lecture, we will discuss theory of microlocale analysis for nonlinear equations

$$F[u] = F(x, u, \cdots, \partial^{\beta} u, \cdots)_{|\beta| \leq m} = 0.$$

If F is linear with respect to u and their derivatives, the theory of microlocale analysis is well-known, see for example [Hö]. But in the nonlinear case, it is well known that the problems of existence of solution, the regularities, and propagation of singularities are very different to linear case. We shall overpass the difficulty of existence of solutions, and assuming that u is a given solution in the whole space, having some regularity ($u \in H^s, s > s_0$). Our problem is to analyze if u belongs to $H^{s'}$ for some $s' > s$ at some point with respect to some direction, that means microlocale analysis for a solution of nonlinear equation $F[u] = 0$.

The Littlewood-Paley decomposition is key tools for this theory, we will give the characterization for Hölder space and Sobolev space. the results of decomposition can be used to define a class of linear operators with non smooth symbol (paradifferential operators), which has a symbolic calculus as that of pseudo-differential operators. And we can also obtain a linear (para-) differential equation from any nonlinear equation if it has a enough regular solution. Then the theory of microlocale analysis for nonlinear equations is carry out.

This lecture is a basic course for this theory, we study in detail the theory of Littelwood-Paley decomposition, and paradifferential operators. We give only some very simple application of this theory.

Chapter 1

Littlewood-Paley theory

1.1. Sobolev space and Hölder space

For $0 < \alpha < 1$, we define the Hölder space by

$$C^\alpha(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n); [u]_\alpha = \sup_{x,y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < +\infty\},$$

with the norm $\|u\|_{C^\alpha} = \|u\|_{L^\infty} + [u]_\alpha$. And for $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}, \alpha = [\alpha] + \beta, 0 < \beta < 1$, we define

$$C^\alpha(\mathbb{R}^n) = \{u \in C^{[\alpha]}(\mathbb{R}^n); D^\lambda u \in C^\beta(\mathbb{R}^n), |\lambda| \le [\alpha]\},$$

with the norm

$$\|u\|_{C^\alpha} = \sum_{|\lambda| \le [\alpha]} \|D^\lambda u\|_{C^\beta}.$$

Then C^α are Banach space, and it is evident that they norm are equivalent to

$$\|u\|'_{C^\alpha} = \sum_{|\lambda| \le [\alpha]} \|D^\lambda u\|_{L^\infty} + \sum_{|\lambda| = [\alpha]} [D^\lambda u]_\beta.$$

For $\alpha = 1$, Zygmund space C^1_* (module the affine functions) is defined by $C^1_* = \{u \in C^0(\mathbb{R}^n); [u]^*_1 < +\infty\}$, where $[u]^*_1 = \sup_{h,x \in \mathbb{R}^n} |u(x+h) + u(x-h) - 2u(x)|/|h|$, then C^1_* is also Banach space with norm $[u]^*_1$. For $m \in \mathbb{N}$, we define $C^m_* = \{u \in C^1_*; D^\lambda u \in C^1_*, |\lambda| \le m-1\}$. From now for $\alpha \in \mathbb{R}_+$, without confusion we denote C^α, if $\alpha \in \mathbb{N}$ it is Zygmund space, if $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ it is hölder space.

We define now Sobolev space $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$,

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}'; (1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^n)\},$$

with the norm $\|f\|_{H^s} = \|(1 + |\xi|^2)^{s/2} \hat{f}\|_{L^2}$, then H^s is Hilbert space.

In this lecture, we will give a characterization of function spaces C^α and H^s by Littlewood-Paley decomposition. We consider now a dyadic decomposition of $\mathbb{R}^n$, for $K > 1$ a fixed constant, and $p \in \mathbb{N}_+$ we set

$$C_p = \{\xi \in \mathbb{R}^n; K^{-1} 2^p \le |\xi| \le K 2^{p+1}\}, \tag{1.1}$$

and $C_{-1} = B(0, K) = \{\xi \in \mathbb{R}^n; |\xi| \le K\}$, then $\{C_p\}_{-1}^{+\infty}$ is a recover of $\mathbb{R}^n$.

Lemma 1.1 *There exist* N_1 *which depends only on* K *such that for any* p

$$\#\{q; C_q \bigcap C_p \neq \emptyset\} \leq N_1.$$

That means $\{C_p\}_{-1}^{+\infty}$ is a uniformly finite recover of $\mathbb{R}^n$.

Proof: Fixed p, and suppose that $C_q \bigcap C_p \neq \emptyset$. The cas $q \geq p$ and $q \leq p$ is similar, we consider only $q \leq p$ and $p \neq -1$. If $\xi \in C_q \bigcap C_p, q \neq -1$, then

$$K^{-1} 2^p \leq |\xi| \leq K 2^{q+1},$$

which give $2^{p-q} \leq 2K^2$, and $p - q \leq [1 + 2\log_2 K]$. Note $N_1 = 2[1 + 2\log_2 K] + 2$, we have proved Lemma.

Lemma 1.2 *There exist* $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$*, with Supp*$\psi \subset C_{-1}$*, Supp*$\varphi \subset C_0$*, such that for any* $\xi \in \mathbb{R}^n$ *and* N_0*, we have*

$$\psi(\xi) + \sum_{p=0}^{\infty} \varphi(2^{-p}\xi) = 1,$$

and

$$\psi(\xi) + \sum_{p=0}^{N_0-1} \varphi(2^{-p}\xi) = \psi(2^{-N_0}\xi).$$

Proof: Take $\theta \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \theta \leq 1, \mathrm{Supp}\theta \subset C_0$, and $\theta(\xi) = 1$ for $1 \leq |\xi| \leq 2$. Set

$$s(\xi) = \sum_{p=-\infty}^{\infty} \theta(2^{-p}\xi), \qquad \xi \in \mathbb{R}^n \setminus 0,$$

then from Lemma 1.1, $s \in C^\infty(\mathbb{R}^n \setminus 0)$. We define now

$$\varphi(\xi) = \theta(\xi)/s(\xi).$$

If $|\xi| \geq K, p \leq -1$, we have $2^{-p}|\xi| = 2^{|p|}|\xi| \geq 2^{|p|}K \geq 2K$, and $2^{-p}\xi \notin C_0$, then $\theta(2^{-p}\xi) = 0$. Therefore, if $|\xi| \geq K$,

$$\sum_{p=0}^{\infty} \varphi(2^{-p}\xi) = \sum_{p=-\infty}^{\infty} \frac{\theta(2^{-p}\xi)}{s(2^{-p}\xi)} = \frac{\sum_{p=-\infty}^{\infty} \theta(2^{-p}\xi)}{\sum_{p=-\infty}^{\infty} \theta(2^{-p}\xi)} = 1.$$

here we have used the fact $s(2^{-p}\xi) = \sum_{q=-\infty}^{\infty} \theta(2^{-(q+p)}\xi) = \sum_{p_1=-\infty}^{\infty} \theta(2^{-p_1}\xi) = s(\xi)$. Take now $\psi(\xi) = 1 - \sum_{p=0}^{\infty} \varphi(2^{-p}\xi)$, then $\psi \in C_0^\infty(\mathbb{R}^n), \mathrm{Supp}\psi \subset C_{-1}$. Now for any N_0,

$$\begin{aligned} & \psi(2^{-N_0}\xi) + \sum_{p=0}^{\infty} \varphi(2^{-p-N_0}\xi) = 1 \\ = \quad & \psi(\xi) + \sum_{p=0}^{N_0-1} \varphi(2^{-p}\xi) + \sum_{p=N_0}^{\infty} \varphi(2^{-p}\xi). \end{aligned}$$

Since $\sum_{p=0}^{\infty} \varphi(2^{-p-N_0}\xi) = \sum_{p_1=N_0}^{\infty} \varphi(2^{-p_1}\xi)$, we have proved Lemma.

1.2. Dyadic decomposition for C^α and H^s

Using the dyadic decomposition of $\mathbb{R}^n$ and the associated partition of unity, we will give a characterization of Hölder space and Sobolev space. For $\psi, \varphi \in C_0^\infty(\mathbb{R}^n)$, we can define the pseudo-differential operators $\psi(D)$ as

$$\psi(D)u(x) = \int e^{i\langle x,\xi\rangle}\psi(\xi)\hat{u}(\xi)d\xi.$$

Definition 1.1 *For $u \in \mathcal{S}'(\mathbb{R}^n)$, we define their Littlewood-Paley decomposition (or dyadic decomposition) $\{u_p\}_{p=-1}^\infty$ as $u_{-1} = \psi(D)u, u_p = \varphi(2^{-p}D)u$.*

It is evident that $u_p \in \mathcal{S}'$ for any $u \in \mathcal{S}'$, and $\text{Supp}\hat{u}_p \subset C_p$, we have

Theorem 1.1 *For $u \in \mathcal{S}'$, we have $u = \sum_{p=-1}^\infty u_p$, in sense of $\mathcal{S}'$.*

Proof: For any $f \in \mathcal{S}$, we have $\hat{f}(\xi) = \sum_{p=-1}^\infty \hat{f}_p$ in the sense of $\mathcal{S}$, then for any $u \in \mathcal{S}'$, using Parseval formula, we have

$$\begin{aligned}(u,f) &= (2\pi)^{-n}(\hat{u},\hat{f}) = (2\pi)^{-n}\sum_{p=-1}^\infty(\hat{u},\hat{f}_p)\\ &= (2\pi)^{-n}\sum_{p=-1}^\infty(\hat{u}_p,\hat{f}) = (2\pi)^{-n}(\sum_{p=-1}^\infty \hat{u}_p,\hat{f})\\ &= (\sum_{p=-1}^\infty u_p, f).\end{aligned}$$

We have proved Theorem.

Since $u_p \in \mathcal{S}', \text{Supp}\hat{u}_p \subset C_p$, Paley- Wienner-Schwartz theorem implies that $u_p \in C^\infty$, we will use this proprieties to characterized Sobolev space and Hölder space.

Theorem 1.2 *For $s > 0$, the following properties are equivalent.*

(a) $u \in H^s(\mathbb{R}^n)$;

(b) $u = \sum_{p=-1}^\infty u_p$, with $\text{Supp}\hat{u}_p \subset C_p$ and

$$\|u_p\|_{L^2} \le c_p 2^{-ps}, \quad \{c_p\} \in l^2;$$

(c) $u = \sum_{p=-1}^\infty u_p$, with $\text{Supp}\hat{u}_p \subset B(0, K_1 2^p)$ and

$$\|u_p\|_{L^2} \le c_p 2^{-ps}, \quad \{c_p\} \in l^2;$$

(d) $u = \sum_{p=-1}^\infty u_p$, with $u_p \in C^\infty$ and for any $\alpha \in \mathbb{N}^n$,

$$\|D^\alpha u_p\|_{L^2} \le c_{p,\alpha} 2^{-p(s-|\alpha|)}, \quad \{c_{p,\alpha}\} \in l^2.$$

Proof: The equivalence of (a) and (b) is for all $s \in \mathbb{R}$, suppose that $u \in H^s(\mathbb{R}^n) \subset \mathcal{S}'$, and Littlewood- Paley decomposition of u as $\sum_{p=-1}^\infty u_p$, then $\text{Supp}\hat{u}_p \subset C_p$, and

$$\begin{aligned}\|u_p\|_{H^s}^2 &= \int(1+|\xi|^2)^s|\hat{u}_p(\xi)|^2 d\xi\\ &= \int(1+|\xi|^2)^s|\varphi(2^{-p}\xi)\hat{u}(\xi)|^2 d\xi\\ &= \int_{C_p}(1+|\xi|^2)^s|\varphi(2^{-p}\xi)\hat{u}(\xi)|^2 d\xi,\end{aligned}$$

since for any $s \in \mathbb{R}$ and $\xi \in C_p$, we have $K_1 2^{2ps} \leq (1+|\xi|^2)^s \leq K_2 2^{2ps}$, we obtain

$$K_1 \|u_p\|_{L^2} \leq 2^{-ps} \|u_p\|_{H^s} \leq K_2 \|u_p\|_{L^2}. \tag{1.2}$$

Set now $c_p = \|u_p\|_{H^s}$, we want to prove $\{c_p\} \in l^2$, since for $|p-q| \geq N_1, C_p \bigcap C_q = \emptyset$, then series

$$S_{q,N_1} u = \sum_{k=0}^{\infty} u_{q+kN_1}$$

is orthogonal, and

$$\begin{aligned} \|S_{q,N_1} u\|^2_{H^s} &= \int (1+|\xi|^2)^s |\widehat{S_{q,N_1}} u|^2 d\xi \\ &= \sum_{k=0}^{\infty} \int (1+|\xi|^2)^s |\hat{u}_{q+kN_1}(\xi)|^2 d\xi \\ &= \sum_{k=0}^{\infty} \|u_{q+kN_1}\|^2_{H^s}, \end{aligned}$$

but

$$\sum_{p=-1}^{\infty} \|u_p\|^2_{H^s} \leq N_1 \sum_{k=0}^{\infty} \|u_{q+kN_1}\|^2_{H^s} = N_1 \|S_{q,N_1} u\|^2_{H^s} \leq \|u\|^2_{H^s} \leq N_1 \|u\|^2_{H^s}.$$

We have proved (a) ⇒ (b). In inverse, if we have (b), then (1.2) give $\|u_p\|_{H^s} \leq K_2 c_p$, and

$$\begin{aligned} \|u\|^2_{H^s} &\leq \sum_{q=0}^{N_1-1} \|S_{q,N_1} u\|^2_{H^s} = \sum_{q=0}^{N_1-1} \left(\sum_{k=0}^{\infty} \|u_{q+kN_1}\|^2_{H^s} \right) \\ &\leq K_2^2 \sum_{p=0}^{\infty} c_p^2 < +\infty. \end{aligned}$$

that means $u \in H^s(\mathbb{R}^n)$, then (a) ⇔ (b).

(b) ⇒ (c) is evident, since $C_p \subset B(0, 2K2^p)$.

Suppose that we have (c), and $u = \sum_{p=-1}^{\infty} u_p$, with $\mathrm{Supp}\hat{u}_p \subset B(0, K_1 2^p)$, then $u_p \in C^\infty$, and for all $\forall \alpha \in \mathbb{N}^n$, we have

$$\begin{aligned} \|D^\alpha u_p\|_{L^2} &= \|\widehat{D^\alpha u_p}\|_{L^2} = \|\xi^\alpha \hat{u}_p(\xi)\|_{L^2} \\ &\leq K_1^{|\alpha|} 2^{p|\alpha|} \|\hat{u}_p\|_{L^2} = K_1^{|\alpha|} 2^{p|\alpha|} \|u_p\|_{L^2} \\ &\leq K_1^{|\alpha|} c_p 2^{-ps+p|\alpha|} = c_{p,\alpha} 2^{-ps+p|\alpha|}, \end{aligned}$$

we have proved (c) ⇒ (d).

We prove now (c) ⇒ (a), it is now necessary $s > 0$. From (c), we have immediately $u = \sum u_p \in L^2$, and the properties $\mathrm{Supp}\hat{u}_p \subset B(0, K_1 2^p)$ implies

$$v_k = \varphi(2^{-k} D) u = \varphi(2^{-k} D) \sum_{|p-k| \leq N_1} u_p.$$

Then, we have

$$\|v_k\|^2_{L^2} = \left\| \sum_{p=k-N_1}^{\infty} \varphi(2^{-p} D) u_p \right\|^2_{L^2} = \int \left| \sum_{p=k-N_1}^{\infty} \varphi(2^{-k} D) u_p(x) \right|^2 dx$$

$$\begin{aligned}
&\leq \int \left(\sum_{p=k-N_1}^{\infty} 2^{2ps} |\varphi(2^{-k}D)u_p(x)|^2 \right) \left(\sum_{p=k-N_1}^{\infty} 2^{-2ps} \right) dx \\
&\leq C2^{-2ks} \sum_{p=-1}^{\infty} 2^{2ps} \|\varphi(2^{-k}D)u_p\|_{L^2}^2.
\end{aligned}$$

Here we have used the fact $s > 0$ to get $\sum_{p=k-N_1}^{\infty} 2^{-2ps} \leq C2^{-2ks}$. As in the proof of (a) $\Rightarrow$ (b), we have

$$\sum_{k=-1}^{\infty} \|\varphi(2^{-k}D)u_p\|_{L^2}^2 \leq C\|u_p\|_{L^2}^2.$$

We set $c_k^2 = \sum_{p=-1}^{\infty} 2^{2ps} \|\varphi(2^{-k}D)u_p\|_{L^2}^2$, then $\{c_k\} \in l^2$, and $u = \sum v_k$ verifies the condition of (b), we have proved (c) $\Rightarrow$ (b) $\Leftrightarrow$ (a).

Now it is enough to prove (d) $\Rightarrow$ (a). Under assumption of (d), we have firstly $u = \sum u_p \in L^2$. Take $\alpha \in \mathbb{N}^n, |\alpha| = s_0 > s > 0$, and $\psi_k(\xi) = \psi(2^{-k}\xi) \in C_0^\infty(\mathbb{R}^n)$ with $\text{Supp}\psi_k \subset B(0, C_2 2^{k+1}), \psi_k(\xi) = 1, |\xi| \leq C_1 2^k$, then

$$\text{Supp}\psi_k(1-\psi_k) \subset \{\xi \in \mathbb{R}^n; C_1 2^k \leq |\xi| \leq C_2 2^{k+1}\}.$$

Set $\hat{u}_k(\xi) = \psi_k(\xi)\hat{u}_k(\xi) + (1-\psi_k(\xi))\hat{u}_k(\xi) = \hat{u}_k^{(1)}(\xi) + \hat{u}_k^{(2)}(\xi)$, we have

$$\begin{aligned}
\|u_k\|_{L^2}^2 &= \|\hat{u}_k\|_{L^2}^2 = \int |\hat{u}_k^{(1)}(\xi) + \hat{u}_k^{(2)}(\xi)|^2 d\xi \\
&= \int |\hat{u}_k^{(1)}(\xi)|^2 d\xi + 2\int \psi_k(\xi)(1-\psi_k(\xi))|\hat{u}_k(\xi)|^2 d\xi + \int |\hat{u}_k^{(2)}|^2 d\xi.
\end{aligned}$$

Since $0 \leq \psi_k(\xi)(1-\psi_k(\xi)) \leq 1$, we have

$$\|u_k^{(1)}\|_{L^2}^2 + \|u_k^{(2)}\|_{L^2}^2 \leq \|u_k\|_{L^2}^2 \leq c_k^2 2^{-2ks}.$$

Similarly,

$$\|u_k^{(1)}\|_{H^{s_0}}^2 + \|u_k^{(2)}\|_{H^{s_0}}^2 \leq \|u_k\|_{H^{s_0}}^2 \leq c_k^2 2^{-2k(s-s_0)}.$$

Set $u^{(1)} = \sum u_k^{(1)}, u^{(2)} = \sum u_k^{(2)}$, then $u = u^{(1)} + u^{(2)}$, and from (c), $u^{(1)} \in H^s$, for $u^{(2)}$, we have

$$\begin{aligned}
\|\varphi(2^{-p}D)u^{(2)}\|_{L^2}^2 &= \int |\sum_{k\leq p+N_0} \varphi(2^{-p}D)u_k^{(2)}|^2 dx \\
&\leq \left(\sum_{k\leq p+N_0} 2^{-2k(s-s_0)} \right) \left(\int \sum_{k\leq p+N_0} 2^{2k(s-s_0)} |\varphi(2^{-p}D)u_k^{(2)}|^2 dx \right) \\
&\leq \frac{1-2^{-2(p+N_0+1)(s-s_0)}}{1-2^{-(s-s_0)}} 2^{-2ps_0} \sum_{k\leq p+N_0} 2^{2k(s-s_0)} \|\varphi(2^{-p}D)u_k^{(2)}\|_{H^{s_0}}^2.
\end{aligned}$$

But for $s_0 > s > 0$, we have

$$\frac{1-2^{-2(p+N_0+1)(s-s_0)}}{1-2^{-(s-s_0)}} 2^{-2ps_0} \leq C2^{-2ps}$$

with C independent on p. Set now $c_p^2 = \sum_{k\leq p+N_0} 2^{2k(s-s_0)} \|\varphi(2^{-p}D)u_k^{(2)}\|_{H^{s_0}}^2$, then

$$\sum_p c_p^2 \leq \sum_k 2^{2k(s-s_0)} \|u_k^{(2)}\|_{H^{s_0}}^2 < +\infty.$$

We have proved $u^{(2)} = \sum_p \varphi(2^{-p}D)u^{(2)} \in H^s$.

Before to study Hölder space C^α, we give a lemma.

Lemma 1.3 *Suppose that* $a \in L^\infty(\mathbb{R}^n)$, *Supp*$\hat{a} \subset B(0,R)$, *then* $a \in C^\infty(\mathbb{R}^n)$, *and for any* $\alpha \in \mathbb{N}^n$ *there exist* $C(n,\alpha) > 0$ *such that*

$$\|D^\alpha a\|_{L^\infty} \leq C(n,\alpha)R^{|\alpha|}\|a\|_{L^\infty}. \tag{1.3}$$

Proof: Choose $\psi \in C_0^\infty(\mathbb{R}^n), \mathrm{Supp}\psi \subset B(0,2), \psi(\xi) = 1$ for $|\xi| \leq 1$, and set $\psi_R(\xi) = \psi(R^{-1}\xi)$, then we have $\hat{a}(\xi) = \psi_R(\xi)\hat{a}(\xi)$ and

$$a(x) = (R^n\hat{\psi}(-R\cdot) * a(\cdot))(x) \in C^\infty(\mathbb{R}^n).$$

We have $D^\alpha a(x) = R^{n+|\alpha|}((D^\alpha\hat{\psi})(-R\cdot) * a(\cdot))(x)$, and $\|R^n(D^\alpha\hat{\psi})(-R\cdot)\|_{L^1} = C(n,\alpha) < +\infty$, which give

$$\|D^\alpha a\|_{L^\infty} \leq \|R^n(D^\alpha\hat{\psi})(-R\cdot)\|_{L^1}R^{|\alpha|}\|a\|_{L^\infty}.$$

For Hölder space we have a similar results as Theorem 1.2.

Theorem 1.3 *For any* $\alpha > 0$, *and* $\alpha = l + \beta, l \in \mathbb{N}, 0 < \beta \leq 1$, *the following properties are equivalents:*

(a) $u \in C^\alpha$*;*

(b) $u = \sum_{p=-1}^\infty u_p$ *with Supp*$\hat{u}_p \subset C_p$, *and* $\|u_p\|_{L^\infty} \leq C2^{-p\alpha}$*;*

(c) $u = \sum_{p=-1}^\infty u_p$ *with Supp*$\hat{u}_p \subset B(0, K_1 2^p)$, *and* $\|u_p\|_{L^\infty} \leq C2^{-p\alpha}$*;*

(d) $u = \sum_{p=-1}^\infty u_p$ *with* $u_p \in C^{l+1}$, *and for all* $\lambda \in \mathbb{N}^n, |\lambda| \leq l+1$, $\|D^\lambda u_p\|_{L^\infty} \leq C_\lambda 2^{-p(\alpha+|\lambda|)}$.

Proof: Suppose that $u \in C^\alpha$, and $u = \sum_{p=-1}^\infty u_p$ his Littlewood-Paley decomposition. Since $\mathrm{Supp}\hat{u}_{-1} \subset B(0,K_1)$, we have

$$\|u_{-1}\|_{L^\infty} \leq C(n)K_1\|u\|_{L^\infty}.$$

For $p > -1$, denote by $\tilde{\varphi}$ the transformation of φ, then $\tilde{\varphi}_p(x) = 2^{np}\tilde{\varphi}(2^p x)$, and for any $\lambda \in \mathbb{N}^n$, we have

$$\int x^\lambda \tilde{\varphi}(x)dx = D_\xi^\lambda \int e^{-ix\xi}\tilde{\varphi}(x)dx|_{\xi=0} = D_\xi^\lambda \varphi(0) = 0. \tag{1.4}$$

Now for any $f \in C^\alpha(\mathbb{R}^n), \alpha = l + \beta$, the Taylor formula give

$$\begin{aligned}
f(x) &= \sum_{|\lambda|<l} \frac{1}{\lambda!}\partial^\lambda f(y)(x-y)^\lambda \\
&+ \frac{l}{\lambda!}(x-y)^\lambda \int \sum_{|\lambda|=l} \partial^\lambda f(y+t(x-y))(1-t)^{l-1}dt \\
&= \sum_{|\lambda|\leq l} \frac{1}{\lambda!}\partial^\lambda f(y)(x-y)^\lambda \\
&+ \frac{l}{\lambda!}(x-y)^\lambda \int \sum_{|\lambda|=l} [\partial^\lambda f(y+t(x-y)) - \partial^\lambda f(y)](1-t)^{l-1}dt,
\end{aligned}$$

then we have

$$|f(x) - \sum_{|\lambda|\leq l} \frac{1}{\lambda!}\partial^\lambda f(y)(x-y)^\lambda| \leq C_l|x-y|^\alpha\|f\|_{C^\alpha}.$$

And for $u_p = \tilde{\varphi}_p * u$, using (1.4), we have

$$\begin{aligned} u_p(x) &= \int \tilde{\varphi}_p(x-y)u(y)dy \\ &= \int \tilde{\varphi}_p(x-y)[u(y) - \sum_{|\lambda|\leq l} \frac{1}{\lambda}(x-y)^\lambda \partial^\lambda u(x)]dy, \end{aligned}$$

hence

$$\begin{aligned} \|u_p\|_{L^\infty} &\leq C\|u\|_{C^\alpha} \int |\tilde{\varphi}_p(x-y)||x-y|^\alpha dy \\ &= C\|u\|_{C^\alpha} 2^{-p\alpha} \int |\tilde{\varphi}(x)||x|^\alpha dx \\ &\leq C_\alpha \|u\|_{C^\alpha} 2^{-p\alpha}. \end{aligned}$$

We have proved (a) ⇒ (b).

(b) ⇒ (c) is evident.

(c) ⇒ (d) is reduced by Lemma 1.3.

We prove now (d) ⇒ (a), take sum we have immediately $u \in C^l(\mathbb{R}^n)$, and for all $|\lambda| = l, p \in \mathbb{N}_+, x, y \in \mathbb{R}^n$,

$$|\partial^\lambda u_p(x) - \partial^\lambda u_p(y)| \leq M 2^{p(1-\beta)}|x-y|.$$

Fixed $|x-y| > 0$ small, take p_0 such that $2^{p_0} \leq |x-y|^{-1} \leq 2^{p_0+1}$, then

$$\begin{aligned} \partial^\lambda u(x) - \partial^\lambda u(y) &= \sum_{p\leq p_0} [\partial^\lambda u_p(x) - \partial^\lambda u_p(y)] \\ &+ \sum_{p>p_0} [\partial^\lambda u_p(x) - \partial^\lambda u_p(y)] \\ &= I + II. \end{aligned}$$

For the first term, we have

$$I \leq M|x-y| \sum_{p\leq p_0} 2^{p(1-\beta)} \leq 2M|x-y|2^{p_0(1-\beta)} \leq 2M|x-y|^\beta.$$

For the second term, we have

$$\begin{aligned} II &\leq \sum_{p>p_0} |\partial^\lambda u_p(x)| + \sum_{p>p_0} |\partial^\lambda u_p(y)| \\ &\leq 2M \sum_{p>p_0} 2^{-p\beta} \leq 4M 2^{-p_0\beta} \leq 8M|x-y|^\beta, \end{aligned}$$

we obtain for $|\lambda| = l$,

$$|\partial^\lambda u(x) - \partial^\lambda u(y)| \leq C|x-y|^\beta.$$

Then $u \in C^\alpha(\mathbb{R}^n)$.

Remark Since the Littlewood-Paley decomposition is well-defined in the distribution space $\mathcal{S}'$. We large the definition of Hölder space for index $\alpha < 0$ as the equivalence of (a) and (b) in Theorem 1.2.

Definition 1.2 *For $\alpha \in \mathbb{R}$, and $u \in \mathcal{S}'$, if his Littlewood-Paley decomposition $u = \sum_{p=-1}^\infty u_p$ satisfy $\|u_p\|_{L^\infty} \leq C2^{-p\alpha}$, then we say $u \in C^\alpha$.*

Using this definition, we have in Theorem 1.3 the equivalence of (a) and (b) for all $\alpha \in \mathbb{R}$, and as for Sobolev space, we have

Theorem 1.4 *Suppose that* $P(D) \in S^m_{1,0}$, *then* $P : C^\alpha \to C^{\alpha-m}$ *is continuous for all* $\alpha \in \mathbb{R}$.

Proof: Without loss general, we can suppose that $P(\xi)$ is homogeneous of degree m for $|\xi| \geq A$. Choose N_0 big enough such that $K^{-1}2^{N_0} \geq A$. For $u \in C^\alpha$ and $u = \sum u_p$ his Littlewood-Paley decomposition, set $v_p = P(D)u_p$. Then $\hat{v}_p(\xi) = P(\xi)\hat{u}_p(\xi)$, and $\text{Supp}\hat{v}_p \subset C_p$, choose now $\Phi \in C^\infty_0(\mathbb{R}^n)$ with $\Phi(\xi) = 1$ on C_0, and $\text{Supp}\Phi \subset C'_0 = \{\xi \in \mathbb{R}^n; K'^{-1} \leq |\xi| \leq 2K'\}$, then $\hat{u}_p(\xi) = \Phi(2^{-p}\xi)\hat{u}_p(\xi)$. Set $\Psi(\xi) = P(\xi)\Phi(\xi)$, and it is Fourier transformation of $h(x)$, then $\Psi_p(\xi) = P(\xi)\Phi(2^{-p}\xi) = 2^{mp}P(2^{-p}\xi)\Phi(2^{-p}\xi)$, and it is the transformation of $2^{(m+n)p}h(2^p x)$, and

$$v_p(x) = 2^{(m+n)p}(h(2^p\cdot) * u_p)(x) = 2^{mp}\int h(t)u_p(x - 2^{-p}t)dt.$$

Since $h \in L^1(\mathbb{R}^n)$, Hausdorff-Young inequality give

$$\|v_p\|_{L^\infty} \leq C2^{mp}\|u_p\|_{L^\infty} \leq C2^{-p(\alpha-m)}.$$

We have proved $v = P(D)u \in C^{\alpha-m}$.

1.3. Sobolev embedding Theorem

Using the Littlewood-Paley decomposition, we have a very simple proof of Sobolev embedding theorem.

Theorem 1.5 *(a) For any* $s \in \mathbb{R}$, *we have continuous embedding* $H^s \subset C^{s-n/2}$;

(b) For any $0 < s < n/2$, *we have continuous embedding* $H^s \subset L^p$ *where* $p = \frac{2n}{n-2s}$;

(c) We have continuous embedding $H^{n/2} \subset VMO$.

Proof: (a) Suppose that $u \in H^s, u = \sum_{p=-1}^\infty u_p$ the Littlewood-Paley decomposition. Take $\Phi(\xi)$ as in the proof of Theorem 1.4, set $\hat{h} = \Phi$, then

$$u_p(x) = 2^{np}[h(2^p\cdot) * u_p](x) = 2^{np}\int u_p(t)h[2^p(x-t)]dt,$$

and Schwartz inequality give

$$\begin{aligned}\|u_p\|_{L^\infty} &\leq \|u_p\|_{L^2}\|2^{np}h(2^p x)\|_{L^2} \\ &= \|u_p\|_{L^2}\left(\int |2^{np}h(2^p x)|^2 dx\right)^{1/2} \\ &= 2^{\frac{1}{2}np}\|u_p\|_{L^2}\|h\|_{L^2}.\end{aligned}$$

We have proved for $u \in H^s$,

$$\|u_p\|_{L^\infty} \leq 2^{-p(s-\frac{n}{2})}c_p\|h\|_{L^2}.$$

Then $u \in C^{\alpha-\frac{n}{2}}$.

(b) We give here a direct proof by Fourier transformation, similar proof can be easily given by use Littlewood-Paley decomposition. For $f \in L^p$, we have

$$\|f\|^p_{L^p} = p\int_0^\infty \lambda^{p-1}m\{|f| > \lambda\}d\lambda,$$

where $m\{|f|>\lambda\}$ is the Lebesgue measure of set $\{x\in \mathbb{R}^n; |f|>\lambda\}$. For $A>0$, set $f_{1,A}=\mathcal{F}^{-1}(\mathbf{1}_{B(0,A)}\hat{f})$, $f=f_{1,A}+f_{2,A}$, we have

$$\begin{aligned}\|f_{1,A}\|_{L^\infty} &\leq \|\hat{f}_{1,A}\|_{L^1}\leq \|f\|_{H^s}(\int_{B(0,A)}\langle\xi\rangle^{-2s}d\xi)^{1/2}\\ &\leq CA^{n/2-s}\|f\|_{H^s}.\end{aligned}$$

For any $A>0$, we have $\{|f|>\lambda\}\subset\{2|f_{1,A}|>\lambda\}\bigcup\{2|f_{2,A}|>\lambda\}$. Choose $A_\lambda=(\lambda/(4C\|f\|_{H^s}))^{p/n}$, then $m\{|f_{1,A_\lambda}|>\lambda/2\}=0$, and

$$\|f\|^p_{L^p}\leq p\int_0^\infty \lambda^{p-1}m\{2|f_{2,A_\lambda}|>\lambda\}d\lambda.$$

but

$$\begin{aligned}m\{|f_{2,A_\lambda}|>\lambda/2\} &= \int_{\{|f_{2,A_\lambda}|>\lambda/2\}}dx\\ &\leq \int_{\{|f_{2,A_\lambda}|>\lambda/2\}}\frac{4|f_{2,A_\lambda}(x)|^2}{\lambda^2}dx\\ &\leq \frac{4}{\lambda^2}\|f_{2,A_\lambda}\|^2_{L^2}.\end{aligned}$$

We obtain

$$\begin{aligned}\|f\|^p_{L^p} &\leq p\int_0^\infty\lambda^{p-3}\|f_{2,A_\lambda}\|^2_{L^2}d\lambda\\ &= p(2\pi)^n\int_{\mathbb{R}_+\times\mathbb{R}^n}\lambda^{p-3}\mathbf{1}_{\{(\lambda,\xi);|\xi|\geq A_\lambda\}}|\hat{f}(\xi)|^2d\xi d\lambda.\end{aligned}$$

By the definition of A_λ, we have $|\xi|\geq A_\lambda\Rightarrow\lambda\leq C_\xi=4C\|f\|_{H^s}\langle\xi\rangle^{n/p}$. Then Fubini theorem implies that

$$\begin{aligned}\|f\|^p_{L^p} &\leq p(2\pi)^n\int_{\mathbb{R}^n}(\int_0^{C_\xi}\lambda^{p-3}d\lambda)|\hat{f}(\xi)|^2d\xi\\ &\leq C_p\|f\|^{p-2}_{H^s}\int_{\mathbb{R}^n}\langle\xi\rangle^{n(p-2)/p}|\hat{f}(\xi)|^2d\xi,\end{aligned}$$

since $n(p-2)/p=2s$, we have proved (b).

(c) By the definition, $u\in$ VMO if and only if $u\in L^1_{\text{loc}}(\mathbb{R}^n)$, and

$$\sup_B|B|^{-1}\int_B|u-u_B|dx<+\infty; \tag{1.5}$$

$$\lim_{\text{diam}B\to 0}|B|^{-1}\int_B|u-u_B|dx=0, \tag{1.6}$$

where $B=B(x_0,R)$, the norm is give by (1.5). Take now $u\in H^{n/2}(\mathbb{R}^n)$, since $\mathcal{S}$ is dense in $H^{n/2}(\mathbb{R}^n)$, we need only to prove (1.5). For any balls $B(x_0,R)$ with $R>0$ small enough, there exist N_0 such that $2^{-N_0-1}\leq R\leq 2^{-N_0}$, we cup the Littlewood-Paley decomposition of u as $u=\sum_{p=-1}^\infty u_p=\sum_{p=-1}^{N_0-1}u_p+\sum_{p=N_0}^\infty u_p=u^{(1)}+u^{(2)}$, then

$$\begin{aligned}&\left(|B|^{-1}\int_B|u-u_B|dx\right)^2\leq|B|^{-1}\int_B|u-u_B|^2dx\\ &\leq 2|B|^{-1}\int_B|u^{(1)}-u^{(1)}_B|^2dx+2|B|^{-1}\int_B|u^{(2)}-u^{(2)}_B|^2dx.\end{aligned}$$

For the first term, using Poincaré inequality and results of (a), we have

$$\begin{aligned} &|B|^{-1}\int_B |u^{(1)}-u_B^{(1)}|^2dx \le CR^2|B|^{-1}\int_B |Du^{(1)}|^2dx \\ &\le CR^2\|Du^{(1)}\|_{L^\infty}^2 \le CR^2\left(\sum_{p=-1}^{N_0-1}C2^p\right)^2 \le CR^2 2^{2N_0}\le C \end{aligned}$$

where $C = C_0\|u\|_{H^{n/2}}^2$ independent on x_0, R. For the second term,

$$\begin{array}{rl} & |B|^{-1}\displaystyle\int_B |u^{(2)}-u_B^{(2)}|^2dx \le 2|B|^{-1}\int_B |u^{(2)}|^2dx \\ \le & CR^{-n}\displaystyle\int_{\mathbb{R}^n} |\hat{u}^{(2)}(\xi)|^2d\xi \le CR^{-n}\int_{|\xi|\ge K^{-1}2^{N_0}} |\hat{u}(\xi)|^2d\xi \\ \le & C\displaystyle\int_{|\xi|\ge K^{-1}2^{N_0}} (1+|\xi|)^n|\hat{u}(\xi)|^2d\xi \le C\|u\|_{H^{n/2}}^2. \end{array}$$

We have proved (1.5) for small R, since $u \in H^{n/2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, (1.5) is evident for $R > R_0$ with constant $C = C(R_0)$.

Using Littlewood-Paley decomposition, we can also define Besov space.

Definition 1.3 *For $s \in \mathbb{R}, p, r \in [1, +\infty]$, we say $u \in B_{p,r}^s$, if $u \in \mathcal{S}'$ and his Littlewood-Paley decomposition $\{u_q\}$ satisfies the condition: $\|u_q\|_{L^p} \le c_q 2^{-qs}$, with $\{c_q\} \in l^r$. The norm is defined by $\|u\|_{B_{p,r}^s} = \|2^{qs}\|u_q\|_{L^p}\|_{l^r}$.*

By the definition, we have $B_{2,2}^s = H^s$, and $B_{\infty,\infty}^\alpha = C^\alpha$. The embedding theorem is now in the form $B_{p_1,r}^s \subset B_{p_2,r}^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}$ for any $p_2 \ge p_1 \ge 1$. The operators T_a is well-defined on Besov space, we have

$$\begin{array}{rcll} \|T_u v\|_{B_{p,r}^s} & \le & C\|u\|_{L^\infty}\|v\|_{B_{p,r}^s}, & \forall s \in \mathbb{R}; \\ \|T_u v\|_{B_{p,r}^{s+\rho}} & \le & C\|u\|_{B_{0,0}^\rho}\|v\|_{B_{p,r}^s}, & \forall s \in \mathbb{R}, \rho < 0; \\ \|R(u,v)\|_{B_{p,r}^{s_1+s_2}} & \le & C\|u\|_{B_{p_1,r_1}^{s_1}}\|v\|_{B_{p_2,r_2}^{s_2}}, & \end{array}$$

for $s_1 + s_2 > 0, \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \le 1, \frac{1}{r} = \min(1, \frac{1}{r_1} + \frac{1}{r_2})$.

Chapter 2

Paradifferential operators

2.1. Paramultipication

There is two essentials problems in the theory of nonlinear partial differential equations, the first one is nonlinearity, it restrict to consider weak solution in some algebra. The second one is that we must work for operators with no regularities coefficients. We will use Littlewood-Paley decomposition of first lecture to study those problems.

Theorem 2.1 *If $s > n/2$, then H^s is an algebra.*

Proof: Take $u, v \in H^s$, denote by $u = \sum u_p, v = \sum v_q$ their Littlewood-Paley decompositions, then we have

$$uv = \sum_{p,q} u_p v_q.$$

For some N_0 big enough, $B(0, 4K2^{-N_0})$ is a very small balls, set

$$C_0' = C_0 + B(0, 4K2^{-N_0}),$$

then $\{C_p'\}$ possess same properties of $\{C_p\}$. We define

$$S_q u = \sum_{-1 \le p \le q - N_0} u_p, \tag{2.1}$$

and

$$\begin{aligned} T_u v &= \sum_q (S_q u) v_q, && (2.2)\\ R(u,v) &= \sum_{|p-q|<N_0} u_p v_q. && (2.3) \end{aligned}$$

Then, we have

$$uv = T_u v + T_v u + R(u, v). \tag{2.4}$$

Since $\mathrm{Supp}\widehat{S_q u} \subset B(0, K2^{q-N_0+1})$, we have $\mathrm{Supp}\widehat{(S_q u)} v_q \subset C_q'$ and

$$\|(S_q u) v_q\|_{L^2} \le \|S_q u\|_{L^\infty} \|v_q\|_{L^2} \le (\sum_{p=-1}^{q-N_0} C 2^{-p(s-n/2)}) c_q 2^{-qs} \le c_q' 2^{-ps}$$

with $\{c'_q\} \in l^2$, using (b) of Theorem 1.2, we have proved $T_u v, T_v u \in H^s$. For $R(u,v)$, we have

$$R(u,v) = \sum_{-N_0<j<N_0} R_j(u,v) = \sum_{-N_0<j<N_0} \sum_{q=j}^{\infty} u_{q-j} v_q.$$

Since

$$\mathrm{Supp}\widehat{u_{q-j}v_q} \subset \mathrm{Supp}\hat{u}_{q-j} + \mathrm{Supp}\hat{v}_q \subset B(0, K'2^q),$$

and

$$\|u_{q-j}v_q\|_{L^2} \le \|u_{q-j}\|_{L^\infty}\|v_q\|_{L^2} \le C2^{-(q-j)(s-n/2)}c_q 2^{-qs} = c'_q 2^{-q(2s-n/2)}.$$

Using (c) of Theorem 1.2, we have proved $R_j(u,v) \in H^{2s-n/2}$, which prove Theorem.

Definition 2.1 *For $a \in L^\infty$ with compact support, we define paramultiplication operators $T_a : \mathcal{S}' \to \mathcal{S}'$ by*

$$T_a u = \sum (S_q a) u_q, \tag{2.5}$$

where $u \in \mathcal{S}', \{a_q\}, \{u_q\}$ the Littlewood-Paley decomposition, and $S_q a = \sum_{p \le q-N_0} a_p$.

Remark that this definition depends on the constant K of Littlewood -Paley decomposition, the partition of unity (defined by function φ), and the constant N_0. then the definition isn't canonic. We will analyze the relation of T_a and (K,φ,N_0) in each cas. We have

Theorem 2.2 *If $a \in L^\infty$ with compact support, then for any $s, \alpha \in \mathbb{R}$, we have that $T_a : H^s \to H^s, T_a : C^\alpha \to C^\alpha$ is continuous and the norms of operators satisfies*

$$\|T_a\|_{\mathcal{L}(H^s,H^s)} \le C_s\|a\|_{L^\infty}; \quad \|T_a\|_{\mathcal{L}(C^\alpha,C^\alpha)} \le C_\alpha\|a\|_{L^\infty}. \tag{2.6}$$

Proof: We prove Theorem only for C^α, since

$$\psi(2^{q-N_0}\xi) = \psi(\xi) + \sum_{p=0}^{N_0-q-1} \varphi(2^{-p}\xi),$$

then

$$\widehat{S_q a}(\xi) = \psi(2^{q-N_0}\xi)\hat{a}(\xi),$$

and for any $u \in C^\alpha$, we have

$$\mathrm{Supp}(\widehat{S_q a)} u_q \subset \mathrm{Supp}\widehat{S_q a} + \mathrm{Supp}\hat{u}_q \subset C'_q.$$

Set $\hat{h}(\xi) = \psi(\xi)$, then

$$S_q a(x) = 2^{n(q-N_0)}[h(2^{q-N_0}\cdot) * a](x),$$

and

$$\|S_q a\|_{L^\infty} \le \|a\|_{L^\infty}\|h\|_{L^1}.$$

Hence we obtain

$$\|(S_q a)u_q\|_{L^\infty} \le \|h\|_{L^1}\|a\|_{L^\infty}\|u_q\|_{L^\infty} \le C\|u\|_{C^\alpha}\|h\|_{L^1}\|a\|_{L^\infty}2^{-q\alpha}.$$

Using Theorem 1.3, we have proved (2.6).

We study now the dependence of T_a on (K,φ,N_0).

Theorem 2.3 *Let $\rho > 0, a \in C^\rho$, and T_a defined by (K, φ, N_0). Suppose that (K', φ') define anther Littlewood-Paley decomposition, and $\{A_q\}$ a sequence in C^∞ verifies $\|a - A_q\|_{L^\infty} \leq C2^{-q\rho}, Supp\hat{A}_q \subset B(0, C2^q)$. For $u \in C^\alpha$, denote by $u = \sum u_p$ decomposition associated with (K, φ), and $u = \sum v_p$ decomposition associated with (K', φ'). Then for any N_0', we have*

$$T_a u - \sum_q A_{q-N_0'} v_q \in C^{\alpha+\rho}. \tag{2.7}$$

Proof: by definition of $T_a u$, we have

$$\begin{aligned}
T_a u - \sum_q A_{q-N_0'} v_q &= \sum_{p \leq q-N_0} a_p u_q - \sum_{p \leq q-N_0} a_p v_q \\
&+ \sum_{p \leq q-N_0} a_p v_q - \sum_q A_{q-N_0'} v_q \\
&= \sum_p a_p [\sum_{q \geq p+N_0} (u_q - v_q)] \\
&+ \sum_q [\sum_{p \leq q-N_0} a_p - A_{q-N_0'}] v_q \\
&= \sum_p a_p \tilde{v}_p + \sum_q f_q.
\end{aligned}$$

Without loss generality, we can suppose that $K' \geq K$, then

$$\mathrm{Supp}\hat{\tilde{v}}_p \subset C'_{p+N_0}, \|\tilde{v}_p\|_{L^\infty} \leq C2^{-p\alpha},$$

and

$$\mathrm{Supp}\widehat{a_p \tilde{v}_p} \subset C''_{p+N_0}, \|a_p \tilde{v}_p\|_{L^\infty} \leq C2^{-p(\alpha+\rho)}.$$

For f_q, it is evident that $\mathrm{Supp}\hat{f}_q \subset C'_q$, and

$$\|f_q\|_{L^\infty} \leq (\|a - \sum_{p \leq q-N_0} a_p\|_{L^\infty} + \|a - A_{q-N_0'}\|_{L^\infty}) \|v_q\|_{L^\infty} \leq C2^{-q(\alpha+\rho)}.$$

We have proved Theorem.

Corollary 2.1 *Suppose that $a \in C^\rho, \rho > 0$, and T_a, T_a' the paramultiplication defined by dyadic decomposition (K, φ, N_0), and (K', φ', N_0'). Then $T_a - T_a' \in \mathcal{L}(C^\alpha, C^{\alpha+\rho})$ and $T_a - T_a' \in \mathcal{L}(H^s, H^{s+\rho})$, and*

$$\|T_a - T_a'\|_{\mathcal{L}(C^\alpha, C^{\alpha+\rho})} \leq C_\alpha \|a\|_\rho; \tag{2.8}$$

$$\|T_a - T_a'\|_{\mathcal{L}(H^s, H^{s+\rho})} \leq C_s \|a\|_\rho. \tag{2.9}$$

The proof of Corollary is directly by use Theorem 2.3 with $A_q = \sum_{p \leq q-N_0'} a_p'$, where $\{a_p'\}$ is the Littlewood-Paley decomposition of a with respect to (K', φ').

We have now for $\rho > 0, a \in C^\rho$,

$$T_a \equiv T_a' (\mathrm{mod}\mathcal{L}(C^\alpha, C^{\alpha+\rho}), \text{and}, \mathrm{mod}\mathcal{L}(H^s, H^{s+\rho})).$$

If an operators in $\mathcal{L}(C^\alpha, C^{\alpha+\rho})$, or $\mathcal{L}(H^s, H^{s+\rho})$, we called an ρ-regularization operators, and denote by $S^{-\rho}$. Then the paramultiplication T_a is well-defined by function a and module $S^{-\rho}$. For paramultiplication, we have also the calculus of operators.

Theorem 2.4 *Let* $a, b \in C^\rho, \rho > 0$, *with compact support, then*

$$T_a \circ T_b - T_{ab} \in S^{-\rho}; \tag{2.10}$$

$$T_a^* - T_{\bar{a}} \in S^{-\rho}, \tag{2.11}$$

their norm can be estimate by $C\|a\|_\rho \|b\|_\rho$ *and* $C\|a\|_\rho$.

The proof of (2.10) is also directly by use Theorem 2.3 with

$$A_q = \sum_{p_2 \le q - N_0} a_{p_2} (S_{q-N_0} b) u_q = \sum v_q.$$

For (2.11), using the formula

$$(T_a^* u, v) = (u, T_a v) = \sum_q \sum_{p \le r - N_0} \int u_q \bar{a}_p \bar{v}_r dx,$$

and

$$(T_{\bar{a}} u, v) = \sum_r \sum_{p \le q - N_0} \int \bar{a}_p u_q \bar{v}_r dx.$$

we can get

$$|((T_a^* - T_{\bar{a}})u, v)| \le C\|a\|_\rho \|u\|_s \|v\|_{-s-\rho}.$$

which prove (2.11).

2.2. Operators with non smooth coefficients and regularization

We have study paramultiplication defined by a non smooth function, in fact this is a regularization of multiplier. We study now the regularization of pseudo-differential operators with non smooth coefficients.

Definition 2.2 *(a) For* $\rho > 0, m \in R$, *we denote by* $l_\rho^m = \{l(x,\xi); l$ *is homogeneous of degree* m, *and belong to* $C^\infty(\mathbb{R}^n \setminus 0)$ *for variables* ξ, *for* x *it is belong to* $C^\rho, \rho > 0$ *with compact support (uniformly respect to* $\xi)\}$.

(b) For any $l \in l_\rho^m$, *denote by* $S_q(l(x,\xi)) = \psi(2^{-q}D_x) l(x,\xi)$, *then* $S_q(l(x,\xi)) \in S_{1,0}^m$. *For any* $u \in \mathcal{S}'$ *we define*

$$T_l u(x) = \sum_q S_{q-N_0}(l(x,D)) u_q(x).$$

If $l(x,\xi) = \sum_j l_j(x,\xi)$ *is a finite sum, we denote by* $T_l = \sum_j T_{l_j}$.

Remark that if $l(x,\xi) = a(x)h(\xi)$, then $T_l = T_a \circ h(D)$, for general $l(x,\xi)$ we can use sphere harmonic decomposition

$$l(x,\xi) = \sum_\nu a_\nu(x) h_\nu(\xi).$$

See [CM] for detail.

Theorem 2.5 *For* $l \in l_\rho^m$, *we have that* $T_l : H^s \to H^{s-m}$ *(or* $C^\alpha \to C^{\alpha-m}$*) is continuous for any* $\alpha, s \in \mathbb{R}$.

Proof: We prove only Theorem for $l(x,\xi) = a(x)h(\xi)$, with $h(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ and homogeneous of degree m, and $a \in C^\rho$ with compact support. Theorem 1.4 give that $h(D) : H^s \to H^{s-m}$ is continuous, and Theorem 2.2 give that $T_a : H^{s-m} \to H^{s-m}$ is also continuous. We have proved Theorem.

As in the Corollary 2.1, we have $T_l - T_l' \in S^{m-\rho}$, then T_l is well-defined by $l(x,\xi)$ modulo $S^{m-\rho}$. We study now symbolic calculus of operators T_l, it is similar to pseudo-differential operators.

Theorem 2.6 *Let* $a \in C^\rho, \rho > 0, \rho \notin \mathbb{N}$, *and with compact support,* $h \in C^\infty(\mathbb{R}^n \setminus 0)$ *homogeneous of degree* m*. Then*

$$R = h(D) \circ T_a - \sum_{|\alpha| \le [\rho]} \frac{1}{\alpha!} T_{D^\alpha a} \circ h^\alpha(D) \in S^{m-\rho},$$

where $h^\alpha(\xi) = \partial_\xi^\alpha(\xi)$.

Proof: Since $\mathrm{Supp}\mathcal{F}(S_{q-N_0}(a)u_q) \subset C_q'$, choose $C_q' \subset C_q''$ and $\varphi_0 \in C_0^\infty(C_0'')$ with $\varphi_0(\xi) = 1$ on C_0', set $\tilde{h}(\xi) = h(\xi)\varphi(\xi)$, then for $\xi \in C_q'$,

$$h(\xi) = 2^{mq}\tilde{h}(2^{-q}\xi).$$

If $\hat{r}(\xi) = \tilde{h}(\xi)$, then $r \in \mathcal{S}$ and for $M > n + \rho/2$,

$$\|(1 + |x|^\rho) r(x)\|_{L^1} \le C\|h\|_{C^{2M}(S^{n-1})}.$$

Now for $u \in H^s$, since for $\xi \in C_q'$, $h^\alpha(\xi) = \tilde{h}^\alpha(\xi)$ and it is Fourier transformation of $(-ix)^\alpha r(x)$, we have

$$\begin{aligned}
R(u) &= \sum_q 2^{mq}[\tilde{h}(2^{-q}D)S_{q-N_0}(a) - \sum_{|\alpha|\le[\rho]} \frac{1}{\alpha!} S_{q-N_0}(D^\alpha a)\tilde{h}^\alpha(2^{-q}D)]u_q \\
&= \sum_q 2^{mq} \int r(t)[S_{q-N_0}(a)(x - 2^{-q}t) \\
&- \sum_{|\alpha|\le[\rho]} \frac{1}{\alpha!} S_{q-N_0}(D^\alpha a)(x)(-i2^{-q}t)^\alpha] u_q(x - 2^{-q}t)dt = \sum_q f_q.
\end{aligned}$$

It is easy to see that $\mathrm{Supp}\hat{f}_q \subset C_q'$, we study now estimation of $\|f_q\|_{L^2}$. Since $a \in C^\rho$, we have

$$\begin{aligned}
|f_q(x)| &\le 2^{mq} \int |r(t)| \|S_{q-N_0}(a)\|_{C^\rho} 2^{-q\rho}|t|^\rho |u_q(x - 2^{-q}t)|dt \\
&\le C2^{q(m-\rho)}\|a\|_{C^\rho} \int |t|^\rho |r(t)||u_q(x - 2^{-q}t)|dt \\
&\le Cc_q 2^{q(m-\rho-s)}\|a\|_{C^\rho}\|h\|_{C^{2M}(S^{n-1})}\|u\|_{H^s} \\
&= c_q' 2^{q(m-\rho-s)}.
\end{aligned}$$

Which prove that $R(u) \in H^{s+\rho-m}$, and

$$\|R\|_{S^{m-\rho}} \le C\|a\|_{C^\rho}\|h\|_{C^{2M}(S^{n-1})}.$$

We study now the composition of two operators.

Theorem 2.7 *Let* $l_j(x,\xi) \in l_\rho^{m_j}, j = 1, 2,$ *set*

$$(l_1 \# l_2)(x,\xi) = \sum_{|\alpha|\leq[\rho]} \frac{1}{\alpha!}\partial_\xi^\alpha l_1(x,\xi) D_x^\alpha l_2(x,\xi). \tag{2.12}$$

Then $T_{l_1} \circ T_{l_2} - T_{l_1 \# l_2} \in S^{m_1+m_2-\rho}$.

Proof: Let $l_1(x,\xi) = a(x)h(\xi), l_2(x,\xi) = b(x)g(\xi)$, then

$$\begin{aligned} T_{l_1} \circ T_{l_2} &= T_a \circ h(D) \circ T_b \circ g(D) \\ &= T_a \circ \sum_{|\alpha|\leq[\rho]} \frac{1}{\alpha!} T_{D^\alpha b} \circ h^\alpha(D) \circ g(D) + T_a \circ R \circ g(D) \\ &= \sum_{|\alpha|\leq[\rho]} T_{a\#D^\alpha b} \circ h^\alpha(D) \circ g(D) + T_a \circ R \circ g(D) + R^\alpha \circ h^\alpha(D) \circ g(D). \end{aligned}$$

using Theorem 2.6, we have $R \in S^{m_1-\rho}$, and Theorem 2.4 implies $R^\alpha \in S^{-(\rho-|\alpha|)}$. We have proved Theorem.

Theorem 2.8 *Let* $l \in l_\rho^m$, *set*

$$l^*(x,\xi) = \sum_{|\alpha|\leq[\rho]} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \bar{l}(x,\xi), \tag{2.13}$$

then $T_l^* \equiv T_{l^*} \quad mod(S^{m-\rho})$.

From those theorems, we sew that the operators T_l is very similar to pseudo-differential operators, but here the symbolic calculus is only to the order $[\rho]$, and the rest terms is in $S^{-\rho}$. On the other hand, for non smooth symbol $l \in l_\rho^m$, we can also define the associated pseudo-differential operators $l(x,D)$.

Theorem 2.9 *Let* $l \in l_\rho^m, \rho > m$, *then for all* $s > m - \rho$,

$$l(x,D) - T_l \in \mathcal{L}(H^s, H^{s'}),$$

where $s' < \min\{\rho, s + \rho - m\}$.

Proof: Let $l(x,\xi) = a(x)h(\xi)$, for $u \in H^s$, we have $v = h(D)u \in H^{s-m}, T_l u = T_a \circ h(D)u = T_a v$, and

$$T_l u - l(x,D)u = T_a v - av = T_v a + R(a,v).$$

Since $s + \rho - m > 0$, we have $R(a,v) \in H^{s+\rho-m}$, for

$$T_v a = \sum_q S_{q-N_0}(v) a_q = \sum_q f_q,$$

we have $\mathrm{Supp}\hat{f}_q \subset C'_q$, and

$$\|f_q\|_{L^2} \leq \|a_q\|_{L^\infty} \|S_{q-N_0}(v)\|_{L^2} \leq C\|a\|_{C^\rho}\|v\|_{H^{s-m}} 2^{-q\rho} \sum_{p\leq q-N_0} 2^{-p(s-m)} c_p,$$

if $s - m > 0$, then $\sum_p 2^{-p(s-m)} c_p \leq C < +\infty$, and we have for any $\varepsilon > 0, T_v a \in H^{\rho-\varepsilon}$; if $s - m < 0$, then

$$\sum_{p\leq q-N_0} 2^{-p(s-m)} c_p \leq C 2^{-q(s-m)},$$

and for any $\varepsilon > 0, T_v a \in H^{s+\rho-m-\varepsilon}$; if $s-m=0$, we have $\|S_{q-N_0}(v)\|_{L^2} \le C\|v\|_{L^2}$, and $T_v a \in H^{\rho-\varepsilon}$. We have proved Theorem.

From this Theorem, we have immediately following results.

Corollary 2.2 *Let $m > 0, l \in l^m_m$, then $l(x,D) - T_l \in \mathcal{L}(H^\varepsilon, L^2)$, for any $\varepsilon > 0$.*

Theorem 2.10 *Let $l \in l^m_\rho, \rho > 0, \rho \notin \mathbb{N}$, then $T_l \in L^m_{1,1}$ with symbol*

$$\sigma(T_l)(x,\xi) = \sum_q S_{q-N_0}(l(x,\xi))\varphi(2^{-q}\xi). \tag{2.14}$$

Proof: It is evident $T_l = \sigma(T_l)(x,D)$, we need only to prove $\sigma(T_l) \in S^m_{1,1}$. Take $l(x,\xi) = a(x)h(\xi)$, then

$$\sigma(T_l)(x,\xi) = \sum_q S_{q-N_0}(a)(x)h(\xi)\varphi(2^{-q}\xi),$$

and for all $\alpha, \beta \in \mathbb{N}^n$,

$$\partial^\alpha_\xi \partial^\beta_x \sigma(T_l)(x,\xi) = \sum_q \partial^\beta_x S_{q-N_0}(a)(x)\partial^\alpha_\xi(h(\xi)\varphi(2^{-q}\xi)).$$

Since $a \in C^\rho_0 \subset L^\infty$, we have

$$\|\partial^\beta_x S_{q-N_0}(a)\|_{L^\infty} \le C(n\beta)2^{q|\beta|}\|a\|_{L^\infty}.$$

Choose now $\tilde\varphi \in C^\infty_0(\mathbb{R}^n)$ such that $\mathrm{Supp}\tilde\varphi \subset C'_0$, and $\tilde\varphi(\xi) = 1$ on $\mathrm{Supp}\varphi, 0 \le \tilde\varphi \le 1$, then

$$\begin{aligned} |\partial^\alpha_\xi h(\xi)\varphi(2^{-q}\xi)| &\le C_\alpha \sum_{\alpha_1+\alpha_2=\alpha} |h^{\alpha_1}(\xi)2^{-q|\alpha_2|}\varphi(2^{-q}\xi)| \\ &\le C_\alpha \|h\|_{C^{2M}(S^{n-1})}2^{-q|\alpha_2|}|\xi|^{m-|\alpha_2|}\tilde\varphi(2^{-q}\xi) \\ &\le C_\alpha 2^{q(m-|\alpha|)}\tilde\varphi(2^{-q}\xi). \end{aligned}$$

Since on $\mathrm{Supp}\tilde\varphi(2^{-q}\xi)$, we have $K'^{-1}2^q \le |\xi| \le K'2^{q+1}$, we obtain

$$|\partial^\alpha_\xi \partial^\beta_x \sigma(T_l)| \le C_{\alpha,\beta}|\xi|^{m-|\alpha|-|\beta|} \sum_{q \ge N_0} \tilde\varphi(2^{-q}\xi) \le C_{\alpha,\beta}|\xi|^{m-|\alpha|-|\beta|}.$$

which prove $\sigma(T_l) \in S^m_{1,1}$.

We know $S^m_{1,1}$ is bad class for symbolic calculus, there isn't asymptotic symbolic calculus in this class. But for his subclass T_l, we have a convenable symbolic calculus as that for pseudo-differential operators.

2.3. Paradifferential operators

We define now symbolic class, and paradifferential operators.

Definition 2.3 *(a) Let $\Omega \subset \mathbb{R}^n$ be an open domain, for $m \in \mathbb{R}, \rho > 0$, we define $\Sigma^m_\rho(\Omega)$ the function class defined on $\Omega \times (\mathbb{R}^n \setminus 0)$ of form*

$$l(x,\xi) = l_m(x,\xi) + l_{m-1}(x,\xi) + \cdots + l_{m-[\rho]}(x,\xi),$$

where $l_{m-k}(x,\xi)$ belong to $C^\infty(\mathbb{R}^n \setminus 0)$ and homogeneous of degree $m-k$ for variables ξ; for variables $x, l_{m-k} \in C^{\rho-k}_{loc}(\Omega)$.

(b) Let $l^j \in \Sigma_\rho^{m_j}(\Omega), j = 1, 2,$ *we define*

$$l^1 \# l^2 = \sum_{|\alpha|+k_1+k_2 \leq [\rho]} \frac{1}{\alpha!} \partial_\xi^\alpha l^1_{m_1-k_1} D_x^\alpha l^2_{m_2-k_2},$$

then $l^1 \# l^2 \in \Sigma_\rho^{m_1+m_2}(\Omega)$.

(c) Let $l \in \Sigma_\rho^m(\Omega)$, *we define*

$$l^* = \sum_{|\alpha|+k \leq [\rho]} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{l}_{m-k},$$

then $l^* \in \Sigma_\rho^m(\Omega)$.

For symbol class $\Sigma_\rho^m(\Omega)$, we define an operators class.

Definition 2.4 *Let* $\Omega \subset \mathbb{R}^n$ *be an open domain, and* $L : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ *a proper supported operators.* L *is called a paradifferential operators of order* m, *if there exist* $l \in \Sigma_\rho^m(\Omega)$, *such that for any* $K \subset\subset \Omega$, *and* $\chi \in C_0^\infty(\Omega), \chi(x) = 1$ *on* K, *we have that*

$$L - \chi T_{\chi l} : H^s_{comp}(K) \to H^{s-m+\rho}_{comp},$$

is continuous. We denote the class of operators by $Op(\Sigma_\rho^m(\Omega))$, *and* $l = \sigma(L)$ *the symbol of operators* L.

It is evident for $L \in \mathrm{Op}(\Sigma_\rho^m(\Omega))$ and any $s \in \mathbb{R}$,

$$L : H^s_{\mathrm{loc}}(\Omega) \to H^{s-m}_{\mathrm{loc}}(\Omega).$$

The following Theorem is essential results for paradifferential operators.

Theorem 2.11 *Let* $\Omega \subset \mathbb{R}^n$ *be an open domain,* $m \in \mathbb{R}, \rho > 0$, *then*

(a) For $L \in Op(\Sigma_\rho^m(\Omega))$, *there exist unique symbol* $\sigma(L) \in \Sigma_\rho^m(\Omega)$, *and*

$$\sigma : Op(\Sigma_\rho^m(\Omega)) \to \Sigma_\rho^m(\Omega)$$

is a surjection, and kerσ *is a continuous maps from* $H^s_{loc}(\Omega)$ *to* $H^{s-m+\rho}_{loc}(\Omega)$.

(b) For $L_j \in Op(\Sigma_\rho^{m_j}(\Omega)), j = 1, 2,$ *then*

$$L_1 \circ L_2 \in Op(\Sigma_\rho^{m_1+m_2}(\Omega)); \quad \sigma(L_1 \circ L_2) = \sigma(L_1) \# \sigma(L_2).$$

(c) For $L \in Op(\Sigma_\rho^m(\Omega))$, *then* $L^* \in Op(\Sigma_\rho^m(\Omega))$, *and*

$$\sigma(L^*) = (\sigma(L))^*.$$

(d) For $L \in L^m_{1,0}$, *a proper supported pseudo-differential operators, and* $\sum_j l_{m-j}(x, \xi)$ *his symbol, then for any* $\rho > 0$ *we have*

$$L \in Op(\Sigma_\rho^m(\Omega)); \quad \sigma(L) = \sum_{0 \leq [\rho]} l_{m-j}.$$

Using definition of paradifferential operators and the properties of operators $T_{\chi l}$, we need only to prove σ surjection. For $l \in \Sigma_\rho^m(\Omega)$, we construct $L \in \mathrm{Op}(\Sigma_\rho^m(\Omega))$ verifies $\sigma(L) = l$. Choose $\{\Omega_j\}$ a local finite recover of Ω with $\Omega_j \subset\subset \Omega$, and $\{\varphi_j\}$ the partition of unity associated with $\{\Omega_j\}$. Take $\chi_j \in C_0^\infty(\Omega_j), \chi_j(x) = 1$ on $\mathrm{Supp}\varphi_j$, we define for $u \in \mathcal{D}'$,

$$Lu = \sum_j \chi_j T_{\chi_j l}(\varphi_j u).$$

Then L is necessary paradifferential operators. The uniqueness of maps σ give by following Theorem.

Theorem 2.12 *Let* $L \in Op(\Sigma_\rho^m(\Omega)), l = l_m + \cdots + l_{m-[\rho]}$ *one of his symbol, if for some* $s \in \mathbb{R}$ *and* $\varepsilon > 0$, $L : H^s \to H^{s-m+\varepsilon}$ *is continuous, then* $l_m = 0$.

Using this theorem, if $L \in \mathrm{Op}(\Sigma_\rho^m(\Omega))$, and $L : H^s \to H^{s-m+\rho}$, then $\sigma(L) = 0$. The proof of Theorem 2.12 give by following symbolic inverse calculus.

Theorem 2.13 *Let* $l \in \Sigma_\rho^m(\Omega), k \in \Sigma_\rho^{m'}(\Omega)$, *and* $l(x,\xi) \neq 0$ *on Suppk, then there exist* $h, h' \in \Sigma_\rho^{m'-m}(\Omega)$ *such that*

$$l\#h = h'\#l = k.$$

We collect some results for paradifferential operators in following corollary.

Corollary 2.3 *(a) For any* $d > 0$, *we have* $Op(\Sigma_\rho^m(\Omega)) \subset Op(\Sigma_{\rho+d}^{m+d}(\Omega))$.

(b) For $L \in Op(\Sigma_\rho^m(\Omega)), \rho > 1$, *if* $\sigma_m(L) = 0$, *then* $L \in Op(\Sigma_{\rho-1}^{m-1}(\Omega))$.

(c) Let $L_j \in Op(\Sigma_\rho^{m-j}(\Omega)), j = 1, 2, \rho > 1$, *then* $[L_1, L_2] \in Op(\Sigma_{\rho-1}^{m_1+m_2-1}(\Omega))$, *and*

$$\sigma_{m_1+m_2-1}([L_1, L_2]) = \frac{1}{i}\{\sigma_{m_1}(L_1), \sigma_{m_2}(L_2)\}.$$

The paradifferential operators defined on function space C^α and Besov space $B_{p,r}^s$ is similarly.

2.4. Tangential paradifferential operators

Since the problems for nonlinear equation is often with boundary value, so we have to study theory of paradifferential calculus near to the boundary of a smooth domain. After localization and a changing of variables, we study only demi-space $\mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^n; x_n > 0\}$, where $x' = \{x_1, \cdots, x_{n-1}\}$. We define the tangential Sobolev space ($s, s' \in \mathbb{R}$),

$$H^{s,s'}(\mathbb{R}^n) = \{u \in \mathcal{S}'; (1 + |\xi|^2)^s (1 + |\xi'|^2)^{s'} |\hat{u}|^2 \in L^1\}.$$

Then $H^{s,s'}(\mathbb{R}^n)$ is an algebra, if $s + s' > n/2, s > 1/2, s + 2s' > 1/2$. Denote $D = (D', D_n), \triangle_p = \varphi(2^{-p}D), \triangle'_p = \varphi(2^{-p}D', 0), \triangle_{p,p'} = \triangle_p \circ \triangle_{p'}$. Then for any $u \in \mathcal{S}'$,

$$u = \sum_{p'=-1}^{N_1} \triangle_{-1,p'} u + \sum_{p \geq 0} \triangle_{p,-1} u + \sum_{p,p' \geq 0} \triangle_{p,p'} u.$$

This is double dyadic decomposition. We can also give a characterization of tangential space $H^{s,s'}$ as for usual Sobolev space.

Theorem 2.14 *$u \in H^{s,s'}(\mathbb{R}^n)$, if and only if*

$$\sum_{p'=-1}^{N_1} 4^{p's'}\|\triangle_{-1,p'}u\|_{L^2}^2 + \sum_{p\geq 0} 4^{ps}\|\triangle_{p,-1}u\|_{L^2}^2 + \sum_{p,p'\geq 0} 4^{ps+p's'}\|\triangle_{p,p'}u\|_{L^2}^2 < +\infty.$$

We have also the similar results as that of Theorem 1.2. We define now tangential paramultiplication

$$T'_a u = \sum_q (S'_q a)\triangle'_q u,$$

where $S'_q a = \sum_{p=-1}^{q-N_0} \triangle'_p u$. We will use also double index paramultiplication

$$\Pi_a u = \sum_{q,q'} S_{q,q'} a \triangle_{q,q'} u,$$

where $S_{q,q'} a = \sum_{p=-1}^{q-N_0} \sum_{p'=-1}^{q'-N_0} \triangle_{p,p'} u$. We have

Theorem 2.15 *(a) If $a \in H^{t,t'}(\mathbb{R}^n), t > 1/2, t+t' > n/2$, then $\Pi_a : H^{s,s'} \to H^{s,s'}$ is continuous for any $s, s' \in \mathbb{R}$.*

(b) If $a \in H^{t,t'}(\mathbb{R}^n), t > 1/2, t+t' > n/2$, then $T'_a : H^{s,s'} \to H^{s,s'}$ is continuous for any $s' \in \mathbb{R}$ and $-t < s \leq t$. And $\Pi_a - T'_a : H^{s,s'} \to H^{s,s'+\rho(t,t')}$, where $\rho(t,t') = \min(s+s'-n/2, s-1/2)$, if $s' \neq (n-1)/2$; $= s-1/2-\varepsilon, \varepsilon > 0$, if $s' = (n-1)/2$.

(c) If $a \in H^{t,t'}(\mathbb{R}^n), t > 1/2, t+t' > n/2$, then $T_a : H^{s,s'} \to H^{s,s'}$ is continuous for any $s \in \mathbb{R}$ and $(t+t'-1/2 < s' \leq (t+t'-1/2)$. And $\Pi_a - T_a : H^{s,s'} \to H^{s,s'+(t+t'-n/2)}$, for any $s \in \mathbb{R}$, and $-(t+t'-1/2) < s' \leq (n-1)/2$.

the operators T'_a is well-defined by modulo $\mathcal{L}(H^{s,s'}, H^{s,s'+t+t'-n/2})$. We have also similar symbolic calculus, and tangential paralinearization.

Theorem 2.16 *Let $F \in C^\infty(\mathbb{R}), F(0) = 0, u \in H^{s,s'}(\mathbb{R}^n), s > 1/2, s+s' > n/2$, then*

$$F(u) - T'_{F'(u)} u \in H^{s,s'+\rho(s,s')}.$$

The proof of $F(u) - \Pi_{F'(u)} u \in H^{s,s'+\rho(s,s')}$ is similar to Theorem 3.3, then Theorem 2.16 reduced by Theorem 2.15.

Remark that for the operators T'_a, variables x_n is only a parameter, then $T'_a|_{\mathbb{R}^n_+}$ is well-defined, so we can use tangential paramultiplication T'_a to study boundary value problems.

Chapter 3

Micolocal analysis for nonlinear equation

3.1. Paralinearization

We study now the theory of linearization of nonlinear partial differential equations, the simplest cas is composition of nonlinear functions.

Theorem 3.1 *Let* $F \in C^\infty(\mathbb{R}^1), F(0) = 0$. *If* $f \in H^s(\mathbb{R}^n), s > n/2$ *is a real function (or* $f \in C^\sigma(\mathbb{R}^n), \sigma > 0$*), then the composition* $F(f) \in H^s(\mathbb{R}^n)$ *(or* $F(f) \in C^\sigma(\mathbb{R}^n)$*).*

Proof: Set $\sigma = s - n/2 > 0, f = \sum_{p=-1}^{\infty} f_p$, the Littlewood-Paley decomposition, then Theorem 1.2 and 1.3 give

$$\|f - S_p(f)\|_{L^\infty} \leq C 2^{-p\sigma} \|f\|_{C^\sigma},$$

that means $S_p(f)$ converge uniformly to f on $\mathbb{R}^n$, and we have

$$F(f) = \sum_{p=-1}^{\infty} [F(S_p(f)) - F(S_{p-1}(f))],$$

here we note $S_{-2}(f) = 0$. Then

$$F(S_p(f)) - F(S_{p-1}(f)) = f_p \int_0^1 F'(S_{p-1}(f) + t f_p) dt = m_p f_p,$$

where $m_p(x)$ depends on f, we call first linearization formula

$$\begin{aligned} F(f) &= \sum_{p=-1}^{\infty} m_p f_p \\ &= \sum_{p=-1}^{\infty} m_p(x) \varphi(2^{-p} D) f = Lf. \end{aligned}$$

We will prove in Theorem 3.2, that $L = \sum_{p=-1}^{\infty} m_p(x)\varphi(2^{-p}D) \in L^0_{1,1}$, then $L : H^s \to H^s$ is continuous for $s > 0$.

Theorem 3.2 *Let* $\{m_p\} \in C^\infty(\mathbb{R}^n)$ *be a functions family verifies*

$$\|\partial^\alpha m_p\|_{L^\infty} \leq C_\alpha 2^{p|\alpha|}. \tag{3.1}$$

Then the operators $L(x,D) = \sum_{p=-1}^{\infty} m_p(x)\varphi(2^{-p}D)$ *is continuous from* H^s *to* H^s*, and* C^α *to* C^α *for any* $s > 0, \alpha > 0$.

Using theorems 1.2 and 1.3, the proof of theorem 3.2 is evident. Then the rest proof of theorem 3.1 is to prove (3.1) for $m_p(x) = \int_0^1 F'(S_{p-1}(f) + tf_p)dt$. Since $\|\partial^\alpha(S_{p-1}(f) + tf_p)\|_{L^\infty} \leq C_\alpha 2^{p|\alpha|}$ for all $\alpha \in \mathbb{N}^n$, then it is easy to get

$$\|\partial^\alpha F'(S_{p-1}(f) + tf_p)\|_{L^\infty} \leq C_\alpha 2^{p|\alpha|}. \tag{3.2}$$

which finish the proof of theorem 3.1.

Theorem 3.3 *Let* $F \in C^\infty(\mathbb{R}^1), F(0) = 0$*, and* f *be a real functions, then we have:*

(i) If $f \in H^s(\mathbb{R}^n), s > n/2$*, then there exist* $g \in H^{2s-n/2}(\mathbb{R}^n)$ *such that*

$$F(f) = T_{F'(f)}f + g.$$

(ii) If $f \in C^\sigma(\mathbb{R}^n), \sigma > 0$*, then there exist* $g \in C^{2\sigma}(\mathbb{R}^n)$ *such that*

$$F(f) = T_{F'(f)}f + g.$$

Proof: Using the first linearization formula, we will prove that there exist $R \in L^{-\sigma}_{1,1}, (\sigma > 0, \sigma = s - n/2 > 0)$, such that

$$L - T_{F'(f)} = R. \tag{3.3}$$

By definition, we have

$$\begin{aligned} \sigma(R) &= \sum_{p \geq N_0} [m_p(x) - S_{p-N_0}(F'(f))(x)]\varphi(2^{-p}\xi) \\ &+ \sum_{p < N_0} m_p(x)\varphi(2^{-p}\xi). \end{aligned}$$

It is evident $\sum_{p<N_0} m_p(x)\varphi(2^{-p}\xi) \in S^{-\infty}$. Using theorems 1.2 and 1.3, for the first terms we need only to prove

$$\|\partial_x^\alpha(m_p(x) - S_{p-N_0}(F'(f))(x))\|_{L^\infty} \leq C_\alpha 2^{p(|\alpha|-\sigma)}.$$

This implies by Taylor formula and (3.2).

The conclusion of Theorem 3.3 is also true for vector, if $F(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N), F(x,0) = 0$, and real functions $u_1, \cdots, u_N \in C^\sigma(\mathbb{R}^n), \sigma > 0$ or in $H^s(\mathbb{R}^n), s > n/2$, then there exist $g \in C^{2\sigma}(\mathbb{R}^n), (H^{2s-n/2}(\mathbb{R}^n))$, such that

$$F(x, u_1, \cdots, u_N) - \sum_{j=1}^{N} T_{F_j} u_j = g, \tag{3.4}$$

where $F_j = \frac{\partial F}{\partial y_j}(x, u_1(x), \cdots, u_N(x))$.

3.2. Paradifferential equations

We consider now nonlinear partial differential equation of order m,

$$F[u]=F(x,u,\cdots,\partial^\beta u,\cdots)_{|\beta|\le m}=0, \tag{3.5}$$

where $F\in C^\infty(\Omega\times \mathbb{R}^N)$ is real function, $\Omega\subset\mathbb{R}^n$ an open domain, $N=\#\{\beta\in\mathbb{N}^n;|\beta|\le m\}$. Eventually, F is linear for some derivation of u, we can rewrite F as following

$$\begin{aligned} F[u] &= \sum_{k_0<k\le m}\sum_{|\alpha|=k} A_\alpha(x,u,\cdots,\partial^\beta u,\cdots)_{|\beta|\le p(k)}\partial^\alpha u \\ &+ A_{k_0}(x,u,\cdots,\partial^\beta u,\cdots)_{|\beta|\le k_0}. \end{aligned}$$

where $p(k)<k$, if A_α depends only on x, we put $p(|\alpha|)=-\infty$. Set

$$d=\max\left(k_0,\frac{k+p(k)}{2}\right).$$

then if F is full nonlinear, $d=m$. If F is quasilinear, $k_0=m-1,p(m)=m-1,k=m,d=m-1/2$. If F is semilinear, $k_0=m-1,k=m,p(m)=0,d=m-1$. If F is only nonlinear for u, $k_0=0,p(k)=-\infty,d=0$. If F is linear, then $d=-\infty$. We can now define the symbol of linearized operators of F on function $u\in C^\rho$.

Theorem 3.4 *Let* $u\in C^\rho_{loc}(\Omega),\rho>\max(k_0,p(k))$. *Set*

$$p(x,\xi)=\sum_{|\beta|>2d-\rho} F_\beta(x,u(x),\cdots)(i\xi)^\beta, \tag{3.6}$$

where $F_\beta=\partial_{u_\beta}F$. *Then* $p(x,\xi)\in\Sigma^m_{\rho+m-2d}(\Omega)$.

Using the symbol (3.6), we can give the paralinearization of nonlinear equation (3.5).

Theorem 3.5 *Let* $u\in C^\rho_{loc}(\Omega),\rho>\max(k_0,p(k))$, *and* P *paradifferential operator with symbol* $p(x,\xi)$ *defined by (3.6), and* u *is a solution of equation (3.5).*

(a) If $\rho>d$, *then* $Pu\in C^{2\rho-2d}_{loc}$.

(b) If $s\ge n/2+\rho,\rho>d-n/4,u\in H^s_{loc}(\Omega)$, *then* $Pu\in H^{s+\rho-2d}_{loc}$.

The proof of this theorem is just as that of Theorem 3.3. In [XU1], we have proved that if $u\in C^\rho(\Omega)\bigcap H^s(\Omega),\rho>\max(k_0,p(k)),s>0$ is a solution of equation (3.5), then

$$Pu\in C^{2\rho-2d}(\Omega)\bigcap H^{s+\rho-2d}(\Omega). \tag{3.7}$$

From a nonlinear equation $F[u]=0$, we have get a linear (paradifferential) equation $Pu=f$, with f more regular than the solution u. Since for the operators P, we have the symbolic calculus as that for pseudo-differential operators, then microlocale analysis for nonlinear equations is carry out.

3.3. Microlocale elliptic regularity

Definition 3.1 *For* $(x_0,\xi_0)\in T^*\mathbb{R}^n\setminus 0$, *we say* $u\in H^s_{x_0,\xi_0}$ *(or* $u\in C^\alpha_{x_0,\xi_0}$*), if there exists a neighborhood* V_{x_0} *of* x_0 *in* $\mathbb{R}^n_x$, *and a conic neighborhood* Γ *of* ξ_0 *in* $\mathbb{R}^n_\xi\setminus 0$, *such that for any* $\varphi\in C^\infty_0(V_{x_0})$, *and* $\psi\in C^\infty(\Gamma)$ *homogeneous of degree 0 in* ξ, *con supp*$\psi\subset\Gamma$, *we have*

$$\psi(D)(\varphi u)\in H^s,(or\quad C^\alpha).$$

We study now microlocale regularity for nonlinear elliptic equations.

Theorem 3.6 *Let $u \in C^{\rho}_{loc}(\Omega), \rho > d$, a solution of equation (3.5), and $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus 0$, $p_m(x_0, \xi_0) \neq 0$. Then $u \in C^{2\rho+m-2d}_{x_0,\xi_0}$.*

Proof: Using Theorem 3.5, there exists $f \in C^{2\rho-2d}_{\text{loc}}(\Omega)$ such that $Pu = f$, where paradifferential operators $P \in Op(\Sigma^m_{\rho+m-2d}(\Omega))$. Since $p_m(x_0, \xi_0) \neq 0$, there exists a conic neighborhood Γ of (x_0, ξ_0), and a classic pseudo-differential operator K of degree 0, with the symbol no vanish on $\Gamma' \subset\subset \Gamma$, and con supp$\sigma(K) \subset \Gamma$. Then the symbolic calculus of paradifferential operators (Theorem 2.13) give $q \in \Sigma^{-m}_{\rho+m-2d}(\Omega)$ such that $q\#p = \sigma(K)$. If Q is the paradifferential operator of symbol q, then $Q \circ P = K + R$ with $R \in S^{-(\rho+m-2d)}$, we have

$$Ku = Qf - Ru \in C^{2\rho+m-2d}_{\text{loc}}(\Omega),$$

which prove Theorem.

3.4. Hypoellipticity for nonlinear equations

For nonlinear hypoellipticity, we consider only second order equation

$$F(x, u, \nabla u, \nabla^2 u) = 0. \tag{3.8}$$

For $u \in C^{\rho}_{\text{loc}}(\Omega), \rho > 2$, we define the linearized operator of F associated with u by

$$L(x, D) = \sum_{j,k=1}^{n} a_{jk}(x)\partial_j\partial_k + \sum_{j=1}^{n} b_j(x)\partial_j + c(x), \tag{3.9}$$

where $a_{jk}(x) = a_{kj}(x) = \partial_{u_{jk}} F(x, u(x), \nabla u(x), \nabla^2 u(x)), b_j(x) = \partial_{u_j} F(x, u(x), \nabla u(x), \nabla^2 u(x))$, and $c(x) = \partial_u F(x, u(x), \nabla u(x), \nabla^2 u(x)) \in C^{\rho-2}_{\text{loc}}(\Omega)$.

Definition 3.2 *We say that the linearized operators L is subelliptic, if for any $x \in \Omega$, we have $(a_{jk}(x)) \geq 0$, and for any $K \subset\subset \Omega$, there exists $\varepsilon > 0, C > 0$ such that for all $\varphi \in C^{\infty}_0(K)$ we have*

$$\|\varphi\|^2_{H^{\varepsilon}} \leq C\{|\langle L\varphi, \varphi\rangle| + \|\varphi\|^2_{L^2}\}. \tag{3.10}$$

Remark that if L is elliptic, then it is subelliptic with $\varepsilon = 1$. There is many sufficient condition for subellipticity, in [XU1], we have study the Hörmander's condition, Oleinik-Radkevic's condition, and Fefferman-Phong's condition.

We prove now nonlinear hypoellipticity.

Theorem 3.7 *Let $u \in C^{\rho}_{loc}(\Omega), \rho \geq 4$ be a real solution of equation (3.8). If the linearized operator L is subelliptic, then $u \in C^{\infty}(\Omega)$.*

The proof of this theorem is using paralinearization theorem to transform the nonlinear equation into a linear paradifferential equation, then paradifferential symbolic calculus give the results as in the classic cas.

Let $P \in Op(\Sigma^2_{\rho-2})$ be the paradifferential operators of symbol of L, then

$$P = \sum_{k=1}^{n} \partial_k G_k + G_0 + P_0,$$

where $\sigma(G_k) = g_k(x,\xi) = \sum_{j=1}^n a_{jk}(x)(i\xi_j), k = 1,\cdots,n$, $\sigma(G_0) = g_0(x,\xi) = \sum_{j=1}^n(b_j(x) - \sum_{k=1}^n \partial_{x_k} a_{jk}(x))(i\xi_j)$, and $\sigma(P_0) = c(x)$. Since $u \in C^\rho_{\text{loc}}(\Omega) = C^\rho_{\text{loc}}(\Omega) \bigcap H^4_{\text{loc}}(\Omega)$, from (3.7), we have

$$Pu = f \in C^\rho_{\text{loc}}(\Omega) \bigcap H^4_{\text{loc}}(\Omega). \tag{3.11}$$

This is a linear equation, we have the following a priori estimates.

Theorem 3.8 *Assume that* $(a_{jk}(x)) \geq 0$, *then for any* $K \subset\subset \Omega, s \in \mathbb{R}$, *there exist* $C > 0$, *such that for all* $v \in C^\infty_0(K)$, *and any* $\sigma > 0$, *we have*

$$\sum_{j=1}^{2n} \|G_j v\|^2_{H^s} + \|G_0 v\|^2_{H^{s-1/2}} \leq C\{\|Pv\|^2_{H^s} + \|v\|^2_{H^{s+\sigma}}\},$$

where $\sigma(G_{n+l}) = \sum_{j,k=1}^n |\xi|^{-1}\partial_{x_l} a_{jk}(x)\xi_j\xi_k, l = 1,\cdots,n$.

Using the positivity of $(a_{jk}(x))$ and the commutators of P with pseudo-differential operators $(1-\triangle)^{s/2}$, the proof of this theorem is evident. Now subellipticity of operators L, and Corollary 2.2 give immediately subelliptic estimate for paradifferential operators

$$\|v\|^2_{H^{s+\varepsilon}} \leq C\{\|Pv\|^2_{H^s} + \|v\|^2_{H^s}\}.$$

By regularization processes, we obtain for any $\varphi \in C^\infty_0(K)$, there exist $\varphi_1, \varphi_2 \in C^\infty_0(\Omega), \varepsilon > 0, C > 0$, such that for $u \in H^s_{\text{loc}}$ solution of equation (3.8), we have

$$\|\varphi u\|^2_{H^{s+\varepsilon}} \leq C\{\|\varphi_1 Pu\|^2_{H^s} + \|\varphi_2 u\|^2_{H^s}\}.$$

From this estimates, we prove immediately theorem 3.7. Since from $u \in C^\rho_{\text{loc}}(\Omega) \subset H^4_{\text{loc}}(\Omega)$, we get $u \in H^{4+\varepsilon}_{\text{loc}}(\Omega)$, then by iteration $u \in H^{4+N\varepsilon}_{\text{loc}}(\Omega)$ for any N, which prove $u \in C^\infty(\Omega)$.

3.5. Propagation of singularity for nonlinear equation

We study now the propagation of singularities for nonlinear hyperbolic equation. If $(x_0,\xi_0) \in T^*\Omega \setminus 0$ such that $p_m(x_0,\xi_0) = 0$, we say that it is a characteristic point, and set

$$\text{Char}p = \{(x,\xi) \in T^*\Omega \setminus 0; p_m(x,\xi) = 0\},$$

where $p(x,\xi)$ is the symbol defined by (3.6). If $\rho > m+2$, the integral curb of Hamiltonnian H_{p_m} is called bicharacteristic for nonlinear equation (3.5) associated with function u. It is evident that if a bicharacteristic contain a characteristic point, then it is full belong to Char$p$. We can now give a result of propagation of singularities for nonlinear equation.

Theorem 3.9 *Let* $u \in H^s_{loc}(\Omega)$ *be a real solution of nonlinear equation (3.5),* $s > n/2 + m + 2, (x_0,\xi_0)$ *a characteristic point,* Γ *a bicharacteristic contain* (x_0,ξ_0). *If for some* $t \leq 2s - n/2 - m - 1$, $u \in H^t_{x_0,\xi_0}$, *then* $u \in H^t_\Gamma$.

The result is similar to linear equations. The proof is very long, we send to [BO1, BO2]. We have studied only very weak singularities, for high frequency singularities, there is also many results for semilinear equations, see the reference of [BO2]. Using tangential paradifferential operators, in [XU2], we have studied the propagation of singularity up to the boundary for nonlinear boundary value problems.

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