

TRACE AND TRACE LIFTING THEOREMS IN WEIGHTED SOBOLEV SPACES

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Résumé *Dans ce travail, nous démontrons des théorèmes de trace et de relèvement pour les espaces de Sobolev associés à un système de champs de vecteurs satisfaisant la condition de Hörmander à l'ordre 2. Le cas des points caractéristiques non dégénérés pour les champs de vecteurs invariants à gauche sur le groupe d'Heisenberg est aussi traité.*

Abstract *We prove in this work the trace and trace lifting theorems for the Sobolev spaces associated with a system of Hörmander's vectors fields of order 2. The case of non degenerate characteristic points for left invariant vector fields on the Heisenberg group is also studied.*

Key words Trace and trace lifting, Heisenberg group, Hörmander condition, Weyl-Hörmander calculus

A.M.S. Classification 35 A, 35 H, 35 S.

1. INTRODUCTION

The main purpose of this paper is to study the problem of restriction of functions that belongs to Sobolev spaces associated to left invariant vector fields for the Heisenberg group \mathbb{H}^d . This group is the space \mathbb{R}^{2d+1} of the (non commutative) law of product

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + (y|x') - (y'|x)).$$

The left invariant vector fields are

$$X_j = \partial_{x_j} + y_j \partial_s \quad \text{and} \quad Y_j = \partial_{y_j} - x_j \partial_s.$$

In all that follows, we shall denote by \mathcal{Z} this family and state $Z_j = X_j$ and $Z_{j+d} = Y_j$.

We associate Sobolev spaces to this system of vector fields through the following definition.

Definition 1.1. Let k be a non negative integer, we denote by $H^k(\mathbb{H}^d)$ the Sobolev space of order k which is the space of the functions u in $L^2(\mathbb{H}^d)$ (for the usual Lebesgue measure on \mathbb{R}^{2d+1}) such that

$$Z_{j_1} \dots Z_{j_\ell} u \in L^2 \quad \text{for any } (j_m)_{1 \leq m \leq \ell} \in \{1, \dots, 2d\}^\ell \quad \text{with } \ell \leq k.$$

When s is any non negative real number, we can, as in the case of classical Sobolev spaces on \mathbb{R}^n , define the space $H^s(\mathbb{H}^d)$ by complex interpolation (see for instance [6]).

As in the usual case, other definitions of Sobolev spaces can be used (and the spaces are the same): the definition using integral and kernel (see [23] and [24]), or the definition using the Littlewood-Paley theory based on Fourier transform on the Heisenberg group (see [1]), or the definition using the Weyl-Hörmander calculus (see [11]).

The key point is that \mathcal{Z} satisfies Hörmander's condition at order 2, which means that the family $(Z_\ell, [Z_\ell, Z_m])$ spans the whole tangent space. Hörmander's sub-elliptic theorem implies that the space $H^s(\mathbb{H}^d)$ is included in the usual Sobolev space $H^{\frac{s}{2}}(\mathbb{R}^{2d+1})$ when s is positive.

These spaces $H^s(\mathbb{H}^d)$ have properties which look very much like the ones of usual Sobolev spaces: Sobolev embeddings are true with exponents where the real dimension $2d+1$ becomes the homogeneous one $2d+2$ (see for instance [12] and [20]), Poincaré's inequality (see [18]), Hardy's inequality, tame estimates (see [2]) and extension properties (see [21]).

In this work, we are interested in problems of trace and trace lifting on a smooth hypersurface of \mathbb{H}^d in the framework of Sobolev spaces. In the framework of Hölder spaces, this problem has been studied by D. Jerison in [19].

First of all, let us point out that the problem of existence of trace appears only when s is less than or equal to 1. The space $H^s(\mathbb{H}^d)$ is included in $H^{\frac{s}{2}}(\mathbb{R}^{2d+1})$. So if s is strictly larger than 1, this implies that the trace on any smooth hypersurface exists and belongs to the usual Sobolev space $H^{\frac{s}{2}-\frac{1}{2}}$ of the hypersurface.

So the existence of traces makes problem when $s \in]1/2, 1]$. Two very different cases then appear: the one when the hypersurface is non characteristic, which means that any point M_0 of the hypersurface Σ is such that $\mathcal{Z}|_{M_0} \not\subset T_{M_0}\Sigma$, and the one when some point M_0 of the hypersurface Σ is characteristic, which means that $\mathcal{Z}|_{M_0} \subset T_{M_0}\Sigma$.

It is well known that a compact hypersurface without boundary has always at least one characteristic point (see for example [4]). Things being what they are, the problem of traces is of course a local one. So it is relevant to assume an hypersurface to be non characteristic. This case is quite well understood. Since the work of M. Derridj (see [14]), it is known that in this non characteristic case, traces of functions in $H^1(\mathbb{H}^d)$ exist in L^2 . Always in this non characteristic case, I. Pesenson (see [22]) has proved trace and trace lifting theorems for the spaces $H^k(\mathbb{H}^d)$ when d greater than or equal to 2. Moreover, these results are also valid for Sobolev spaces defined with L^p for p greater than 1 (see also [13] and [15]). The reason why this hypothesis $d \geq 2$ is required is that the proof uses in a crucial way the fact that $\mathcal{Z} \cap T\Sigma$ is of rank $2d-1$ and satisfies Hörmander's condition at order 2. This is of course not the case when $d=1$, because, in this case, $\mathcal{Z} \cap T\Sigma$ is of rank 1. In the characteristic case, even the existence of traces is a problem.

The aim of this work is first to prove a theorem of trace and trace lifting in the non characteristic case (so valid also for $d=1$), and then to prove a theorem of trace and trace lifting in the characteristic case up to $d=1$. The set of characteristic points of a compact hypersurface may have a complicated structure. For instance, this set may contain curves that may be or may be not integral curves of one of the vector fields Z . The result we shall prove here demands an hypothesis about the nature of the set of characteristic points. Let us state it.

Definition 1.2. Let M_0 be a point of a smooth hypersurface Σ of \mathbb{H}^d . This point M_0 is said to be a non degenerate characteristic point if and only if

- the point M_0 is characteristic, which means that $\mathcal{Z}|_{M_0} \subset T_{M_0}\Sigma$,
- for any 1-form θ of $T^*\mathbb{R}^{2d+1}$ that vanishes on $T\Sigma$ and such that $\theta(M_0) \neq 0$, the system $(\mathcal{L}_{Z_\ell}\theta|_{T_{M_0}\Sigma})_{1 \leq \ell \leq 2d}$ spans the cotangent bundle $T_{M_0}^*\Sigma$, where \mathcal{L}_Z denotes the Lie derivative with respect to Z .

Let us notice that a non degenerate characteristic point is necessarily isolated. The non degeneracy condition can be interpreted by the fact that the matrix $(Z_\ell Z_m g(M_0))_{1 \leq \ell, m \leq 2d}$ is invertible if g is a local defining function of the hypersurface Σ (this is the aim of Proposition 4.1). Moreover, as $\mathcal{Z}|_{M_0}$ is of rank $2d$, it spans the tangent space $T_{M_0}\Sigma$.

Two simple examples of non degenerate characteristic points are the origin for the hyperplane $\Sigma_0 \stackrel{\text{def}}{=} (s = 0)$ and the two poles $(0, 0, \pm 1)$ of the Heisenberg sphere.

Now, let us define the module of tangent vector fields on Σ which are going to be the differentiations on which the Sobolev regularity of traces will be based.

Definition 1.3. Let Σ be a smooth hypersurface of \mathbb{H}^d with a finite number of non degenerate characteristic points $\mathcal{M} \stackrel{\text{def}}{=} (M_j)_{1 \leq j \leq N}$. Let us denote by $C_{\mathcal{M}}^\infty$ the set of smooth functions a on $\Sigma \setminus \mathcal{M}$ such that for any integer k and for any point M_j of \mathcal{M} , a constant C_k exists such that in a neighborhood of M_j , we have

$$|D^k a(M)| \leq C_k |M - M_j|^{-k}.$$

Let us notice that functions of $C_{\mathcal{M}}^\infty$ have singularities of type "homogeneous of degree 0" at the points of \mathcal{M} .

Definition 1.4. Let Σ be a smooth hypersurface of \mathbb{H}^d with a finite number of non degenerate characteristic points $\mathcal{M} \stackrel{\text{def}}{=} (M_j)_{1 \leq j \leq N}$. Let us denote by $\mathcal{Z}_{\Sigma, \mathcal{M}}$ the $C_{\mathcal{M}}^\infty$ -module of vector fields spanned by the set of all vector fields of $\mathcal{Z} \cap T\Sigma$ that vanish on \mathcal{M} .

This module $\mathcal{Z}_{\Sigma, \mathcal{M}}$ is spanned by a finite number of smooth vector fields \mathcal{R}_Σ . If g is a local defining function of Σ , a possible choice for \mathcal{R}_Σ is the family

$$R_{j,k} \stackrel{\text{def}}{=} Z_j(g)Z_k - Z_k(g)Z_j \quad \text{for } 1 \leq j < k \leq 2d. \quad (1.1)$$

This will be proved in lemma 4.1. Let us mention now that the non degeneracy condition is crucial for the proof of this property.

For sake of simplicity of the notations, we reorder the family $\mathcal{R}_\Sigma \stackrel{\text{def}}{=} (R_j)_{1 \leq j \leq N}$. In all that follows we shall denote by Δ_Σ the differential operator on Σ defined by

$$\Delta_\Sigma \stackrel{\text{def}}{=} - \sum_{j=1}^N R_j^* R_j. \quad (1.2)$$

Let us point that in the case when $d = 1$, there is only one vector field in the family \mathcal{R}_Σ which we shall denote by R .

When k is a non negative integer, let us define the Sobolev spaces associated to this module of vector fields.

Definition 1.5. If $d \geq 2$, we denote by $TH^k(\mathcal{R}_\Sigma)$ the space of functions u of $L^2(\Sigma)$ such that

$$R_{j_1} \dots R_{j_\ell} u \in L^2 \quad \text{for any } (j_m)_{1 \leq m \leq \ell} \in \{1, \dots, d(2d-1)\}^\ell \quad \text{with } \ell \leq k.$$

If $d = 1$ and if $k = 2j$, we denote by $TH^k(\mathcal{R}_\Sigma)$ the space of functions u of $L^2(\Sigma)$ such that

$$R^\ell u \in L^2 \quad \text{for any } \ell \leq k.$$

and

$$T_{j_1} \dots T_{j_\ell} u \in L^2 \quad \text{for any } (j_m)_{1 \leq m \leq \ell} \in \{1, \dots, 4\}^\ell \quad \text{with } \ell \leq k$$

where $(T_\ell)_{1 \leq \ell \leq 4}$ is a family of generators of the module (on the ring of smooth functions on Σ) of the vector fields tangent to Σ that vanish at the points of \mathcal{M} .

Now, we can define the spaces $TH^s(\mathcal{R}_\Sigma)$ for positive real numbers by complex interpolation. We shall prove the following theorem.

Theorem 1.1. *Let Σ be an hypersurface with a finite number of non degenerate characteristic points. The restriction to Σ map, denoted by γ_Σ (defined on $\mathcal{D}(\mathbb{H}^d)$), can be extended in an onto continuous map from $H^1(\mathbb{H}^d)$ to $TH^{\frac{1}{2}}(\mathcal{R}_\Sigma)$.*

Remarks Let us consider the case when $d \geq 2$. As proved in Proposition 4.4, the operator Δ_Σ defined by (1.2) is selfadjoint with domain $TH^2(\mathcal{R}_\Sigma)$. So the space $TH^{\frac{1}{2}}(\mathcal{R}_\Sigma)$ can be understood as the domain of the operator $(-\Delta_\Sigma)^{\frac{1}{4}}$.

As previously said, the fact that we deal with the case when $d = 1$ impose us to reconsider the non characteristic case. To do this, let us state the following definition which is a modification of the classical one.

Definition 1.6. *Let $\mathcal{P}_\Sigma = (P_{\Sigma,j})_{1 \leq j \leq N}$ be a family of smooth vector fields tangent to a submanifold Σ of \mathbb{R}^n and k a positive integer. We denote by $H^k(\mathcal{P}_\Sigma)$ the set of functions u in $H^{\frac{k}{2}}(\Sigma)$ such that*

$$P_{\Sigma,j_1} \dots P_{\Sigma,j_\ell} u \in L^2(\Sigma) \quad \text{for any } (j_m)_{1 \leq m \leq \ell} \in \{1, \dots, N\}^\ell \quad \text{with } \ell \leq k.$$

When s is a positive real number, the space $H^s(\mathcal{P}_\Sigma)$ is defined by complex interpolation.

Here the submanifold Σ will be either \mathbb{R}^n itself or a smooth hypersurface.

Remark When the system of vector fields \mathcal{P}_Σ satisfies the Hörmander condition at order 2, the above definition is the same if we substitute L^2 to $H^{\frac{k}{2}}$.

Let Σ be a smooth non characteristic hypersurface for a system $\mathcal{P} \stackrel{\text{def}}{=} (P_j)_{1 \leq j \leq N}$ (this means that \mathcal{P} is not tangent to Σ). The C^∞ -module spanned by $\mathcal{P} \cap T\Sigma$ is of finite type. Let us denote by $\mathcal{P}_\Sigma \stackrel{\text{def}}{=} (P_{\Sigma,j})_{1 \leq j \leq N}$ a finite system of generators of the system $\mathcal{P} \cap T\Sigma$. The trace theorem is the following.

Theorem 1.2. *Let Σ be a non characteristic hypersurface for the system \mathcal{P} ; the restriction to Σ map, denoted by γ_Σ , can be extended in an onto continuous map from $H^s(\mathcal{P})$ to $H^{s-\frac{1}{2}}(\mathcal{P}_\Sigma)$ when s is greater than $1/2$.*

Let us point out that it implies in particular that γ_Σ is continuous from the space $H^s(\mathcal{P})$ into the space $H^{\frac{s}{2}-\frac{1}{4}}(\Sigma)$. A similar theorem has been proved by S. Berhanu and I. Pesenson in [7] in the case when the system \mathcal{P}_Σ still satisfies the Hörmander condition at order 2 on the hypersurface Σ and for $s = 1$.

Now let us explain the structure of the paper together with the main ideas of the proofs.

The first case we study is the non characteristic one. Let us give a flavor of the proof by looking at a very simple case. Let us consider when in \mathbb{R}^2 the system \mathcal{P} is $(\partial_{x_1}, x_1 \partial_{x_2})$ and the hypersurface Σ is $\{x \in \mathbb{R}^2 / x_1 = 0\}$. The system \mathcal{P}_Σ then reduces to 0. The above Theorem 1.2 tells us that the map γ_Σ is continuous and onto from $H^k(\mathcal{P})$ to $H^{\frac{k}{2}-\frac{1}{4}}(\Sigma)$. Let

us point out that this case is as different as possible from the case of the Heisenberg group when $d \geq 2$.

Let us denote by \widehat{u}^2 the Fourier transform of u with respect to the second variable. So, we can write that, for any u in $\mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned} \|\gamma_\Sigma(u)\|_{H^{\frac{k}{2}-\frac{1}{4}}(\mathbb{R})}^2 &= \int \langle \xi_2 \rangle^{k-\frac{1}{2}} |\widehat{u}^2(0, \xi_2)|^2 d\xi_2 \\ &= 2 \int \langle \xi_2 \rangle^{k-\frac{1}{2}} \int_{-\infty}^0 \left(\frac{\partial \widehat{u}^2}{\partial x_1}(x_1, \xi_2) |\widehat{u}^2(x_1, \xi_2)| \right) dx_1 d\xi_2. \end{aligned}$$

Using Cauchy-Schwarz inequality, we get that

$$\begin{aligned} \|\gamma_\Sigma(u)\|_{H^{\frac{k}{2}-\frac{1}{4}}(\mathbb{R})}^2 &\leq 2 \int \langle \xi_2 \rangle^{k-\frac{1}{2}} \left(\int_{-\infty}^{\infty} \left| \frac{\partial \widehat{u}^2}{\partial x_1}(x_1, \xi_2) \right|^2 dx_1 \right)^{1/2} \\ &\quad \times \left(\int_{-\infty}^{\infty} |\widehat{u}^2(x_1, \xi_2)|^2 dx_1 \right)^{1/2} d\xi_2 \\ &\leq 2 \|\partial_{x_1} u\|_{L^2(\mathbb{R}, H^{\frac{k}{2}-\frac{1}{2}}(\mathbb{R}))} \|u\|_{L^2(\mathbb{R}, H^{\frac{k}{2}}(\mathbb{R}))}. \end{aligned}$$

But, using Hörmander's subelliptic theorem, we infer that $H^{k-1}(\mathcal{P}) \subset H^{\frac{k-1}{2}}(\mathbb{R}^2)$. So $\partial_{x_1} u$ belongs to $H^{\frac{k}{2}-\frac{1}{2}}(\mathbb{R}^2)$. As $k \geq 1$, we have $H^{\frac{k}{2}-\frac{1}{2}}(\mathbb{R}^2) \subset L^2(\mathbb{R}, H^{\frac{k}{2}-\frac{1}{2}}(\mathbb{R}))$. So the map γ_Σ is continuous.

Let us now consider a function v in $H^{\frac{k}{2}-\frac{1}{4}}(\mathbb{R})$ and let us state

$$u(x_1, x_2) = \mathcal{F}_{\xi_2}^{-1}(\chi(x_1 < \xi_2 >^{1/2}) \widehat{v}(\xi_2))$$

where χ is a function of $\mathcal{D}(\mathbb{R})$ with value 1 in a neighbourhood of 0. This is obvious that $\gamma_\Sigma(u) = v$. Moreover, when $k_1 + 2k_2 + k_3 \leq k$, we have

$$\begin{aligned} N_{k_1, k_2, k_3}(u) &\stackrel{\text{def}}{=} \|(x_1 \partial_{x_2})^{k_1} \partial_{x_2}^{k_2} \partial_{x_1}^{k_3} u\|_{L^2(\mathbb{R}^2)}^2 \\ &= \int |(x_1 \partial_{x_2})^{k_1} (\partial_{x_2})^{k_2} (\partial_{x_1})^{k_3} u(x_1, x_2)|^2 dx_1 dx_2 \\ &= (2\pi)^{-1} \int |x_1^{k_1} \xi_2^{k_1+k_2} \langle \xi_2 \rangle^{\frac{k_3}{2}} \chi^{(k_3)}(x_1 < \xi_2 >^{1/2}) \widehat{v}(\xi_2)|^2 dx_1 d\xi_2 \\ &\leq (2\pi)^{-1} \int |y_1^{k_1} \chi^{(k_3)}(y_1) \langle \xi_2 \rangle^{\frac{k_1}{2}+k_2+\frac{k_3}{2}} \widehat{v}(\xi_2)|^2 \langle \xi_2 \rangle^{-1/2} dy_1 d\xi_2 \\ &\leq (2\pi)^{-1} \int |y_1^{k_1} \chi^{(k_3)}(y_1)|^2 \langle \xi_2 \rangle^{k-\frac{1}{2}} |\widehat{v}(\xi_2)|^2 dy_1 d\xi_2 \\ &\leq c \|v\|_{H^{\frac{k}{2}-\frac{1}{4}}(\mathbb{R})}^2. \end{aligned}$$

So the map γ_Σ is onto. When s is an integer, the proof of Theorem 1.2 consists in substituting a weight for an Hörmander's metric to $\langle \xi_2 \rangle$ in the above model case.

The second section of this paper consists first in recalling basic facts about Hörmander's pseudodifferential calculus and about Sobolev spaces in this framework. Afterwards we explain the relation between the Sobolev spaces defined in this introduction and the Weyl-Hörmander calculus.

The third section is devoted to the proof of Theorem 1.2 in the case when s is an integer. We follow the same lines as in the above proof.

The fourth section is devoted to the study of the characteristic case for the Heisenberg group. The method used, as possibly inferred by the introduction of the ring C_M^∞ is a "blow

up" type method. Let us explain it in the simple case when Σ is the hyperplane $\Sigma_0 \stackrel{\text{def}}{=} (s = 0)$ and the characteristic point is the origin.

Let us consider a function φ of $\mathcal{D}(\mathbb{R} \setminus \{0\})$ such that

$$\forall t, \quad 0 < |t| \leq 1, \quad \sum_{p=0}^{\infty} \varphi(2^p t) = 1.$$

Let us introduce the Heisenberg distance to the origin ρ which is

$$\rho(x, y, s) = ((x^2 + y^2)^2 + s^2)^{\frac{1}{4}}.$$

If $u \in \mathcal{D}(\mathbb{R}^{2d+1})$ supported in $\{\rho(x, y, s) \leq 1\}$, we have

$$u = \sum_{p=0}^{\infty} \varphi_p u$$

where $\varphi_p(x, y, s) = \varphi(2^p \rho(x, y, s))$. Now let us compute $Z(\varphi_p u)$ for $Z \in \mathcal{Z}$. The Leibnitz formula implies that

$$Z(\varphi_p u) = \varphi_p Z u + u Z \varphi_p.$$

But $Z \varphi_p = 2^p \varphi'(2^p \rho) Z \rho$. Moreover, when $\rho \leq 1$, we have $|Z \rho| \leq c$. So it turns out that

$$\|Z(\varphi_p u)\|_{L^2}^2 \leq C 2^{2p} \|\varphi_p' u\|_{L^2}^2 + 2 \|\varphi_p Z u\|_{L^2}^2.$$

By definition of φ_p , we have $2^{2p} \sim \rho^{-2}$ on the support of $\varphi_p' u$. From this, we deduce that

$$2^{2p} \|\varphi_p' u\|_{L^2}^2 \leq C \int \frac{|\varphi_p' u|^2}{\rho^2} dx dy ds.$$

As the functions $\varphi_p' u$ and $\varphi_{p'} u$ have disjoint supports as soon as $|p - p'|$ is large enough, we have

$$\sum_{p=0}^{\infty} 2^{2p} \|\varphi_p' u\|_{L^2}^2 \leq C \int \frac{|u|^2}{\rho^2} dx dy ds.$$

Let us recall the well known Hardy's inequality on the Heisenberg group.

Lemma 1.1. *Let k be an integer in the interval $]0, d + 1[$. A constant C exists such that, for any function u in $H^k(\mathbb{H}^d)$ supported in the Heisenberg ball $B(0, 1)$, we have*

$$\int_{\mathbb{R}^{2d+1}} \frac{|u|^2}{\rho^{2k}} dx dy ds \leq C \sum_{(j_m) \in \{1, \dots, 2d\}^k} \|Z_{j_1} \cdots Z_{j_k} u\|_{L^2}^2.$$

So it turns out that

$$\sum_{p=0}^{\infty} \sum_{Z \in \mathcal{Z}} \|Z(\varphi_p u)\|_{L^2}^2 \leq C \|u\|_{H^1(\mathbb{H}^d)}^2. \quad (1.3)$$

Now, let us use the dilation of parameter 2^p on the Heisenberg group and let us state

$$u_p(x, y, s) = (\varphi_p u)(2^{-p} x, 2^{-p} y, 2^{-2p} s).$$

A simple computation shows that

$$\|Z u_p\|_{L^2} = 2^{pd} \|Z(\varphi_p u)\|_{L^2}.$$

Moreover the support of the function u_p is included in a ring $\mathcal{C} = \{\rho(x, y, s) \sim 1\}$ of the Heisenberg distance ρ . This ring \mathcal{C} is a compact subset of \mathbb{H}^d which is independent of p and

that does not cross the origin. So the hypersurface Σ_0 is non characteristic on the support of u_p and results of Theorem 1.2 can be applied to u_p . So, we deduce in particular that

$$\begin{aligned} \|\gamma_{\Sigma_0}(u_p)\|_{L^2}^2 &\leq C \left(\sum_{Z \in \mathcal{Z}} \|Zu_p\|_{L^2}^2 + \|u_p\|_{L^2}^2 \right) \\ &\leq C 2^{2pd} \|\varphi_p u\|_{H^1(\mathbb{H}^d)}^2. \end{aligned}$$

Then, a dilatation on Σ_0 tells us that

$$\|\gamma_{\Sigma_0}(\varphi_p u)\|_{L^2} \leq C \|\varphi_p u\|_{H^1(\mathbb{H}^d)}.$$

inequality (1.3) implies that

$$\sum_{p=0}^{\infty} \|\gamma_{\Sigma_0}(\varphi_p u)\|_{L^2}^2 \leq C \|u\|_{H^1(\mathbb{H}^d)}^2.$$

The fact that the family $\gamma_{\Sigma_0}(\varphi_p u)$ is almost orthogonal in L^2 allows us to conclude that the map γ_{Σ_0} can be continuously extended from $H^1(\mathbb{H}^d)$ to $L^2(\Sigma_0)$. The complete proof of Theorem 1.1 will be the purpose of the forth section.

The fifth section is devoted to the case when s is not integer. We prove Theorem 1.2 using Weyl-Hörmander calculus.

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2. WEYL-HÖRMANDER CALCULUS AND SOBOLEV SPACES

In this section, we want first to present some basic properties of the Weyl-Hörmander calculus and of abstract Sobolev spaces defined with this theory. Then, we shall show the link between the Sobolev spaces defined in the introduction in Definition 1.6 and these abstract Sobolev spaces.

2.1. Basic Weyl-Hörmander calculus. We follow [8], [9] and [17]. Weyl quantization procedure is a way to associate an operator a^w , acting on $L^2(\mathbb{R}^n)$ to a function a (the symbol of a^w) defined on the phase space $T^*\mathbb{R}^n = \mathbb{R}^{2n}$. This operator a^w is defined by

$$a^w u(x) = (2\pi)^{-n} \int_{T^*\mathbb{R}^n} e^{i\langle x-z, \xi \rangle} a\left(\frac{x+z}{2}, \xi\right) u(z) dz d\xi.$$

Let us denote by $[X, Y]$ the standard symplectic form on $T^*\mathbb{R}^n$. We have the following composition formula $a^w \circ b^w = (a\#b)^w$ with

$$(a\#b)(X) = \pi^{-2n} \int_{T^*\mathbb{R}^n \times T^*\mathbb{R}^n} e^{-2i[X-Y_1, X-Y_2]} a(Y_1) b(Y_2) dY_1 dY_2.$$

Let us define the concept of Hörmander's metric.

Definition 2.1. *Let g be a measurable map from $T^*\mathbb{R}^n$ into the set of positive quadratic forms on $T^*\mathbb{R}^n$. This map g is an Hörmander's metric if and only if the following three conditions are satisfied:*

Slowness *A positive constant c_0 exists such that*

$$g_X(X - Y) \leq \frac{1}{c_0} \Rightarrow c_0^{-1} g_X \leq g_Y \leq c_0 g_X.$$

Uncertainty principle

$$g_X \leq g_X^\sigma \quad \text{with} \quad g_X^\sigma(T) = \sup_{W \neq 0} \frac{[T, W]^2}{g_X(W)}$$

Temperance A positive constant c_0 and an integer N_0 exist such that for any (X, Y) in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$

$$c_0^{-1}(1 + g_Y^\sigma(X - Y))^{-N_0} g_X \leq g_Y \leq c_0(1 + g_Y^\sigma(X - Y))^{N_0} g_X.$$

Moreover, we shall denote by λ_g the function defined by

$$\lambda_g^2(X) \stackrel{\text{def}}{=} \inf_{T \in T^*\mathbb{R}^n} \frac{g_X^\sigma(T)}{g_X(T)}.$$

Let us consider an Hörmander's metric g and let us choose a positive real number r strictly less than c_0^{-1} . In all that follows, we shall denote by U_X the ball of center X and radius $r^{1/2}$ for the metric g_X . Let us define $\Delta(X, Y)$ by

$$\Delta(X, Y) \stackrel{\text{def}}{=} 1 + \max\{g_X^\sigma(U_X - U_Y), g_Y^\sigma(U_X - U_Y)\}$$

where

$$g_X^\sigma(U_X - U_Y) = \inf_{(X', Y') \in U_X \times U_Y} g_X^\sigma(X' - Y').$$

The function $\Delta(X, Y)$ measures how far away (for the metric g^σ) are the two balls U_X and U_Y . We shall omit to note that this function Δ depends on r . As proved in [9], we can substitute slowness and temperance by

$$\frac{1}{c_0} \Delta(X, Y)^{-N_0} g_X \leq g_Y \leq c_0 \Delta(X, Y)^{N_0} g_X.$$

One of the key properties of the function Δ , obviously symmetric, is described in the following lemma, proved in [9].

Lemma 2.1. *An integer N_1 exists such that*

$$\sup_{X \in T^*\mathbb{R}^n} \int_{T^*\mathbb{R}^n} \Delta(X, Y)^{-N_1} |g_Y|^{1/2} dY < \infty,$$

where $|g_Y|^{1/2}$ denotes the determinant of the quadratic form g_Y in any symplectic basis of $T^*\mathbb{R}^n$.

An Hörmander's metric can be understood as an admissible procedure of localization in the phase space.

Let us define the concept of g -weights, of g -symbols.

Definition 2.2. *Let g be an Hörmander's metric. A measurable function m defined on $T^*\mathbb{R}^n$ with value in \mathbb{R}_+^* is a g -weight if and only if a constant \tilde{C} and an integer \tilde{N} exist such that*

$$\left(\frac{m(X)}{m(Y)} \right)^{\pm 1} \leq \tilde{C} \Delta(X, Y)^{\tilde{N}}.$$

Let m be a g -weight. Let us denote by $S(m, g)$ the set of smooth functions a on $T^*\mathbb{R}^n$ such that for any integer k ,

$$\|a\|_{k, S(m, g)} = \sup_{\substack{j \leq k, X \in T^*\mathbb{R}^n \\ g_X(T_j) \leq 1}} \frac{|\partial_{T_1} \dots \partial_{T_j} a(X)|}{m(X)} < \infty.$$

Now, let us remark that if a and b are two smooth compactly supported functions on $T^*\mathbb{R}^n$, there is no reason why $a \# b$ could be so. The relevant notion is the following one.

Definition 2.3. Let γ be a positive quadratic form on $T^*\mathbb{R}^n$ such that $\gamma^\sigma \geq \gamma$ and Y a point of $T^*\mathbb{R}^n$. We equip $\mathcal{S}(T^*\mathbb{R}^n)$ with the following semi-norms (denoted by semi-norms of confinement)

$$\|a\|_{k, \text{conf}(\gamma, Y)} = \sup_{\substack{X \in T^*\mathbb{R}^n \\ j \leq k, \gamma(T_j) \leq 1}} (1 + \gamma^\sigma(X - B_\gamma(Y, r)))^{j/2} |\partial_{T_1} \dots \partial_{T_j} a(X)|.$$

Let g be an Hörmander's metric and $(a_Y)_{Y \in T^*\mathbb{R}^n}$ a family of functions of $\mathcal{S}(T^*\mathbb{R}^n)$. This family is uniformly confined if and only if, for any integer k ,

$$\|(a_Y)\|_{k, \text{conf}(g)} = \sup_{Y \in T^*\mathbb{R}^n} \|a_Y\|_{k, \text{conf}(g_Y, Y)} < \infty.$$

The main interest of this concept is the following two lemmas, proved in [9].

Lemma 2.2. Let g be an Hörmander's metric and a and b two functions of $\mathcal{S}(T^*\mathbb{R}^n)$. For any couple of integer (k, N) , an integer ℓ and a constant C exist such that, for any couple (Y, Z) of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, we have

$$\|a\#b\|_{k, \text{conf}(g_Y, Y)} + \|a\#b\|_{k, \text{conf}(g_Z, Z)} \leq C \Delta(Y, Z)^{-N} \|a\|_{\ell, \text{conf}(g_Y, Y)} \|b\|_{\ell, \text{conf}(g_Z, Z)}$$

Lemma 2.3. For any integer N , a constant C and an integer M exist such that, if the family $(a_Y)_{Y \in T^*\mathbb{R}^n}$ is a uniformly confined family of symbols, if b belongs to $S(m, g)$, then

$$\begin{aligned} & \| (m^{-1}(Y)a_Y\#b) \|_{N, \text{conf}(g)} + \| (m^{-1}(Y)a_Y b) \|_{N, \text{conf}(g)} \\ & + \| (\lambda_g(Y)m^{-1}(Y)(a_Y\#b - a_Y b)) \|_{N, \text{conf}(g)} \leq C \| (a_Y) \|_{N, \text{conf}(g)} \| b \|_{k, S(m, g)}. \end{aligned}$$

It will be convenient here to assume that the metric g is strongly tempered (see [8] for a precise definition). All the metrics used here are so. This property implies in particular the following lemma, proved in [8].

Lemma 2.4. Two uniformly confined families (φ_Y) and (ψ_Y) exist such that for any X in $T^*\mathbb{R}^n$, we have

$$\int_{T^*\mathbb{R}^n} \varphi_Y(X) |g_Y|^{1/2} dY = \int_{T^*\mathbb{R}^n} (\psi_Y\#\varphi_Y)(X) |g_Y|^{1/2} dY = 1.$$

We can also assume that (φ_Y) is such that the support of φ_Y is included in the ball of center Y and radius r (for some fixed positive number r) for the metric g_Y .

Let us define the concept of Sobolev spaces associated to a g -weight.

Definition 2.4. Let m be a g -weight. The space $H(m, g)$ is the set of tempered distributions u such that

$$\|u\|_{H(m, g)} = \left(\int_{T^*\mathbb{R}^n} m^2(Y) \|\varphi_Y^w u\|_{L^2}^2 |g_Y|^{1/2} dY \right)^{1/2} < \infty.$$

The space $H(m, g)$ is also the set of tempered distributions u on \mathbb{R}^n such that, for any $a \in S(m, g)$, we have $a^w u \in L^2$. Moreover $H(1, g) = L^2$. As proved in [8], the space $H(m, g)$ is ‘‘almost independent’’ of the metric g . So in all that follows, we shall drop the metric g and shall denote the Sobolev space $H(m)$. The study of these Sobolev spaces has been developed in [5], [8], [9], [11] and [12].

The Weyl quantization associates to a symbol a in $S(m_1, g)$ an operator on $\mathcal{S}'(\mathbb{R}^n)$. Those operators act on the Sobolev spaces in the following way.

Proposition 2.1. *Let a be in $S(m_1, g)$ for some g -weight m_1 . Then for any g -weight m , a constant C and an integer k exist (depending only on the constants that appear in Definitions 2.1 and 2.2 for g , m and m_1) such that for any u in $H(m)$,*

$$\|a^w u\|_{H(m_1^{-1}m)} \leq C \|a\|_{k, S(m_1, g)} \|u\|_{H(m)}.$$

In the next section, we shall need the following technical lemma.

Lemma 2.5. *Let a be a complex valued measurable function on $T^*\mathbb{R}^n$, (θ_Y) a uniformly confined family and v a measurable function from $T^*\mathbb{R}^n$ to $L^2(\mathbb{R}^n)$. A constant C and an integer k exist (depending only on the constants related to g and m) such that, if*

$$u \stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^n} a(Y) \theta_Y^w(v_Y) |g_Y|^{\frac{1}{2}} dY,$$

then we have

$$\|u\|_{H(m)}^2 \leq C \|(\theta_Y)\|_{k, Conf(g)}^2 \int_{T^*\mathbb{R}^n} |a(Y)|^2 m^2(Y) \|v_Y\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY.$$

To prove this lemma, let us first observe that by definition of the $H(m)$ norm, we have

$$\|u\|_{H(m)}^2 = \int_{(T^*\mathbb{R}^n)^3} m^2(Z) a(Y) \overline{a(Y')} (\varphi_Z^w \theta_Y^w v_Y | \varphi_Z^w \theta_{Y'}^w v_{Y'})_{L^2} |g_Y|^{\frac{1}{2}} |g_{Y'}|^{\frac{1}{2}} |g_Z|^{\frac{1}{2}} dY dY' dZ.$$

Using Lemma 2.2 we get that, for any integer N , a constant C and an integer k exist such that

$$|(\varphi_Z^w \theta_Y^w v_Y | \varphi_Z^w \theta_{Y'}^w v_{Y'})_{L^2}| \leq C \|(\theta_Y)\|_{k, Conf(g)}^2 \Delta(Y, Z)^{-N} \Delta(Y', Z)^{-N} \|v_Y\|_{L^2} \|v_{Y'}\|_{L^2}.$$

As m is a g -weight, we have

$$m^2(Z) \leq C m(Y) m(Y') \Delta(Y, Z)^{\tilde{N}} \Delta(Y', Z)^{\tilde{N}}.$$

So for any integer N a constant C exists such that

$$\begin{aligned} m^2(Z) |(\varphi_Z^w \theta_Y^w v_Y | \varphi_Z^w \theta_{Y'}^w v_{Y'})_{L^2}| &\leq C \|(\theta_Y)\|_{k, Conf(g)}^2 \\ &\times m(Y) m(Y') \Delta(Y, Z)^{-N} \Delta(Y', Z)^{-N} \|v_Y\|_{L^2} \|v_{Y'}\|_{L^2}. \end{aligned}$$

Using the Cauchy-Schwarz inequality with respect to the measure

$$\Delta(Y, Z)^{-N} \Delta(Y', Z)^{-N} |g_Y|^{\frac{1}{2}} |g_{Y'}|^{\frac{1}{2}} |g_Z|^{\frac{1}{2}} dY dY' dZ$$

we get that

$$\begin{aligned} \|u\|_{H(m)}^2 &\leq C \|(\theta_Y)\|_{k, Conf(g)}^2 \int_{(T^*\mathbb{R}^n)^3} |a(Y)|^2 m^2(Y) \|v_Y\|_{L^2}^2 \\ &\times \Delta(Y, Z)^{-N} \Delta(Z, Y')^{-N} |g_Y|^{\frac{1}{2}} |g_{Y'}|^{\frac{1}{2}} |g_Z|^{\frac{1}{2}} dY dY' dZ. \end{aligned}$$

Using Lemma 2.1 and integrating first in Y' , and in Z , we find that

$$\|u\|_{H(m)}^2 \leq C \|(\theta_Y)\|_{k, Conf(g)}^2 \int_{T^*\mathbb{R}^n} |a(Y)|^2 m^2(Y) \|v_Y\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY.$$

So the lemma is proved.

Proposition 2.2. *Let m_1 and m_2 be two weights for an Hörmander's metric g . If $[A, B]_\theta$ denotes the complex interpolation space between A and B associated to $0 < \theta < 1$, then*

$$[H(m_1), H(m_2)]_\theta = H(m_1^{1-\theta} m_2^\theta)$$

and the norms are equivalent with constants that only depend on the constants that appear in Definitions 2.1 and 2.2 for the metric g and the two weights m_1 and m_2 .

The proof is done in [8] page 89.

2.2. The link with classical Sobolev spaces. Here we mainly follow [11] (see also [25]). Let us consider a family of smooth vector fields $\mathcal{P} = (P_j)_{1 \leq j \leq N}$ on \mathbb{R}^n which are supposed to be bounded as well as all their derivatives. Let us recall Lemma 1.2.1 and corollary 1.2.2 of [11].

Lemma 2.6. *Let a be defined by*

$$a(X) = a(x, \xi) \stackrel{\text{def}}{=} \sum_{j=1}^N (P_j(x)|\xi)^2.$$

The function m and the metric g defined by

$$\begin{aligned} m(X) &= (\langle \xi \rangle + a(X))^{\frac{1}{2}} \quad \text{and} \\ g_X(dx, d\xi) &= m^{-2}(X)(\langle \xi \rangle^2 dx^2 + d\xi^2) \end{aligned}$$

have the following properties. The metric g is an Hörmander metric on $T^*\mathbb{R}^n$. The function m is g -weight on $T^*\mathbb{R}^n$. The function a belongs to $S(m, g)$ and for any k , the seminorms $\|a\|_{k, S(m^2, g)}$ depends only of the supremum of a finite number of derivative of the coefficients of the vector fields P_j .

Corollary 2.1. *The function m is a weight for the metric $g_{\frac{1}{2}, \frac{1}{2}}$ defined by*

$$g_{\frac{1}{2}, \frac{1}{2}}(dx^2, d\xi^2) = \langle \xi \rangle dx^2 + \frac{1}{\langle \xi \rangle} d\xi^2.$$

The main theorem in this section is the following.

Theorem 2.1. *Let \mathcal{P} be a family of vector fields. Then for any positive s , we have that the spaces $H^s(\mathcal{P})$ and $H(m^s)$ are equal. Moreover, a constant C exists such that*

$$C^{-1} \|u\|_{H(m^s)} \leq \|u\|_{H^s(\mathcal{P})} \leq C \|u\|_{H(m^s)}.$$

The above constant C depends only of the supremum of a finite number of derivative of the coefficients of the vector fields P_j .

Let us prove this theorem. By Definition 1.6 and Lemma 2.2, it is enough to prove this theorem for integer index. The fact that $H(m^k)$ is embedded in $H^k(\mathcal{P})$ follows immediately from Corollary 4.4 of [8] and from the fact that $P_{j_1} \cdots P_{j_\ell}$ with $\ell \leq k$ is an operator whose symbols belongs to $S(m^k, g)$.

Let us point out that in all results of [8], the constants of continuity and the constants of equivalence between norms depend only on the constants that appear in Definitions 2.1 and 2.2.

The proof of the opposite inequality is a little bit more delicate. The proof is similar to the proof of Theorem 2.1 of [11]. The idea is to decompose the phase space $T^*\mathbb{R}^n$ in regions where the vector fields are the "main terms". More precisely, for any positive real number A let us define

$$\mathcal{E} \stackrel{\text{def}}{=} \left\{ Y / \sum_{j=1}^N (P_j(y)|\eta)^2 \geq A^2 \langle \eta \rangle \right\} \quad \text{and} \quad \mathcal{E}_j \stackrel{\text{def}}{=} \mathcal{E} \cap \left\{ Y / (P_j(y)|\eta)^2 \geq \frac{1}{2N} \sum_{j=1}^N (P_j(y)|\eta)^2 \right\}.$$

The region \mathcal{E}_j can be understood as the region where the vector field P_j is elliptic. Moreover, in the union of the \mathcal{E}_j , which of course contains \mathcal{E} , the function λ_g is greater than $(2N)^{-1}A$.

Now, for any compactly supported smooth function on \mathbb{R}^n we have, for any family (φ_Y) which satisfies Lemma 2.4 with the metric g defined in Lemma 2.6,

$$\begin{aligned} \|u\|_{H(m^k)}^2 &= \int_{T^*\mathbb{R}^n} m^{2k}(Y) \|\varphi_Y^w u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY \\ &\leq \sum_{j=0}^N I_{j,k}(u) \quad \text{with} \\ I_{0,k}(u) &\stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^n \setminus \mathcal{E}} m^{2k}(Y) \|\varphi_Y^w u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY \quad \text{and} \\ I_{j,k}(u) &\stackrel{\text{def}}{=} \int_{\mathcal{E}_j} m^{2k}(Y) \|\varphi_Y^w u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY. \end{aligned}$$

As in $T^*\mathbb{R}^n \setminus \mathcal{E}$, we have

$$m(Y) \leq (A+1)\langle \eta \rangle^{\frac{1}{2}},$$

we can write that

$$I_{0,k}(u) \leq (A+1)^{2k} \int_{Y \in T^*\mathbb{R}^n} \langle \eta \rangle^k \|\varphi_Y^w u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY.$$

Theorem 6.9 of [8] implies that

$$I_{0,k}(u) \leq C(A+1)^{2k} \|u\|_{H^{\frac{k}{2}}(\mathbb{R}^n)}^2.$$

So we have

$$\|u\|_{H(m^k)}^2 \leq C(A+1)^{2k} \|u\|_{H^{\frac{k}{2}}(\mathbb{R}^n)}^2 + \sum_{j=1}^N I_{j,k}(u). \quad (2.1)$$

Let us admit for a while the following lemma, very close to Lemma 2.2 of [11].

Lemma 2.7. *For any j in $\{1, \dots, N\}$, two uniformly confined families $(\delta_{j,Y})$ and $(R_{j,Y})$ exist such that for any $Y \in \mathcal{E}_j$,*

$$\varphi_Y^w = m^{-k}(Y) \delta_{j,Y}^w P_j^k + \lambda_g^{-1}(Y) R_{j,Y}^w.$$

Using the fact that on the set \mathcal{E}_j the function λ_g is greater than $(2N)^{-1}A$, we get from this lemma that

$$\begin{aligned} I_{j,k}(u) &\leq \int_{T^*\mathbb{R}^n} \|\delta_{j,Y}^w P_j^k u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY + \int_{\mathcal{E}_j} m^{2k}(Y) \lambda_g(Y)^{-2} \|R_{j,Y}^w u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY \\ &\leq \int_{T^*\mathbb{R}^n} \|\delta_{j,Y}^w P_j^k u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY + \left(\frac{2N}{A}\right)^2 \int_{\mathcal{E}_j} m^{2k}(Y) \|R_{j,Y}^w u\|_{L^2}^2 |g_Y|^{\frac{1}{2}} dY. \end{aligned}$$

As the families $(\delta_{j,Y}^w)$ and $(R_{j,Y}^w)$ are uniformly confined, we have, applying Lemma 2.3. of [11],

$$I_{j,k}(u) \leq C \|P_j^k u\|_{L^2}^2 + CA^{-2} \|u\|_{H(m^k)}^2.$$

So if we choose A large enough, Theorem 2.1 is proved.

Now let us prove Lemma 2.7. For any $Y \in \mathcal{E}_j$, let us write

$$\varphi_Y(X) = \frac{\varphi_Y(X)}{(P_j(x)|\xi)^k} (P_j(x)|\xi)^k.$$

The support of φ_Y is included in a g_Y -ball of center Y and radius r . So, using Taylor inequality we get for any $X = (x, \xi)$ in the support of φ_Y that

$$|(P_j(x)|\xi) - (P_j(y)|\eta)| \leq Cr\langle\eta\rangle^{\frac{1}{2}}.$$

As Y is supposed to be in \mathcal{E}_j , we deduce that

$$\left| \frac{(P_j(x)|\xi)}{(P_j(y)|\eta)} - 1 \right| \leq \frac{Cr}{A}.$$

So, if A is large enough, we have that, for any X belonging to the support of φ_Y ,

$$\left| \frac{(P_j(x)|\xi)}{(P_j(y)|\eta)} \right|^{\pm 1} \leq 1 + r. \quad (2.2)$$

Let us define

$$\delta_{j,Y} \stackrel{\text{def}}{=} \begin{cases} \frac{m^k(Y)\varphi_Y(X)}{(P_j(x)|\xi)^k} & \text{if } Y \in \mathcal{E}_j \\ 0 & \text{if } Y \notin \mathcal{E}_j. \end{cases}$$

As $Y \in \mathcal{E}_j$ we have thanks to Inequality (2.2) and Lemma 2.3 that the family $(\delta_{j,Y})$ is uniformly confined. So we get that

$$\begin{aligned} \varphi_Y^w &= m^{-k}(Y)\delta_{j,Y}^w \circ P_j^k + \lambda_g^{-1}(Y)R_{j,Y}^w \quad \text{with} \\ R_{j,Y}^w &\stackrel{\text{def}}{=} \lambda_g(Y)m^{-k}(Y)\left(\left(\delta_{j,Y}(P_j(x)|\xi)^k\right)^w - \delta_{j,Y}^w \circ P_j^k\right). \end{aligned}$$

Lemma 2.3 ensures that the family $(R_{j,Y}^w)$ is uniformly confined. So Lemma 2.7 is proved.

3. THE NON CHARACTERISTIC CASE WHEN s IS AN INTEGER

3.1. Some geometrical properties. In the non characteristic case, we consider any system \mathcal{P} . The intersection $\mathcal{P} \cap T\Sigma$ can be described in a very simple way.

Proposition 3.1. *Let Σ be a non characteristic hypersurface for a C^∞ -module \mathcal{P} of vector fields of finite rank (as a C^∞ -module). Let us consider a vector field P in \mathcal{P} transverse to Σ . Then for any point M of Σ , the space $T_M\Sigma \cap \mathcal{P}|_M$ is the projection of $\mathcal{P}|_M$ on $T_M\Sigma$ in the direction of P .*

If a vector field Q belongs to \mathcal{P} , then

$$Q = \sum_{j=1}^N \alpha_j P_j + \alpha_0 P$$

where the α_j are smooth functions and the P_j are vector fields of \mathcal{P} . Let us assume that Q is also in $T\Sigma$. We are proving an obviously local property. So let us consider a local defining function g of Σ . Then, we have

$$\sum_{j=1}^N \alpha_j P_j \cdot g + \alpha_0 P \cdot g = 0.$$

As P is transverse to Σ , $P \cdot g$ does not vanish. So, we have that

$$\alpha_0 = -\frac{1}{P \cdot g} \sum_{j=1}^N \alpha_j P_j \cdot g.$$

So it turns out that

$$Q = \sum_{j=1}^N \alpha_j \left(P_j - \frac{P_j \cdot g}{P \cdot g} P \right).$$

So the proposition is proved.

In the particular case of the system related to Heisenberg group, we have the following property, used in the works of I. Pesenson (see [22]) and S. Berhanu and I. Pesenson (see [7]).

Lemma 3.1. *If \mathcal{Z} is the module of the vector fields associated to the Heisenberg group \mathbb{H}^d with $d \geq 2$, then the system $\mathcal{Z} \cap T\Sigma$ satisfies also the Hörmander condition at order 2.*

To prove this, let us first observe that the family $(Z_\ell)_{1 \leq \ell \leq 2d}$ is so that, for any $j \leq d$

$$\text{rank} \{(Z_\ell)_{1 \leq \ell \leq 2d}, [Z_j, Z_{j+d}]\} = 2d + 1 \quad \text{and} \quad \forall \ell \neq j + d, [Z_j, Z_\ell] = 0.$$

This property is invariant by diffeomorphism. Let us assume that Σ is the hyperplane Σ_0 whose equation is $x_1 = 0$ and that $Z_1 = \partial_{x_1}$. Without any lose of generality, we can now assume that $j = 2$. Let us denote by Π the projection over $T\Sigma_0$ with respect to ∂_{x_1} . Then the rank of

$$\{(\Pi(Z_\ell))_{2 \leq \ell \leq 2d}, \Pi([Z_2, Z_{2+d}])\}$$

is $2d$. The only thing we have to prove now is that

$$\Pi([Z_2, Z_{2+d}]) = [\Pi(Z_2), \Pi(Z_{2+d})].$$

We have

$$Z_j = \sum_{\ell=2}^{2d+1} z_j^\ell(x_1, x') \partial_{x_\ell} + z_j^1(x_1, x') \partial_{x_1}.$$

The fact that $[Z_1, Z_2] = [Z_1, Z_{2+d}] = 0$ implies that for any $\ell \geq 2$, we have

$$z_2^\ell(x_1, x') = z_2^\ell(x') \quad \text{and} \quad z_{2+d}^\ell(x_1, x') = z_{2+d}^\ell(x').$$

So, a straightforward computation shows that

$$[Z_2, Z_{2+d}] = [\Pi(Z_2), \Pi(Z_{2+d})] + \beta(x_1, x') \partial_{x_1}$$

for some function β . So the lemma is proved.

3.2. Weight associated to a system of vector fields. The problem of restriction and trace lifting is obviously a local problem. So, near a non characteristic point of an hypersurface, we may suppose that the hypersurface is $x_1 = 0$ and that the family \mathcal{P} contains the vector field ∂_{x_1} . For technical reasons that will appear clearly in the next section, we shall have to consider a family of systems of vector fields.

In all this paragraph, we shall denote by $\mathcal{P}_\theta \stackrel{\text{def}}{=} (P_{j,\theta})_{1 \leq j \leq N, \theta \in \Theta}$ a family of vector fields whose coefficients depend on a parameter $\theta \in \Theta$ and such that

$$P_{1,\theta} = \partial_{x_1} \quad \text{and} \quad P_{j,\theta} = \sum_{\ell=2}^n P_{j,\theta}^\ell(x_1, x') \partial_{x_\ell} \quad \text{for} \quad j \in \{2, \dots, N\} \quad (3.1)$$

Moreover we assume that the coefficients and all their derivatives are bounded uniformly with respect to the parameter θ . In all this paragraph, we shall denote by C a generic constant which depends only on a finite number of the C_k defined by

$$C_k = \sup_{\substack{|\alpha| \leq k \\ \theta \in \Theta}} \sup_{\substack{1 \leq j \leq N \\ 1 \leq \ell \leq n}} \|\partial^\alpha P_{j,\theta}^\ell\|_{L^\infty}.$$

and $\Sigma_t = \{(x_1, x') \in \mathbb{R}^n ; x_1 = t\}$. So, it is obvious that $P_{\Sigma_t, \theta} = (P_{2,\theta}(t, \cdot), \dots, P_{N,\theta}(t, \cdot))$.

Lemma 3.2. Let $(a_\theta)_{\theta \in \Theta}$ be the family of functions defined by

$$a_\theta(t, X') = a_\theta(t, x', \xi') \stackrel{\text{def}}{=} \sum_{j=2}^N (P_{j,\theta}(t, x')|\xi')^2.$$

The functions m_θ and the metrics $g_{\theta,t}$ defined by

$$\begin{aligned} m_\theta(t, X') &= (a_\theta(t, X') + \langle \xi' \rangle)^{\frac{1}{2}} \quad \text{and} \\ g_{\theta,t,X'}(dx', d\xi') &= m_\theta^{-2}(t, X')(\langle \xi' \rangle^2 dx'^2 + d\xi'^2). \end{aligned}$$

have the following properties.

- The metrics $g_{\theta,t}$ are Hörmander metrics on $T^*\mathbb{R}^{n-1}$ uniformly with respect to the parameter (θ, t) in $\Theta \times \mathbb{R}$.
- The functions $m_\theta(t, \cdot)$ are $g_{\theta,t}$ -weights on $T^*\mathbb{R}^{n-1}$ uniformly with respect to the parameter (θ, t) in $\Theta \times \mathbb{R}$.
- The functions $a_\theta(t, \cdot)$ belongs to $S(m_\theta^2(t, \cdot), g_{\theta,t})$ with, for any k ,

$$\sup_{\substack{\theta \in \Theta \\ t \in \mathbb{R}}} \|a_\theta(t, \cdot)\|_{k, S(m_\theta^2(t, \cdot), g_{\theta,t})} < \infty.$$

- The functions $(P_{j,\theta}(t, x')|\xi')$ belongs to $S(m_\theta(t, \cdot), g_{\theta,t})$ with, for any k ,

$$\sup_{\substack{\theta \in \Theta \\ t \in \mathbb{R}}} \|(P_{j,\theta}(t, x')|\xi')\|_{k, S(m_\theta(t, \cdot), g_{\theta,t})} < \infty.$$

Here, uniformly with respect to the parameter (θ, t) in $\Theta \times \mathbb{R}$ means that the constants that appear in Definitions 2.1 and 2.2 do not depend on (θ, t) .

This is simply Lemma 1.2.1 of [11]. As in [11], we have the following corollary.

Corollary 3.1. The functions $m_\theta(t, \cdot)$ are g weights uniformly with respect to (θ, t) in $\Theta \times \mathbb{R}$ where g is the metric on $T^*\mathbb{R}^{n-1}$ defined by

$$g_{(x', \xi')}(dx'^2, d\xi'^2) = \langle \xi' \rangle dx'^2 + \frac{1}{\langle \xi' \rangle} d\xi'^2.$$

Again as in [11], we have the following proposition.

Proposition 3.2. For any positive s , the space $H^s(\mathcal{P}_{\Sigma_t, \theta})$ is equal to the space $H(m_\theta^s(t, \cdot))$ uniformly in the following sense; a constant C exists such that for any (θ, t) in $\Theta \times \mathbb{R}$,

$$C^{-1} \|u\|_{H(m_\theta^s(t, \cdot))} \leq \|u\|_{H^s(\mathcal{P}_{\Sigma_t, \theta})} \leq C \|u\|_{H(m_\theta^s(t, \cdot))}.$$

Now we can prove the following lemma.

Lemma 3.3. For any integer k , a constant C exists such that for any u in $H^k(\mathcal{P}_\theta)$ and for any θ in Θ we have

$$\frac{1}{C} \|u\|_{H^k(\mathcal{P}_\theta)}^2 \leq \sum_{j=0}^k \int_{\mathbb{R}} \|\partial_t^j u(t, \cdot)\|_{H(m_\theta^{k-j}(t, \cdot))}^2 dt \leq C \|u\|_{H^k(\mathcal{P}_\theta)}^2.$$

Using Proposition 3.2, we immediately get that

$$\sum_{j=0}^k \int_{\mathbb{R}} \|\partial_t^j u(t, \cdot)\|_{H(m_\theta^{k-j}(t, \cdot))}^2 dt \leq C \|u\|_{H^k(\mathcal{P}_\theta)}^2.$$

In an other way, $P_{\Sigma_t, \theta, j}$ are tangential vector fields. So, for $j_\ell \in \{2, \dots, N\}, \ell = 1, \dots, k$, we have

$$\|[\partial_t, [P_{\Sigma_t, \theta, j_1}, \dots [P_{\Sigma_t, \theta, j_{k-1}}, P_{\Sigma_t, \theta, j_k}] \dots]]u(t, \cdot)\|_{L^2(\mathbb{R}^{n-1})} \leq C \|u(t, \cdot)\|_{H^1(\mathbb{R}^{n-1})}.$$

As $m_\theta^2(t, x', \xi') \geq \langle \xi' \rangle$, we have

$$\|[\partial_t, [P_{\Sigma_t, \theta, j_1}, \dots [P_{\Sigma_t, \theta, j_{k-1}}, P_{\Sigma_t, \theta, j_k}] \dots]]u(t, \cdot)\|_{L^2(\mathbb{R}^{n-1})} \leq C \|u(t, \cdot)\|_{H(m_\theta^2(t, \cdot))}.$$

So we have

$$\|P_{\Sigma_t, \theta, j_1}^{J_1} \partial_t^{k_1} \dots P_{\Sigma_t, \theta, j_\ell}^{J_\ell} \partial_t^{k_\ell} u(t, \cdot)\|_{L^2(\mathbb{R}^{n-1})} \leq C \sum_{i=1}^k \|\partial_t^i u(t, \cdot)\|_{H(m_\theta^{k+j-i}(t, \cdot))},$$

where $k = k_1 + \dots + k_\ell, j = |J_1| + \dots + |J_\ell|$. Then the lemma is proved.

3.3. An "elementary" proof of Theorem 1.2 for integer index. Basically, we are going to follow the lines of the proof given in the introduction just after Theorem 1.2. As in the next section we need a uniform version of the restriction theorem, we are going to do the proof with a parameter θ .

The continuity of the restriction operator is an immediate consequence of the following proposition.

Proposition 3.3. *A constant C exists such that for any θ in Θ and any u in $H^k(\mathcal{P}_\theta)$, we have*

$$\sup_{t \in \mathbb{R}} \|u(t, \cdot)\|_{H(m_\theta^{k-\frac{1}{2}}(t, \cdot))} \leq C \|u\|_{H^k(\mathcal{P}_\theta)}.$$

Let us consider a g -partition of unity $(\theta_Y)_{Y \in T^*\mathbb{R}^{n-1}}$ and let us assume that u is in $\mathcal{D}(\mathbb{R}^n)$. Let us state

$$I_Y(u)(t) \stackrel{\text{def}}{=} m_\theta^{2k-1}(t, Y) \|\theta_Y^w u(t, \cdot)\|_{L^2(\mathbb{R}^{n-1})}^2.$$

As in the introduction, we write that this function is the integral of its derivative. So, it turns out that

$$\begin{aligned} I_Y(u)(t) &= (2k-1) \int_{-\infty}^t \partial_t m_\theta(t', Y) m_\theta(t', Y)^{2k-2} \|\theta_Y^w u(t', \cdot)\|_{L^2(\mathbb{R}^{n-1})}^2 dt' \\ &\quad + 2 \int_{-\infty}^t m_\theta(t', Y)^{2k-1} (\theta_Y^w \partial_t u(t', \cdot) | \theta_Y^w u(t', \cdot))_{L^2(\mathbb{R}^{n-1})}^2 dt'. \end{aligned}$$

We have that

$$\partial_t m_\theta(t, X') = \frac{\partial_t a_\theta(t, X')}{2m_\theta(t, X')}.$$

As the function a_θ is non negative, we have

$$(\partial_t a_\theta(t, X'))^2 \leq 2a_\theta(t, X') \sup_{t \in \mathbb{R}} |\partial_t^2 a_\theta(t, X')|.$$

As a constant C (which does not depend on (θ, t)) exists such that

$$\sup_{t \in \mathbb{R}} |\partial_t^2 a_\theta(t, X')| \leq C \langle \xi' \rangle^2,$$

we deduce from the fact that $a_\theta(t, X') + \langle \xi' \rangle = m_\theta^2(t, X')$ that

$$|\partial_t a_\theta(t, Y)| \leq C m_\theta(t, Y) \langle \xi' \rangle \leq C m_\theta^3(t, Y). \quad (3.2)$$

So we infer that $|\partial_t m_\theta(t, X')| \leq C m_\theta^2(t, X')$ So, we have that

$$I_Y(u)(t) \leq C \int_{-\infty}^t m_\theta(t', Y)^{2k} \|\theta_Y^w u(t', \cdot)\|_{L^2(\mathbb{R}^{n-1})}^2 dt' \\ + 2 \int_{-\infty}^t m_\theta(t', Y)^{2k-1} (\theta_Y^w \partial_{t'} u(t', \cdot) | \theta_Y^w u(t', \cdot))_{L^2(\mathbb{R}^{n-1})}^2 dt'.$$

Cauchy-Schwarz Inequality implies that

$$I_Y(u)(t) \leq C \int_{-\infty}^t m_\theta(t', Y)^{2k} \|\theta_Y^w u(t', \cdot)\|_{L^2(\mathbb{R}^{n-1})}^2 dt' \\ + C \int_{-\infty}^t m_\theta(t', Y)^{2k-2} \|\theta_Y^w \partial_{t'} u(t', \cdot)\|_{L^2(\mathbb{R}^{n-1})}^2 dt'.$$

So it turns out that

$$\sup_{t \in \mathbb{R}} m_\theta(t, Y)^{2k-1} \|\theta_Y^w u(t, \cdot)\|_{L^2(\mathbb{R}^{n-1})}^2 \in L^1(dY).$$

Moreover for any $Y \in T^*\mathbb{R}^{n-1}$, the function $t \rightarrow m_\theta^{k-\frac{1}{2}}(t, Y) \theta_Y^w u(t, \cdot)$ is continuous. The application of Lebesgue's convergence theorem concludes the proof.

But applying this method to ∂_t^j for $j \leq k-1$, we have in fact proved the first part of the following theorem.

Theorem 3.1. *The map Γ defined by*

$$\Gamma \begin{cases} H^k(\mathcal{P}_\theta) & \rightarrow \bigoplus_{j=0}^{k-1} H(m_\theta^{k-j-1/2}(0, \cdot)) \\ u & \mapsto (\gamma_j(u))_{0 \leq j \leq k-1} \quad \text{with} \quad \gamma_j(u) \stackrel{\text{def}}{=} \gamma_\Sigma(\partial_{x_1}^j u). \end{cases}$$

is continuous and onto.

As the trace problem is a local one, using Proposition 3.2 this theorem implies Theorem 1.2 in the case of integer index.

3.4. Trace lifting theorem. Let us prove in this paragraph the following theorem which obviously implies the second part of Theorem 3.1.

Theorem 3.2. *A map \mathcal{R} exists such that*

$$\mathcal{R} : \bigoplus_{j=0}^{k-1} H(m_\theta^{k-j-1/2}(0, \cdot)) \rightarrow H^k(\mathcal{P}_\theta)$$

is continuous and such that $\Gamma \circ \mathcal{R} = \text{Id}$.

Let us consider the families (ψ_Y^w) and (ϕ_Y^w) given by Lemma 2.4 and a function $\chi \in \mathcal{D}(\mathbb{R})$ with value 1 in the neighborhood of 0. If (v_0, \dots, v_{k-1}) belongs to $\bigoplus_{j=0}^{k-1} H(m_\theta^{k-j-1/2}(0, \cdot))$, let us state

$$u = \sum_{j=0}^{k-1} \mathcal{R}_j(v_j) \quad \text{with} \quad \mathcal{R}_j(v)(x_1, x') \stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^{n-1}} \frac{x_1^j}{j!} \chi(x_1 m_\theta(x_1, Y)) (\psi_Y^w \phi_Y^w)(v)(x') dY.$$

It is obvious that $\gamma_j(u) = v_j$. The only thing we have to prove is that \mathcal{R}_j is continuous from $H(m_\theta^{k-j-\frac{1}{2}}(0, \cdot))$ into $H^k(\mathcal{P}_\theta)$. First of all let us write that

$$\mathcal{R}_j(v)(x_1, x') = \int_{T^*\mathbb{R}^{n-1}} m_\theta(x_1, Y)^{-j} \chi_{Y,j}(x_1) \psi_Y^w \phi_Y^w(v)(x') dY$$

with

$$\chi_{Y,j}(x_1) \stackrel{\text{def}}{=} \frac{(x_1 m_\theta(x_1, Y))^j}{j!} \chi(x_1 m_\theta(x_1, Y)).$$

By definition of the $H(m_\theta^k(x_1, \cdot))$ norm, we have

$$\begin{aligned} \rho_{j,k} &\stackrel{\text{def}}{=} \int_{\mathbb{R}} dt \|\mathcal{R}_j(v)(t, \cdot)\|_{H(m_\theta^k(t, \cdot))}^2 \\ &= \int_{\mathbb{R}} dt \int_{T^*\mathbb{R}^{n-1}} m_\theta^{2k}(t, Y) \|\phi_Y^w \mathcal{R}_j(v)(t, \cdot)\|_{L^2(\mathbb{R}^{n-1})}^2 dY. \end{aligned}$$

The Taylor inequality implies that

$$|m_\theta(t, Y) - m_\theta(0, Y)| \leq |t| \sup_t |\partial_t m_\theta(t, Y)|.$$

By definition of m_θ and using Inequality (3.2), we have that

$$|m_\theta(t, Y) - m_\theta(0, Y)| \leq C|t|\langle \eta \rangle.$$

Because the function $t \mapsto m_\theta(t, Y)$ is greater than $\langle \eta \rangle^{\frac{1}{2}}$, this implies that

$$\left(\frac{m_\theta(t, Y)}{m_\theta(0, Y)} \right)^\pm \leq C(1 + \langle \eta \rangle t^2)^{\frac{1}{2}}, \quad (3.3)$$

and by definition of \mathcal{R}_j and by Lemma 2.5, we can write

$$\begin{aligned} \rho_{j,k} &\leq \int_{\mathbb{R}} dt \int_{T^*\mathbb{R}^{n-1}} m_\theta^{2k-2j}(t, Y) \chi_{Y,j}^2(t) \|\phi_Y^w(v)\|_{L^2(\mathbb{R}^{n-1})}^2 dY \\ &\leq C \int_{\mathbb{R}} dt \int_{T^*\mathbb{R}^{n-1}} m_\theta^{2k-2j}(0, Y) (1 + \langle \eta \rangle t^2)^{2N_0} \chi_{Y,j}^2(t) \|\phi_Y^w(v)\|_{L^2(\mathbb{R}^{n-1})}^2 dY. \end{aligned}$$

The support of the function $\chi_{Y,j}$ is included in the set of t such that $|t| \leq C m_\theta(t, Y)^{-1}$. But as $m_\theta(t, Y)$ is greater than $\langle \eta \rangle^{\frac{1}{2}}$, the support of $\chi_{Y,j}$ is included in $[-C\langle \eta \rangle^{-\frac{1}{2}}, C\langle \eta \rangle^{-\frac{1}{2}}]$. So the function $(1 + \langle \eta \rangle t^2)^{2N_0} \chi_{Y,j}^2(t)$ is bounded. Using again the inequality (3.3), it turns out that, on the support of $\chi_{Y,j}$, we have

$$\left(\frac{m_\theta(t, Y)}{m_\theta(0, Y)} \right)^\pm \leq C.$$

So the support of $\chi_{Y,j}$ is included in the ball of center 0 and radius $C m_\theta^{-1}(0, Y)$. As the function $\chi_{Y,j}$ is bounded, we have that

$$\int_{\mathbb{R}} (1 + \langle \eta \rangle t^2)^{N_0} \chi_{Y,j}^2(t) dt \leq C m_\theta^{-1}(0, Y).$$

So, by definition of the $H(m_\theta^{k-j-1/2}(0, \cdot))$ norm, we get

$$\begin{aligned} \rho_{j,k} &\leq C \int_{T^*\mathbb{R}^{n-1}} m_\theta^{2k-2j}(0, Y) m_\theta^{-1}(0, Y) \|\phi_Y^w(v)\|_{L^2(\mathbb{R}^{n-1})}^2 dY \\ &= \|v\|_{H(m_\theta^{k-j-1/2}(0, \cdot))}^2. \end{aligned}$$

Following exactly the same lines, we have, for $1 \leq i \leq j$,

$$\int_{\mathbb{R}} \|\partial_t^i \mathcal{R}_j(v)(t, \cdot)\|_{H(m_\theta^{k-j-i}(t, \cdot))}^2 dt \leq C \|v\|_{H(m_\theta^{k-j-1/2}(0, \cdot))}^2.$$

So the operator \mathcal{R} is continuous thanks to Lemma 3.3, the theorem is proved.

4. THE CHARACTERISTIC CASE IN THE CURVED SITUATION

In this section, we go back to the case when the vector fields are those associated to the Heisenberg group.

4.1. Some geometric properties of non degenerate characteristic points. The aim of this paragraph is the proof of some propositions that will ensure the geometric nature and so the invariance through the action of diffeomorphism of the objects we are going to work with.

Proposition 4.1. *Let Σ be an hypersurface of \mathbb{H}^d and g one of its local defining function. A characteristic point of Σ is non degenerate if and only if the matrix*

$$(Z_i \cdot Z_j \cdot g(M_0))_{1 \leq i, j \leq 2d}$$

is invertible.

Let g be a local defining function of Σ . Of course, dg vanishes on $T\Sigma$. As $Z_i(M_0)$ belongs to $T_{M_0}\Sigma$, we have $\mathcal{L}_{Z_i}(dg)(M_0) = d(Z_i g)(M_0)$. But, the system $\mathcal{Z}|_{M_0}$ spans $T_{M_0}\Sigma$. So the fact that the system $(\mathcal{L}_{Z_i}(dg)(M_0))_{1 \leq i \leq 2d}$ spans $T_{M_0}^*\Sigma$ is equivalent to the fact that the matrix $\langle d(Z_i \cdot g), Z_j \rangle = Z_i \cdot Z_j \cdot g$ is invertible at point M_0 . Conversely, let θ be a 1-form that vanishes on $T\Sigma$ and such that $\theta(M_0) \neq 0$. A function a that does not vanish at M_0 exists such that $\theta = adg$. Thanks to Leibnitz formula that $\mathcal{L}_{Z_j}(\theta)(M_0)|_{T_{M_0}\Sigma} = a(M_0)d(Z_j g)(M_0)|_{T_{M_0}\Sigma}$. The fact that the function a does not vanish at point M_0 implies the proposition.

Remark Thanks to implicit function theorem, we can choose as defining function for the hypersurface Σ a function of the type $s - h(x, y) = 0$. The non degeneracy condition turns out to be

$$\det(J + D^2 h(x_0, y_0)) \neq 0$$

where J denote the matrix of the standard symplectic form on \mathbb{R}^{2d} .

The following fact is simple but very important.

Proposition 4.2. *Let us consider the function G defined by*

$$G \begin{cases} \Sigma & \rightarrow \mathbb{R}^{2d} \\ M & \mapsto (Z_j \cdot g(M))_{1 \leq j \leq 2d}. \end{cases} \quad (4.1)$$

The function G is a diffeomorphism near M_0 .

The fact that the point M_0 is characteristic means that $G(M_0) = 0$. As the system $\mathcal{Z}|_{M_0}$ spans the tangent space at point M_0 the invertibility of the matrix $(Z_i Z_j g(M_0))_{1 \leq i, j \leq 2d}$ means that $DG(M_0)$ is invertible. So the local inverse theorem implies the result.

From this proposition, we can immediately deduce the following corollary:

Corollary 4.1. *Let Σ be an hypersurface of \mathbb{H}^d .*

- *A non degenerate characteristic point M_0 of Σ is isolated.*
- *Let g be an defining function of Σ near the point M_0 . A constant c exists such that for any $M \in \Sigma$,*

$$\sum_{j=1}^{2d} |Z_j \cdot g(M)|^2 \geq c|M - M_0|^2.$$

The following lemma describes the structure of the family $\mathcal{Z}_{\Sigma, \mathcal{M}}$ introduced in Definition 1.4 in the introduction.

Lemma 4.1. *Let Σ be a smooth hypersurface of \mathbb{H}^d with a finite number of non degenerate characteristic points $\mathcal{M} \stackrel{\text{def}}{=} (M_j)_{1 \leq j \leq N}$. A family $\mathcal{R}_\Sigma = (R_\ell)_{1 \leq \ell \leq N}$ of vectors fields of $\mathcal{Z} \cap T\Sigma$ that vanish on \mathcal{M} exists such that \mathcal{R}_Σ spans $\mathcal{Z}_{\Sigma, \mathcal{M}}$ as a $C_{\mathcal{M}}^\infty$ -module, which means that, for any vector fields Z of $\mathcal{Z}_{\Sigma, \mathcal{M}}$, a family of functions $(a_j)_{1 \leq j \leq N}$ of $C_{\mathcal{M}}^\infty$ exists such that*

$$Z(M) = \sum_{\ell=1}^N a_\ell(M) R_\ell(M).$$

To prove this, it is enough by Definition 1.4 of $\mathcal{Z}_{\Sigma, \mathcal{M}}$ to prove the property near a point M_0 . Let us consider, for some small ϵ , the two families $(\underline{\chi}_j)_{1 \leq j \leq 2d}$ and $(\tilde{\chi}_j)_{1 \leq j \leq 2d}$ of smooth functions on $\mathbb{R}^{2d} \setminus \{0\}$ homogeneous of degree 0 with value in $[0, 1]$ such that

$$\text{Supp } \underline{\chi}_j \subset \{\zeta \in \mathbb{R}^{2d}, |\zeta_j| \geq \epsilon|\zeta|\}, \quad \text{Supp } \tilde{\chi}_j \subset \{\zeta \in \mathbb{R}^{2d}, |\zeta_j| \geq \epsilon/2|\zeta|\}$$

and

$$\tilde{\chi}_j \equiv 1 \text{ near } \text{Supp } \underline{\chi}_j \quad \text{and} \quad \sum_{j=1}^{2d} \underline{\chi}_j(\zeta) = 1. \quad (4.2)$$

Now let us consider a local defining function g for the hypersurface Σ and let us consider the function G defined in (4.1). Let us state

$$\chi_j(M) \stackrel{\text{def}}{=} \underline{\chi}_j(G(M)). \quad (4.3)$$

Using the fact that G is a local diffeomorphism from Σ to \mathbb{R}^{2d} , we have, for any point M in the support of χ_j ,

$$|Z_j \cdot g(M)|^2 \geq c|M - M_0|^2. \quad (4.4)$$

Let us consider X in $\mathcal{Z}_{\Sigma, M_0}$. By definition, X vanishes at M_0 . This implies that

$$X = \sum_{\ell=1}^{2d} \alpha_\ell Z_\ell \quad \text{with} \quad \alpha_\ell(M_0) = 0.$$

The fact that X is tangent to Σ implies that

$$\sum_{k=1}^{2d} \alpha_k Z_k \cdot g = 0.$$

On the support of χ_j , we have

$$\alpha_j = - \sum_{k \neq j} \alpha_k \frac{Z_k \cdot g}{Z_j \cdot g}.$$

From this, we deduce that

$$\begin{aligned} \chi_j X &= \sum_{k \neq j} \chi_j \alpha_k \left(Z_k - \frac{Z_k \cdot g}{Z_j \cdot g} Z_j \right) \\ &= \sum_{k \neq j} \frac{\chi_j \alpha_k}{Z_j \cdot g} ((Z_j \cdot g) Z_k - (Z_k \cdot g) Z_j). \end{aligned}$$

Inequality (4.4) and the fact that $\alpha_k(M_0) = 0$ ensure that

$$\alpha_{j,k} \stackrel{\text{def}}{=} \frac{\chi_j \alpha_k}{Z_j \cdot g} \in C_{M_0}^\infty.$$

So we have

$$X = \sum_{1 \leq j < k \leq 2d} \alpha_{j,k} ((Z_j \cdot g) Z_k - (Z_k \cdot g) Z_j)$$

and the lemma is proved.

4.2. Blow up procedure in the curved situation. First, let us take $s - h(x, y) = 0$ as a defining function of the hypersurface Σ near a non degenerate characteristic point M_0 of Σ . Let us start by straightening the hypersurface. Then

- the hypersurface Σ is $\Sigma_0 = \{(x, y, s) / s = 0\}$ and the point M_0 is the origin;
- the vector fields X_j and Y_j are

$$\begin{aligned}\tilde{X}_j &= \frac{\partial}{\partial x_j} + a_j(x, y) \frac{\partial}{\partial s} \quad \text{with} \quad a_j(x, y) = y_j + y_j^0 - \frac{\partial h}{\partial x_j}(x + x_0, y + y_0) \quad \text{and} \\ \tilde{Y}_j &= \frac{\partial}{\partial y_j} + a_{j+d}(x, y) \frac{\partial}{\partial s} \quad \text{with} \quad a_{j+d}(x, y) = -x_j - x_j^0 - \frac{\partial h}{\partial y_j}(x + x_0, y + y_0).\end{aligned}$$

For sake of simplicity of the notations we still state $\tilde{Z}_j = \tilde{X}_j$ and $\tilde{Z}_{j+d} = \tilde{Y}_j$. The family $\tilde{\mathcal{R}}_\Sigma$ can be choose as

$$\tilde{R}_{j,k} \stackrel{\text{def}}{=} a_k(x, y) \tilde{Z}_j - a_j(x, y) \tilde{Z}_k \quad \text{for} \quad 1 \leq j < k \leq 2d.$$

The inequality of Corollary 4.1 can be rewritten as

$$\sum_{j=1}^{2d} a_j^2(x, y) \geq c(|x|^2 + |y|^2). \quad (4.5)$$

Now, let us follow as closely as possible the lines of the proof presented in the introduction. As in the introduction, using dilatations adapted to the Heisenberg group, let us state

$$u = \sum_p \varphi_p u \quad \text{and} \quad u_p(\underline{x}, \underline{y}, \underline{s}) = \varphi_0(\underline{x}, \underline{y}, \underline{s}) u(2^{-p}\underline{x}, 2^{-p}\underline{y}, 2^{-2p}\underline{s})$$

We have that

$$\begin{aligned}\tilde{X}_j(\varphi_p u)(x, y, s) &= 2^p (X_{j,p} u_p)(2^p x, 2^p y, 2^{2p} s) \quad \text{and} \\ \tilde{Y}_j(\varphi_p u)(x, y, s) &= 2^p (Y_{j,p} u_p)(2^p x, 2^p y, 2^{2p} s).\end{aligned}$$

In the model case studied in the introduction, we had $X_{j,p} = \tilde{X}_j = X_j$ and $Y_{j,p} = \tilde{Y}_j = Y_j$. Here, we have

$$\begin{aligned}X_{j,p}(x, y) &= \frac{\partial}{\partial x_j} + a_{j,p}(x, y) \frac{\partial}{\partial s} \quad \text{and} \\ Y_{j,p}(x, y) &= \frac{\partial}{\partial y_j} + a_{j+d,p}(x, y) \frac{\partial}{\partial s} \quad \text{with} \\ a_{j,p}(x, y) &= y_j + 2^p \left(y_j^0 - \frac{\partial h}{\partial x_j}(2^{-p}x + x_0, 2^{-p}y + y_0) \right) \quad \text{and} \\ a_{j+d,p}(x, y) &= -x_j - 2^p \left(x_j^0 + \frac{\partial h}{\partial y_j}(2^{-p}x + x_0, 2^{-p}y + y_0) \right).\end{aligned}$$

Again, let us state $Z_{j,p} = X_{j,p}$, $Z_{j+d,p} = Y_{j,p}$ and

$$R_{j,k,p} \stackrel{\text{def}}{=} a_{k,p}(x, y) Z_{j,p} - a_{j,p}(x, y) Z_{k,p} \quad \text{for} \quad 1 \leq j < k \leq 2d.$$

These vector fields do depend on p . This is due to the fact that we have started with a curved hypersurface. The fact that the point M_0 is supposed to be characteristic implies that the coefficients of the vector fields $Z_{j,p}$ are locally bounded uniformly with respect to p . Moreover, all the derivatives of the coefficients of these vector fields are obviously locally bounded uniformly with respect to p .

Now we can define the space $T^{\frac{1}{2}}$ which describes the trace space in terms of Weyl-Hörmander calculus and state the restriction theorem.

Definition 4.1. *The space $T^{\frac{1}{2}}$ is the space of the functions in $L^2(\Sigma_0)$ supported in the set $B(0, 1) \cap \Sigma_0$ such that*

$$\|v\|_{T^{\frac{1}{2}}}^2 \stackrel{\text{def}}{=} \sum_{p=0}^{\infty} 2^{-2pd} \|v_p\|_{H(m_p^{\frac{1}{2}})}^2 < \infty$$

where $v_p = \varphi_{0|\Sigma_0}(\cdot)v(2^{-p}\cdot)$ and m_p is the weight defined by

$$m_p^2(x, y, \xi, \eta) \stackrel{\text{def}}{=} \langle (\xi, \eta) \rangle + \tilde{\varphi}_0^2(x, y) \sum_{1 \leq j < k \leq 2d} (R_{j,k,p}(x, y)|(\xi, \eta))^2$$

where $\tilde{\varphi}_0(x, y) \stackrel{\text{def}}{=} \tilde{\varphi}(|x|^2 + |y|^2)$ and $\tilde{\varphi}$ is in $\mathcal{D}(\mathbb{R}_+^*)$ with value 1 near the support of φ .

Now let us state the trace and trace lifting theorem in this context.

Theorem 4.1. *The restriction to Σ_0 map can be extended in a continuous and onto map from $H_B^1(\tilde{\mathcal{Z}})$ onto $T^{\frac{1}{2}}$ where $H_B^1(\tilde{\mathcal{Z}})$ denotes the space of functions in $H^1(\tilde{\mathcal{Z}})$ whose support is included in $B(0, 1)$.*

4.3. Proof of Theorem 4.1. To prove this theorem, we want to apply Theorem 3.1 and Theorem 3.2 to each function u_p with the family $(Z_{j,p})_{1 \leq j \leq 2d}$ because this family is non characteristic on $\mathcal{C} \cap \Sigma_0$. To do this, we need to be in the situation where the hypersurface is Σ_0 and one of the vector field of the family is ∂_s . Unfortunately, this is not the case for the family $\mathcal{Z}_p \stackrel{\text{def}}{=} (Z_{j,p})_{1 \leq j \leq 2d}$. So we have to straighten one of those vector fields. It is not possible to do this globally on $\mathcal{C} \cap \Sigma_0$.

Let us consider the two families $(\chi_j)_{1 \leq j \leq 2d}$ and $(\tilde{\chi}_j)_{1 \leq j \leq 2d}$ defined during the proof of Lemma 4.1 by (4.2) and let us state, as in (4.3),

$$\chi_{j,p} \stackrel{\text{def}}{=} \chi_j((Z_{k,p} \cdot s)_{1 \leq k \leq 2d}) \quad \text{and} \quad \tilde{\chi}_{j,p} \stackrel{\text{def}}{=} \tilde{\chi}_j((Z_{k,p} \cdot s)_{1 \leq k \leq 2d}).$$

Thanks to the non degeneracy condition, a constant c exists which does not depend on p such that on the set $\text{Supp } \chi_{j,p} \cap \mathcal{C}$ we have $|Z_{j,p} \cdot s| \geq c$. So on the set $\text{Supp } \chi_{j,p} \cap \mathcal{C}$, we can substitute to the family $(Z_{j,p})_{1 \leq j \leq 2d}$ the family

$$\{\tilde{\chi}_{j,p} \tilde{\varphi}_0 Z_{j,p}, \tilde{\varphi}_0 R_{k,\ell,p} \quad \text{with} \quad 1 \leq k < \ell \leq 2d\}.$$

In order to apply Theorem 3.1, we first extend the vector field $\tilde{\chi}_{j,p} \tilde{\varphi}_0 Z_{j,p}$ on all \mathbb{R}^{2d+1} . Let us define

$$\begin{aligned} \bar{Z}_{j,p} &\stackrel{\text{def}}{=} \tilde{\chi}_{j,p} \tilde{\varphi}_0 Z_{j,p} + (1 - \tilde{\chi}_{j,p} \tilde{\varphi}_0) \partial_s \quad \text{if} \quad \tilde{\varphi}_0 Z_{j,p} \cdot s > 0 \quad \text{and} \\ \bar{Z}_{j,p} &\stackrel{\text{def}}{=} \tilde{\chi}_{j,p} \tilde{\varphi}_0 Z_{j,p} - (1 - \tilde{\chi}_{j,p} \tilde{\varphi}_0) \partial_s \quad \text{if} \quad \tilde{\varphi}_0 Z_{j,p} \cdot s < 0. \end{aligned}$$

If $\bar{Z}_{j,p}$ is the family

$$\{\bar{Z}_{j,p}, \tilde{\varphi}_0 R_{k,\ell,p} \quad \text{with} \quad 1 \leq k < \ell \leq 2d\}.$$

it is obvious that $\|\chi_{j,p} u_p\|_{H^1(\bar{Z}_{j,p})} \leq C \|\chi_{j,p} u_p\|_{H^1(Z_{j,p})}$. Now let us straighten the vector field $\bar{Z}_{j,p}$ near Σ_0 . In quite a standard way, we state

$$\Psi_{j,p}(s, x, y) = (s, \psi_{j,p}(s, x, y))$$

where $\psi_{j,p}$ is defined by

$$\begin{aligned}\frac{d}{ds}\psi_{j,p}(s, x, y) &= \bar{Z}_{j,p}(\psi_{j,p}(s, x, y)) \\ \psi_{j,p}(0, x, y) &= (x, y).\end{aligned}$$

Classical theory of ordinary differential equations combined with the fact that, on all Σ_0 , the function $\bar{Z}_{j,p} \cdot s$ is greater than a constant independent of p implies that a positive ϵ exists such that $\Psi_{j,p}$ is a diffeomorphism from $U \stackrel{\text{def}}{=} \Sigma_0 \times]-\epsilon, \epsilon[$ onto $\Psi_{j,p}(U)$. Moreover it is obvious that $\Psi_{j,p|_{\Sigma_0}}$ is the identity and that $\Psi_{j,p}^*(\bar{Z}_{j,p}) = \partial_s$. The fact that $\bar{Z}_{j,p} \cdot s$ is greater than a positive constant independent of p implies also that the family $\Psi_{j,p}^*(\bar{Z}_{j,p})$ is a family of vector fields whose coefficients are bounded independently of p . If we state

$$u_{j,p} \stackrel{\text{def}}{=} (\chi_{j,p} u_p) \circ \Psi_{j,p}^{-1},$$

it is obvious that $u_{j,p}$ belongs to $H^1(\Psi_{j,p}^*(\bar{Z}_{j,p}))$. Moreover, the chain rule implies that a constant C (independent of p) exists such that for any p

$$\|u_{j,p}\|_{H^1(\Psi_{j,p}^*(\bar{Z}_{j,p}))} \leq C \|u_p\|_{H^1(\mathcal{Z}_p)}.$$

As $\Psi_{j,p|_{\Sigma_0}}$ is the identity, Theorem 3.1 implies that a constant C exists such that for any p

$$\|\gamma_{\Sigma_0}(\chi_{j,p} u_p)\|_{H(m_p^{\frac{1}{2}})} = \|\gamma_{\Sigma_0}(u_{j,p})\|_{H(m_p^{\frac{1}{2}})} \leq C \|u_{j,p}\|_{H^1(\Psi_{j,p}^*(\bar{Z}_{j,p}))} \quad (4.6)$$

So we get that

$$\|\gamma_{\Sigma_0}(u_p)\|_{H(m_p^{\frac{1}{2}})} \leq C \|u_p\|_{H^1(\mathcal{Z}_p)}.$$

As $\|Z_p u_p\|_{L^2} = 2^{pd} \|\tilde{Z}(\varphi_p u)\|_{L^2}$, we have, using Inequality (1.3) that

$$\sum_p 2^{-2pd} \|\gamma_{\Sigma_0}(u_p)\|_{H(m_p^{\frac{1}{2}})}^2 \leq C \|u\|_{H^1(\tilde{\mathcal{Z}})}^2.$$

So we have proved that the restriction map to Σ_0 can be extended to a continuous map from $H_B^1(\tilde{\mathcal{Z}})$ into $T^{\frac{1}{2}}$. Now let us prove that this map is onto.

Let us consider a function v in $T^{\frac{1}{2}}$. Let us state $v_{j,p} \stackrel{\text{def}}{=} \chi_{j,p} v_p$. As the function $\tilde{\varphi}$ has value 1 near the support of φ , we have that $v_{j,p} = \chi_{j,p} \tilde{\varphi}_0 v_p$. But the function $\chi_{j,p} \tilde{\varphi}_0$ is a compactly supported smooth function with (uniformly) bounded derivatives, this is a symbol in $S(1, g)$ and moreover the semi-norms $\|\chi_{j,p} \tilde{\varphi}_0\|_{S(1, g)}$ are bounded independently of p . Thanks to Proposition 2.1 we have

$$\|v_{j,p}\|_{H(m_p^{\frac{1}{2}})} \leq C \|v_p\|_{H(m_p^{\frac{1}{2}})}.$$

Using Theorem 3.2, a function $u_{j,p}$ exists in $H^1(\Psi_{j,p}^*(\bar{Z}_{j,p}))$ (which we can assumed to be supported in a compact subset of $\Psi_{j,p}(U_{j,p})$ after cut off) such that

$$\gamma_{\Sigma_0}(u_{j,p}) = v_{j,p} \quad \text{and} \quad \|u_{j,p}\|_{H^1(\Psi_{j,p}^*(\bar{Z}_{j,p}))} \leq C \|v_{j,p}\|_{H(m_p^{\frac{1}{2}})}.$$

Then let us define

$$u \stackrel{\text{def}}{=} \sum_p u_p \quad \text{with} \quad u_p \stackrel{\text{def}}{=} \sum_{j=1}^{2d} (u_{j,p} \circ \Psi_{j,p})(2^p x).$$

We have

$$\begin{aligned} \|u_p\|_{H^1(\tilde{\mathcal{Z}})} &\leq \sum_{j=1}^{2d} \|(u_{j,p} \circ \Psi_{j,p})(2^p \cdot)\|_{H^1(\tilde{\mathcal{Z}})}^2 \\ &\leq C 2^{-2pd} \sum_{j=1}^{2d} \|u_{j,p}\|_{H^1(\Psi_{j,p}^*(\tilde{\mathcal{Z}}_{j,p}))}^2 \\ &\leq C 2^{-2pd} \|v_p\|_{H(m_p^{\frac{1}{2}})}^2. \end{aligned}$$

But as the support of u_p and $u_{p'}$ are disjoint when $|p - p'|$ is large enough, then the sequence (u_p) is almost orthogonal in $H^1(\tilde{\mathcal{Z}})$. So

$$u \in H^1(\tilde{\mathcal{Z}}) \quad \text{and} \quad \|u\|_{H^1(\tilde{\mathcal{Z}})} \leq C \|v\|_{T^{\frac{1}{2}}}$$

and Theorem 4.1 is proved.

4.4. Conclusion of the proof of Theorem 1.1. To prove Theorem 1.1, we have to prove that the definition of the trace space given by complex interpolation in the introduction is the same that the one defined with the Weyl-Hörmander calculus.

Proposition 4.3. *We have the following equality between spaces*

$$T^{\frac{1}{2}} = [L^2, TH^2(\mathcal{R}_{\Sigma_0})]_{\frac{1}{4}}$$

where $[L^2, TH^2(\mathcal{R}_{\Sigma_0})]_{\frac{1}{4}}$ denotes the complex interpolation space between L^2 and $TH^2(\mathcal{R}_{\Sigma_0})$ of index $1/4$.

In order to prove this proposition, let us first recall the definition of the complex interpolation in our particular case. Let \mathcal{F} be the space of holomorphic functions f from the strip $0 < \Re z < 1$ into L^2 such that $f(\alpha + it)$ is continuous and vanishes at infinity in A_α (with $A_0 = L^2$ and $A_1 = TH^2(\mathcal{R}_{\Sigma_0})$). Then, for $\theta \in]0, 1[$, the space $[A_0, A_1]_\theta$ is

$$[A_0, A_1]_\theta \stackrel{\text{def}}{=} \{v \in L^2 / \exists f \in \mathcal{F} / f(\theta) = v\}$$

equipped with the norm

$$\|v\|_{[A_0, A_1]_\theta} \stackrel{\text{def}}{=} \inf_{f \in \mathcal{F}} \max_{t \in \mathbb{R}} \{\sup \|f(it)\|_{A_0}, \|f(1+it)\|_{A_1}\}.$$

For further details about this theory, we refer to [6].

First let us prove that $T^{\frac{1}{2}} \subset [L^2, TH^2(\mathcal{R}_{\Sigma_0})]_{\frac{1}{4}}$. For any function v in $T^{\frac{1}{2}}$, let us define

$$\begin{aligned} f(z) &\stackrel{\text{def}}{=} \exp\left(\epsilon z^2 - \frac{\epsilon}{16}\right) \sum_p f_p(z) \quad \text{with} \\ f_p(z) &\stackrel{\text{def}}{=} \tilde{\varphi}_p \int_{T^* \mathbb{R}^{2d}} m_p^{\frac{1}{2}-2z}(Y) (\psi_Y^w \varphi_Y^w v_p)(2^p \cdot) dY. \end{aligned}$$

By Lemma 2.4, we have that

$$f_p\left(\frac{1}{4}\right) = \varphi_p v.$$

Moreover, it is obvious that f_p is an holomorphic function on $0 < \Re z < 1$ with value in L^2 and that, thanks to the almost orthogonality in L^2 , the series (f_p) converges in L^2 . So, for

any θ between 0 and 1, we have

$$\|f(\theta + it)\|_{L^2}^2 \leq C \sum_p \|f_p(\theta + it)\|_{L^2}^2.$$

Thanks to Lemma 2.5, we have, for any real number t

$$\|f_p(\theta + it)\|_{L^2}^2 \leq C 2^{-2pd} \int_{T^*\mathbb{R}^{2d}} m_p^{1-4\theta}(Y) \|\varphi_Y^w v_p\|_{L^2}^2 dY.$$

As the weight m_p is greater than 1, we have, for any θ between 0 and 1,

$$\|f_p(\theta + it)\|_{L^2}^2 \leq C 2^{-2pd} \int_{T^*\mathbb{R}^{2d}} m_p(Y) \|\varphi_Y^w v_p\|_{L^2}^2 dY.$$

So by the definition of the $H(m)$ norms, we get that

$$\|f_p(\theta + it)\|_{L^2}^2 \leq C 2^{-2pd} \|v_p\|_{H(m_p^{\frac{1}{2}})}^2.$$

So by almost orthogonality, we infer that

$$\|f(\theta + it)\|_{L^2}^2 \leq c \|v\|_{T^{\frac{1}{2}}}^2. \quad (4.7)$$

Now, we are going to estimate $\|R_j R_k f(1 + it)\|_{L^2}$ for R_j and R_k in \mathcal{R}_{Σ_0} . Using again the almost orthogonality, we have

$$\|R_j R_k f(1 + it)\|_{L^2}^2 \leq C e^{\epsilon(1-t^2)} \sum_p \|R_j R_k f_p(1 + it)\|_{L^2}^2.$$

We have

$$R_j R_k f_p(1 + it) = \int_{T^*\mathbb{R}^{2d}} m_p^{\frac{1}{2}-2-2it}(Y) R_j R_k \left(\tilde{\varphi}_p(\psi_Y^w \varphi_Y^w v_p)(2^p \cdot) \right) dY.$$

The Leibnitz formula implies that

$$\begin{aligned} R_j R_k f_p(1 + it) &= - \sum_{m=1}^3 \Delta_p^m \quad \text{with} \\ \Delta_p^1 &\stackrel{\text{def}}{=} R_j R_k(\tilde{\varphi}_p) \int_{T^*\mathbb{R}^{2d}} m_p^{-\frac{3}{2}-2it}(Y) (\psi_Y^w \varphi_Y^w v_p)(2^p \cdot) dY, \\ \Delta_p^2 &\stackrel{\text{def}}{=} R_j(\tilde{\varphi}_p) \int_{T^*\mathbb{R}^{2d}} m_p^{-\frac{3}{2}-2it}(Y) R_k \left((\psi_Y^w \varphi_Y^w v_p)(2^p \cdot) \right) dY \\ &\quad + R_k(\tilde{\varphi}_p) \int_{T^*\mathbb{R}^{2d}} m_p^{-\frac{3}{2}-2it}(Y) R_j \left((\psi_Y^w \varphi_Y^w v_p)(2^p \cdot) \right) dY \quad \text{and} \\ \Delta_p^3 &\stackrel{\text{def}}{=} \tilde{\varphi}_p \int_{T^*\mathbb{R}^{2d}} m_p^{-\frac{3}{2}-2it}(Y) R_j R_k \left((\psi_Y^w \varphi_Y^w v_p)(2^p \cdot) \right) dY. \end{aligned}$$

As the vector fields R_j vanish in M_0 , for any smooth function θ whose support is included in a ball, we have

$$\sup_{\substack{x \in \Sigma_0 \\ k, \ell}} \{ |R_k(\theta(2^p x))| + |R_k R_\ell(\theta(2^p x))| \} \leq C \quad (4.8)$$

where C is independent of p . Using the estimate (4.8), the estimate about Δ_p^1 is strictly similar to (4.7).

Using the estimate (4.8) we can write that

$$\int_{T^*\mathbb{R}^{2d}} m_p^{-\frac{3}{2}-2it}(Y) R_\ell \left((\psi_Y^w \varphi_Y^w v_p)(2^p \cdot) \right) dY = \int_{T^*\mathbb{R}^{2d}} m_p^{-\frac{1}{2}-2it}(Y) \left(\Theta_{\ell,p,Y}^w \varphi_Y^w v_p \right) (2^p \cdot) dY$$

with

$$\Theta_{\ell,p,Y}^w \stackrel{\text{def}}{=} \frac{1}{m_p(Y)} R_{\ell,p} \psi_Y^w.$$

By Lemma 3.2, the family $(R_{\ell,p}(x,y)|(\xi,\eta))_{1 \leq \ell \leq N}$ belongs to $S(m_p, g_{\frac{1}{2}, \frac{1}{2}})$ uniformly with respect to p . So by Lemma 2.3 the family $(\Theta_{\ell,p,Y})_{Y \in T^*\mathbb{R}^{2d}}$ is uniformly confined with constants that do not depend on p . So, using the fact that m_p is greater than 1, Lemma 2.5 implies that

$$\begin{aligned} \|\Delta_p^2\|_{L^2}^2 &\leq C 2^{-2pd} \int_{T^*\mathbb{R}^{2d}} m_p(Y) \|\varphi_Y^w v_p\|_{L^2}^2 dY \\ &\leq C 2^{-2pd} \|v_p\|_{H(m_p^{\frac{1}{2}})}^2 \end{aligned}$$

Exactly along the same lines, we write that

$$\int_{T^*\mathbb{R}^{2d}} m_p^{-\frac{3}{2}-2it}(Y) R_j R_k \left((\psi_Y^w \varphi_Y^w v_p)(2^p \cdot) \right) dY = \int_{T^*\mathbb{R}^{2d}} m_p^{\frac{1}{2}-2it}(Y) \left(\tilde{\Theta}_{j,k,p,Y}^w \varphi_Y^w v_p \right) (2^p \cdot) dY$$

with

$$\tilde{\Theta}_{j,k,p,Y}^w \stackrel{\text{def}}{=} \frac{1}{m_p^2(Y)} R_{j,p} R_{k,p} \psi_Y^w.$$

Again Lemmas 3.2, 2.3 and 2.5 imply that

$$\|\Delta_p^3\|_{L^2}^2 \leq C 2^{-2pd} \|v_p\|_{H(m_p^{\frac{1}{2}})}^2.$$

So, we infer that

$$\begin{aligned} \|R_j R_k f(1+it)\|_{L^2}^2 &\leq C e^{\epsilon(1-t^2)} \sum_p 2^{-2pd} \|v_p\|_{H(m_p^{\frac{1}{2}})}^2 \\ &\leq C e^\epsilon \|v\|_{T^{\frac{1}{2}}}^2. \end{aligned} \tag{4.9}$$

In the case when $d = 1$, we have that

$$\|T_j f(1+it)\|_{L^2}^2 \leq C \|v\|_{T^{\frac{1}{2}}}^2. \tag{4.10}$$

The proof is along the same lines as before and we omit it. The two estimates (4.7) and (4.9) imply that $T^{\frac{1}{2}}$ is included in $[L^2, TH^2(\mathcal{R}_{\Sigma_0})]_{\frac{1}{4}}$.

Now, let us prove the opposite inclusion. To do this, let us consider a function v in the space $[L^2, TH^2(\mathcal{R}_{\Sigma_0})]_{\frac{1}{4}}$ and any function f of \mathcal{F} such that $f(1/4) = v$. It is enough to prove that

$$\|v\|_{T^{\frac{1}{2}}}^2 \leq C \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{L^2}^2, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{TH^2(\mathcal{R}_{\Sigma_0})}^2 \right\}.$$

To prove this inequality, let us consider a compact K of $T^*\mathbb{R}^{2d}$ and an integer N and let us introduce the function $F_{K,N}$ defined by

$$F_{K,N}(z) \stackrel{\text{def}}{=} \exp\left(\epsilon z^2 - \frac{\epsilon}{16}\right) \sum_{p \leq N} 2^{-2pd} \int_K m_p^{\frac{1}{2}+2z}(Y) (\varphi_Y^w v_p | \varphi_Y^w f_p(z))_{L^2} dY$$

with $f_p \stackrel{\text{def}}{=} \varphi_p f$.

Let us estimate this function. Its regularity properties are the same as the ones of $f \in \mathcal{F}$. Let us also remark that

$$F_{K,N}\left(\frac{1}{4}\right) = \sum_{p \leq N} 2^{-2pd} \int_K m_p(Y) \|\varphi_Y^w v_p\|_{L^2}^2 dY.$$

Now, let us estimate $|F_{K,N}(it)|$ and $|F_{K,N}(1+it)|$. Cauchy-Schwarz inequality implies that

$$|F_{K,N}(it)|^2 \leq \left| F_{K,N}\left(\frac{1}{4}\right) \right| \sum_{p \leq N} 2^{-2pd} \int_{T^*\mathbb{R}^{2d}} \|\varphi_Y^w f_p(it)\|_{L^2}^2 dY.$$

We clearly have that

$$\int_{T^*\mathbb{R}^{2d}} \|\varphi_Y^w f_p(z)\|_{L^2}^2 dY \leq C \|f_p(z)\|_{L^2}^2.$$

As (φ_p) is a dyadic partition of unity, we can deduce by dilatation that

$$\sum_{p \leq N} 2^{-2pd} \int_{T^*\mathbb{R}^{2d}} \|\varphi_Y^w f_p(it)\|_{L^2}^2 dY \leq C \|f(it)\|_{L^2}^2.$$

So it turns out that

$$|F_{K,N}(it)|^2 \leq \left| F_{K,N}\left(\frac{1}{4}\right) \right| \|f(it)\|_{L^2}^2. \quad (4.11)$$

Now, let us estimate $|F_{K,N}(1+it)|$. Using again Cauchy-Schwarz inequality we get that

$$|F_{K,N}(1+it)|^2 \leq e^\epsilon \left| F_{K,N}\left(\frac{1}{4}\right) \right| \sum_{p \leq N} 2^{-2pd} \int_{T^*\mathbb{R}^{2d}} m_p^4(Y) \|\varphi_Y^w f_p(1+it)\|_{L^2}^2 dY.$$

By definition of the $H(m)$ norms, we get that

$$|F_{K,N}(1+it)|^2 \leq e^\epsilon \left| F_{K,N}\left(\frac{1}{4}\right) \right| \sum_{p \leq N} 2^{-2pd} \|f_p(1+it)\|_{H(m_p^2)}^2.$$

Let us estimate the sum which appears in the right hand side of the above inequality.

If $d \geq 2$, by Lemma 3.1, we know that $\tilde{\mathcal{R}}_{\Sigma,p}$ satisfies the Hörmander condition at order 2 uniformly with respect to p . Using Proposition 3.2, we have that

$$\|f_p(1+it)\|_{H(m_p^2)}^2 \leq C \left(\|f_p(1+it)\|_{L^2}^2 + \sum_{1 \leq j,k \leq 2d} \|R_{j,p} R_{k,p} f_p(1+it)\|_{L^2}^2 \right).$$

By dilatation, we have that

$$\|R_j R_k (\varphi_p f(1+it))\|_{L^2}^2 = 2^{-2pd} \|R_{j,p} R_{k,p} f_p(1+it)\|_{L^2}^2.$$

By Leibnitz formula, we infer

$$R_j R_k (\varphi_p v) - \varphi_p R_j R_k v = (R_j \varphi_p)(R_k v) + (R_k \varphi_p)(R_j v) + (R_j R_k \varphi_p)v.$$

So we have, using Inequality (4.8), the almost orthogonality in L^2 that

$$\|R_j R_k (\varphi_p v) - \varphi_p R_j R_k v\|_{L^2} \leq C c_p (\|R_j v\|_{L^2} + \|R_k v\|_{L^2} + \|v\|_{L^2}) \quad (4.12)$$

with as in all that follows $\sum_p c_p^2 = 1$. So, we have that

$$\|R_j R_k(\varphi_p v)\|_{L^2} \leq C c_p (\|R_j R_k v\|_{L^2} + \|R_j v\|_{L^2} + \|R_k v\|_{L^2} + \|v\|_{L^2}).$$

This obviously implies that

$$|F_{K,N}(1+it)|^2 \leq e^\epsilon \left| F_{K,N}\left(\frac{1}{4}\right) \right| \|f(1+it)\|_{TH^2(\mathcal{R}_{\Sigma_0})}^2. \quad (4.13)$$

If $d = 1$, Proposition 3.2 implies that

$$\|f_p(1+it)\|_{H(m_p^2)}^2 \leq C (\|R_p^2 f_p(1+it)\|_{L^2}^2 + \|f_p(1+it)\|_{H^1(\Sigma_0)}^2).$$

As in the above proof, we have

$$\sum_p 2^{-2p} \|R_p^2 f_p(1+it)\|_{L^2}^2 \leq \|R^2 f(1+it)\|_{L^2}^2 + \|f(1+it)\|_{L^2}^2.$$

Let us estimate $\|w_p\|_{H^1(\Sigma_0)}^2$ for any w in $H^1(\Sigma_0)$. It is clearly enough to estimate $\|\partial w_p\|_{L^2}^2$. By dilation and Leibnitz's formula, we have

$$2^{-2p} \|\partial w_p\|_{L^2}^2 \leq \int_{\mathbb{R}^2} |(\partial\varphi)_p|^2 |w|^2 dx dy + \int_{\mathbb{R}^2} 2^{-2p} |\varphi_p|^2 |\partial w|^2 dx dy.$$

As on the support of φ_p , we have $2^{-p} \leq C(|x| + |y|)$, we have

$$\int_{\mathbb{R}^2} 2^{-2p} |\varphi_p|^2 |\partial w|^2 dx dy \leq C \int_{\mathbb{R}^2} |\varphi_p|^2 |x \partial w|^2 dx dy + C \int_{\mathbb{R}^2} |\varphi_p|^2 |y \partial w|^2 dx dy.$$

Again by almost orthogonality, we have that

$$\sum_p 2^{-2p} \|f_p(1+it)\|_{H^1(\Sigma_0)}^2 \leq C \|f(1+it)\|_{TH^2(\mathcal{R}_{\Sigma_0})}^2.$$

So Inequality (4.13) is also valid for $d = 1$. Using the Phragmen-Lindelöf principle and Inequality (4.11), we get that for any compact K of $T^*\mathbb{R}^{2d}$,

$$\left| F_{K,N}\left(\frac{1}{4}\right) \right|^2 \leq C e^\epsilon \left| F_{K,N}\left(\frac{1}{4}\right) \right| \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{L^2}^2, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{TH^2(\mathcal{R}_{\Sigma_0})}^2 \right\}.$$

So we have

$$\left| F_{K,N}\left(\frac{1}{4}\right) \right| \leq C e^\epsilon \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{L^2}^2, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{TH^2(\mathcal{R}_{\Sigma_0})}^2 \right\}.$$

As the above estimate is true for any function f in \mathcal{F} such that $f(1/4) = v$, this conclude the proof of Proposition 4.3 by passing to the limit.

As complex interpolation is often understood in the frame of domain of selfadjoint operators, we shall prove the following proposition.

Proposition 4.4. *If $d \geq 2$, the operator Δ_Σ is a selfadjoint operator with domain $TH^2(\mathcal{R}_\Sigma)$.*

It is enough to prove that, for any smooth compactly supported function v , we have

$$\sum_{j=1}^N \|R_j v\|_{L^2}^2 + \sum_{j,k=1}^N \|R_j R_k v\|_{L^2}^2 \leq C (\|v\|_{L^2}^2 + \|\Delta_\Sigma v\|_{L^2}^2). \quad (4.14)$$

First of all, it is obvious that

$$\sum_{j=1}^N \|R_j v\|_{L^2}^2 = \sum_{j=1}^N (R_j^* R_j v | v)_{L^2} = -(\Delta_\Sigma v | v)_{L^2}.$$

So we have that

$$\sum_{j=1}^N \|R_j v\|_{L^2}^2 \leq \|\Delta_\Sigma v\|_{L^2}^2 + \|v\|_{L^2}^2.$$

Of course, the problem appears only near the characteristic points where the vector fields R_j vanish. Away from the characteristic point, Lemma 3.1 tells us that the Hörmander condition is satisfied. The maximal estimate (see for instance [16]) implies Inequality (4.14).

As Inequality (4.14) is invariant under the action of diffeomorphism, we can, near the non degenerate characteristic point M_0 , straighten the hypersurface Σ to Σ_0 . Let (φ_p) a dyadic partition of unity near M_0 . By Inequality (4.12), we have that

$$\|\varphi_p R_j R_k v\|_{L^2} \leq \|R_j R_k (\varphi_p v)\|_{L^2} + C c_p (\|R_j v\|_{L^2} + \|R_k v\|_{L^2} + \|v\|_{L^2}).$$

Now let us write

$$R_j R_k (\varphi_p v) = (R_{j,p} R_{k,p} (\varphi v (2^{-p} \cdot))) (2^p \cdot).$$

Thanks to Lemma 3.1 and to the non degeneracy condition, we have that the system of vector fields $(R_{j,p})$ satisfies on the ring \mathcal{C} the Hörmander condition uniformly with respect to p .

So using again the maximal estimate we claim that a constant C (independent of p) exists such that for any smooth function w whose support is included in \mathcal{C}

$$\|R_{j,p} R_{k,p} w\|_{L^2} \leq C \left\| \sum_{j=1}^N R_{j,p}^* R_{j,p} w \right\|_{L^2} + C \|w\|_{L^2}.$$

By dilatation, we have using the inequality (4.12)

$$\|\varphi_p R_j R_k v\|_{L^2} \leq C \|\Delta_\Sigma (\varphi_p v)\|_{L^2} + C c_p (\|R_j v\|_{L^2} + \|R_k v\|_{L^2} + \|v\|_{L^2}).$$

Similar computations as above tell us that

$$\|\Delta_\Sigma (\varphi_p v)\|_{L^2} \leq \|\varphi_p \Delta_\Sigma v\|_{L^2} + C c_p \left(\sum_{j=1}^N \|R_j v\|_{L^2} + \|v\|_{L^2} \right).$$

So, we have that

$$\|\varphi_p R_j R_k v\|_{L^2}^2 \leq C \|\varphi_p \Delta_\Sigma v\|_{L^2}^2 + C c_p^2 \left(\sum_{j=1}^N \|R_j v\|_{L^2}^2 + \|v\|_{L^2}^2 \right)$$

and Proposition 4.4 is proved.

Corollary 4.2. *If $d \geq 2$, then*

$$C^{-1} \|v\|_{TH^2(\mathcal{R}_\Sigma)}^2 \leq \|\Delta_\Sigma v\|_{L^2}^2 + \|v\|_{L^2}^2 \leq C \|v\|_{TH^2(\mathcal{R}_\Sigma)}^2.$$

5. THE PROOF OF THEOREM 1.2 IN THE GENERAL CASE

As trace and trace lifting problems are local ones, we can assume as in paragraph 3.2 that the hypersurface Σ is $\Sigma_0 \stackrel{\text{def}}{=} \{x / x_1 = 0\}$ and the system \mathcal{P} of vector fields is

$$P_1 = \partial_{x_1} \quad \text{and} \quad P_j = \sum_{\ell=2}^n P_j^\ell(x_1, x') \partial_{x_\ell} \quad \text{for } j \in \{2, \dots, N\}.$$

We shall denote by \mathcal{P}_Σ the family $(P_j(0, \cdot))_{2 \leq j \leq N}$. Let us introduce some notations.

- For $\tilde{X} \in T^* \mathbb{R}^n$, we shall write

$$\tilde{X} = (x_1, \xi_1, X) \quad \text{with} \quad X = (x, \xi) \in T^* \mathbb{R}^{n-1} \quad \text{and} \quad \pi(\tilde{X}) \stackrel{\text{def}}{=} \tilde{X}' \stackrel{\text{def}}{=} X.$$

- We state now

$$M(\tilde{X}) \stackrel{\text{def}}{=} \left(\xi_1^2 + m_{x_1}^2(X) \right)^{\frac{1}{2}} \quad \text{with} \quad m_{x_1}(X) \stackrel{\text{def}}{=} \left(\langle \xi \rangle + \sum_{j=2}^N (P_j(x_1, x') | \xi \rangle)^2 \right)^{\frac{1}{2}}.$$

- For $X \in T^*\mathbb{R}^{n-1}$, we also state $m(X) \stackrel{\text{def}}{=} m_0(X)$ and

$$\begin{aligned} \tilde{g}_{\tilde{X}}(d\tilde{x}, d\tilde{\xi}) &\stackrel{\text{def}}{=} \langle \tilde{\xi} \rangle d\tilde{x}^2 + \frac{1}{\langle \tilde{\xi} \rangle} d\tilde{\xi}^2 \quad \text{and} \\ g_X(dx, d\xi) &\stackrel{\text{def}}{=} \langle \xi \rangle dx^2 + \frac{1}{\langle \xi \rangle} d\xi^2. \end{aligned}$$

By Lemma 2.6, we have that M (resp. m) is a \tilde{g} (resp. g)-weight. By Theorem 2.1, we have that

$$H^s(\mathcal{P}) = H(M^s) \quad \text{and} \quad H^s(\mathcal{P}_\Sigma) = H(m^s).$$

The proof of Theorem 1.2 reduces to prove that the restriction operator on Σ_0 can be extended in a continuous and onto operator from $H(M^s)$ onto $H(m^{s-\frac{1}{2}})$ for any s greater than $1/2$.

5.1. Continuity of the trace operator. The proof of continuity of the trace operator consists in the proof of the following inequality:

$$\|\gamma(u)\|_{H(m^{s-\frac{1}{2}})} \leq C \|u\|_{H(M^s)}. \quad (5.1)$$

Let $(\phi_Y, \theta_Y)_{Y \in T^*\mathbb{R}^{n-1}}$ (resp. $(\varphi_{\tilde{Y}}, \psi_{\tilde{Y}})_{\tilde{Y} \in T^*\mathbb{R}^n}$) a partition of unity of $T^*\mathbb{R}^{n-1}$ (resp. of $T^*\mathbb{R}^n$) given by Lemma 2.4 for the metric g (resp. \tilde{g}). The above inequality (5.1) is equivalent to

$$\int_{T^*\mathbb{R}^{n-1}} m^{2s-1}(Y) \|\theta_Y^{w_\Sigma} \gamma(u)\|_{L^2(\mathbb{R}^{n-1})}^2 dY \leq C^2 \int_{T^*\mathbb{R}^n} M^{2s}(\tilde{Y}) \|\psi_{\tilde{Y}}^w u\|_{L^2(\mathbb{R}^n)}^2 d\tilde{Y}. \quad (5.2)$$

Let us estimate now $\|\theta_Y^{w_\Sigma} \gamma(u)\|_{L^2(\Sigma)}^2$. To do this, let us use the partition of unity of $T^*\mathbb{R}^n$ and let us write that

$$\theta_Y^{w_\Sigma} \gamma(u) = \int_{T^*\mathbb{R}^n} \theta_Y^{w_\Sigma} \gamma(\varphi_{\tilde{Y}}^w \psi_{\tilde{Y}}^w u) d\tilde{Y}.$$

The key lemma is the following.

Lemma 5.1. *For any N , a constant C exists such that, for any function v of $L^2(\mathbb{R}^n)$, we have*

$$\|\theta_Y^{w_\Sigma} \gamma(\varphi_{\tilde{Y}}^w v)\|_{L^2(\mathbb{R}^{n-1})} \leq C \langle \tilde{\eta}' \rangle^{\frac{1}{4}} \Delta(Y, \pi(\tilde{Y}))^{-N} (1 + \langle \tilde{\eta}' \rangle |y_1|^2)^{-N} \|v\|_{L^2(\mathbb{R}^n)}.$$

Let us compute, for a given function v of $\mathcal{S}(\mathbb{R}^n)$, the function

$$\theta_Y^{w_\Sigma} \gamma(\varphi_{\tilde{Y}}^w v).$$

By Weyl quantization formula, we have

$$\varphi_{\tilde{Y}}^w v(0, x') = (2\pi)^{-n} \int_{\mathbb{R}^2} dz_1 d\xi_1 e^{-iz_1 \xi_1} \int_{T^*\Sigma} e^{i\langle \xi', x' - z' \rangle} \varphi_{\tilde{Y}} \left(\frac{z_1}{2}, \frac{x' + z}{2}, \xi_1, \zeta \right) v(z_1, z) dZ.$$

Using integrations by parts with respect to $\langle \tilde{\eta}' \rangle^{\frac{1}{2}} \partial_{\xi_1}$, which is a vector of $\tilde{g}_{\tilde{Y}}$ -length 1, we find that, for any positive integer N ,

$$\begin{aligned} \varphi_{\tilde{Y}}^w v(0, x') &= (2\pi)^{-n} \int_{\mathbb{R}^2} dz_1 d\xi_1 (1 + \langle \tilde{\eta}' \rangle |z_1|^2)^{-N} e^{-iz_1 \xi_1} \\ &\quad \int_{T^*\Sigma} e^{i\langle \xi', x' - z' \rangle} \varphi_{\tilde{Y}}^{(N)} \left(\frac{z_1}{2}, \frac{x' + z}{2}, \xi_1, \zeta \right) v(z_1, z) dZ, \end{aligned}$$

with

$$\varphi_{\tilde{Y}}^{(N)}(z_1, z, \xi_1, \zeta) \stackrel{\text{def}}{=} (1 - \langle \tilde{\eta}' \rangle \partial_{\xi_1}^2)^N \varphi_{\tilde{Y}}\left(\frac{z_1}{2}, z, \xi_1, \zeta\right).$$

This equality can be written as

$$\varphi_{\tilde{Y}}^w v(0, x') = (2\pi)^{-1} \int_{\mathbb{R}^2} (1 + \langle \tilde{\eta}' \rangle |z_1|^2)^{-N} e^{-iz_1 \xi_1} \left(\varphi_{\tilde{Y}}^{(N)}\left(\frac{z_1}{2}, \cdot, \xi_1, \cdot\right) \right)^{w_\Sigma} v(z_1, \cdot) dz_1 d\xi_1.$$

So it turns out that the operator $\theta_Y^{w_\Sigma} \gamma \varphi_{\tilde{Y}}^w$ can be seen as a superposition of operators of the type $\theta_Y \#_{T^*\Sigma} \psi$. More precisely, we have

$$\begin{aligned} (\theta_Y^{w_\Sigma} \gamma \varphi_{\tilde{Y}}^w v)(x) &= (2\pi)^{-1} \int_{\mathbb{R}^2} (1 + \langle \tilde{\eta}' \rangle z_1^2)^{-N} \\ &\quad \times e^{-iz_1 \xi_1} (\theta_Y \#_{T^*\Sigma} \varphi_{\tilde{Y}}^{(N)}\left(\frac{z_1}{2}, \cdot, \xi_1, \cdot\right))^{w_\Sigma} v(z_1, \cdot) dz_1 d\xi_1. \end{aligned} \quad (5.3)$$

Now, let us deal with the following problem: let a be a function of $\mathcal{S}(T^*\mathbb{R}^n)$; how is the function $a(z_1, \cdot, \xi_1, \cdot)$ confined on $T^*\Sigma$? By Definition 2.3 of the semi-norms of confinement, we have, because $\tilde{g}_{\tilde{Y}}^g = \tilde{g}_{\tilde{Y}}$,

$$|a(z_1/2, z, \xi_1, \zeta)| \leq \text{Conf}_{2M, \tilde{Y}, \tilde{g}_{\tilde{Y}}} (a) (1 + \tilde{g}_{\tilde{Y}}((z_1/2, z, \xi_1, \zeta) - \tilde{Y}))^{-M}.$$

By definition of the two metrics \tilde{g} and g , we have

$$\tilde{g}_{\tilde{Y}}((z_1/2, z, \xi_1, \zeta) - \tilde{Y}) = \langle \tilde{\eta}' \rangle \left(\frac{z_1}{2} - \tilde{y}_1 \right)^2 + \frac{(\xi_1 - \tilde{\eta}_1)^2}{\langle \tilde{\eta}' \rangle} + g_{\pi(\tilde{Y})}(Z - \pi(\tilde{Y})).$$

So we infer that

$$\begin{aligned} |\varphi_{\tilde{Y}}^{(N)}(z_1/2, z, \xi_1, \zeta)| &\leq \text{Conf}_{2N+2M, \tilde{Y}, \tilde{g}_{\tilde{Y}}}(\varphi_{\tilde{Y}}) \left(1 + \langle \tilde{\eta}' \rangle \left(\frac{z_1}{2} - \tilde{y}_1 \right)^2 + \frac{(\xi_1 - \tilde{\eta}_1)^2}{\langle \tilde{\eta}' \rangle} \right)^{-\frac{M}{2} - \frac{N}{2}} \\ &\quad \times \left(1 + g_{\pi(\tilde{Y})}(Z - \pi(\tilde{Y})) \right)^{-\frac{M}{2} - \frac{N}{2}}. \end{aligned}$$

The estimates are of course analogous for the derivatives of $g_{\pi(\tilde{Y})}$ -length less than 1. This means that for any integer k , a constant C exists such that for any N we have

$$\begin{aligned} \text{Conf}_{k, \pi(\tilde{Y}), g_{\pi(\tilde{Y})}}(\varphi_{\tilde{Y}}^{(N)}(z_1/2, \cdot, \zeta_1, \cdot)) \\ \leq C \left(1 + \langle \tilde{\eta}' \rangle \left(\frac{z_1}{2} - \tilde{y}_1 \right)^2 + \frac{(\xi_1 - \tilde{\eta}_1)^2}{\langle \tilde{\eta}' \rangle} \right)^{-\frac{N}{2}} \text{Conf}_{k+2N, \tilde{Y}, \tilde{g}_{\tilde{Y}}}(\varphi_{\tilde{Y}}). \end{aligned} \quad (5.4)$$

Biconfinement Lemma 2.2 implies that for any N_1 and N_2

$$\begin{aligned} \left\| \left((\theta_Y \#_{T^*\Sigma} \varphi_{\tilde{Y}}^{(N)}\left(\frac{z_1}{2}, \cdot, \xi_1, \cdot\right)) \right)^{w_\Sigma} v(z_1, \cdot) \right\|_{L^2(\mathbb{R}^{n-1})} \\ \leq C \Delta(Y, \pi(\tilde{Y}))^{-N_1} \left(1 + \langle \tilde{\eta}' \rangle \left(\frac{z_1}{2} - \tilde{y}_1 \right)^2 + \frac{(\xi_1 - \tilde{\eta}_1)^2}{\langle \tilde{\eta}' \rangle} \right)^{-N_2} \|v(z_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})}. \end{aligned}$$

From Formula (5.3), we deduce that, for any couple (N_1, N_2) of integers, a constant C exists such that

$$\begin{aligned} \|\theta_Y^{w_\Sigma} \gamma(\varphi_{\tilde{Y}}^w v)\|_{L^2(\mathbb{R}^{n-1})} &\leq C \Delta(Y, \pi(\tilde{Y}))^{-N_1} \int_{\mathbb{R}^2} (1 + \langle \tilde{\eta}' \rangle |z_1|^2)^{-N_1} \\ &\quad \times \left(1 + \langle \tilde{\eta}' \rangle \left(\frac{z_1}{2} - \tilde{y}_1 \right)^2 + \frac{(\xi_1 - \tilde{\eta}_1)^2}{\langle \tilde{\eta}' \rangle} \right)^{-N_2} \|v(z_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})} dz_1 d\xi_1. \end{aligned}$$

As we have

$$(1 + \langle \tilde{\eta}' \rangle \tilde{y}_1^2)^{N_1} \leq 2^{N_1} (1 + \langle \tilde{\eta}' \rangle (\tilde{y}_1 - z_1/2)^2)^{\frac{N_1}{2}} (1 + \langle \tilde{\eta}' \rangle z_1^2)^{\frac{N_1}{2}},$$

we infer that, for any integer N , a constant C exists such that

$$\begin{aligned} \|\theta_Y^{w\Sigma} \gamma(\varphi_Y^w v)\|_{L^2(\mathbb{R}^{n-1})} &\leq C \Delta(Y, \pi(\tilde{Y}))^{-N} (1 + \langle \tilde{\eta}' \rangle |\tilde{y}_1|^2)^{-N} \\ &\quad \times \int_{\mathbb{R}^2} (1 + \langle \tilde{\eta}' \rangle |z_1|^2)^{-1} \left(1 + \frac{(\xi_1 - \tilde{\eta}_1)^2}{\langle \tilde{\eta}' \rangle}\right)^{-1} \|v(z_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})} dz_1 d\xi_1. \end{aligned}$$

An integration in ξ_1 gives

$$\begin{aligned} \|\theta_Y^{w\Sigma} \gamma(\varphi_Y^w v)\|_{L^2(\mathbb{R}^{n-1})} &\leq C \Delta(Y, \pi(\tilde{Y}))^{-N} (1 + \langle \tilde{\eta}' \rangle |\tilde{y}_1|^2)^{-N} \langle \tilde{\eta}' \rangle^{\frac{1}{2}} \\ &\quad \times \int_{\mathbb{R}^2} (1 + \langle \tilde{\eta}' \rangle |z_1|^2)^{-1} \|v(z_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})} dz_1. \end{aligned}$$

Cauchy-Schwarz inequality implies that

$$\int_{\mathbb{R}^2} (1 + \langle \tilde{\eta}' \rangle |z_1|^2)^{-1} \|v(z_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})} dz_1 \leq \langle \tilde{\eta}' \rangle^{-\frac{1}{4}} \|v\|_{L^2(\mathbb{R}^n)}.$$

This concludes the proof of Lemma 5.1.

Now, let us estimate $\|\gamma(u)\|_{H(m^{s-\frac{1}{2}})}$. By definition of the norm $H(m^{s-\frac{1}{2}})$, we have to estimate

$$\int_{T^*\mathbb{R}^{n-1}} m^{2s-1}(Y) \|\theta_Y^{w\Sigma} \gamma(u)\|_{L^2(\Sigma)}^2 dY.$$

Using the fact that

$$\theta_Y^{w\Sigma} \gamma(u) = \int_{T^*\mathbb{R}^n} \theta_Y^{w\Sigma} \gamma(\varphi_Y^w \psi_Y^w u) d\tilde{Y},$$

and applying Lemma 5.1 with $v = \psi_Y^w u$, we found that for any integer N , a constant C exists such that

$$\begin{aligned} \|\gamma(u)\|_{H(m^{s-\frac{1}{2}})}^2 &\leq C \int_{T^*\mathbb{R}^{n-1}} m^{2s-1}(Y) \\ &\quad \times \left(\int_{T^*\mathbb{R}^n} \langle \tilde{\eta}' \rangle^{\frac{1}{4}} \Delta(Y, \pi(\tilde{Y}))^{-N} (1 + \langle \tilde{\eta}' \rangle |\tilde{y}_1|^2)^{-N} \|\psi_Y^w u\|_{L^2(\mathbb{R}^n)} d\tilde{Y} \right)^2 dY. \end{aligned}$$

The fact that M is a \tilde{g} -weight implies that, for any real s , a constant C and an integer N_0 exist such that

$$1 \leq C (1 + \langle \tilde{\eta}' \rangle |\tilde{y}_1|^2)^{-N_0} \frac{M^s(\tilde{Y})}{M^s(0, \tilde{y}', \tilde{\eta}_1, \tilde{\eta}')}.$$

From this, we deduce that for any integer N , a constant C exists such that

$$\begin{aligned} \|\gamma(u)\|_{H(m^{s-\frac{1}{2}})}^2 &\leq C \int_{T^*\mathbb{R}^{n-1}} m^{2s-1}(Y) \mathcal{I}_N^2(Y) dY \quad \text{with} \\ \mathcal{I}_N(Y) &\stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^n} \frac{\langle \tilde{\eta}' \rangle^{\frac{1}{4}} (1 + \langle \tilde{\eta}' \rangle |\tilde{y}_1|^2)^{-N}}{M^s(0, \tilde{y}', \tilde{\eta}_1, \tilde{\eta}')} M^s(\tilde{Y}) \|\psi_Y^w u\|_{L^2(\mathbb{R}^n)} \Delta(Y, \pi(\tilde{Y}))^{-N} d\tilde{Y}. \end{aligned}$$

Cauchy-Schwarz inequality for the measure $\Delta(Y, \pi(\tilde{Y}))^{-N} d\tilde{Y}$ implies that

$$\begin{aligned} \mathcal{I}_N^2(Y) &\leq C \mathcal{J}_N(Y) \mathcal{K}_N(Y) \quad \text{with} \\ \mathcal{J}_N(Y) &\stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^n} \frac{\langle \tilde{\eta}' \rangle^{\frac{1}{2}} (1 + \langle \tilde{\eta}' \rangle |\tilde{y}_1|^2)^{-2N}}{M^{2s}(0, \tilde{y}', \tilde{\eta}_1, \tilde{\eta}')} \Delta(Y, \pi(\tilde{Y}))^{-N} d\tilde{Y} \quad \text{and} \\ \mathcal{K}_N(Y) &\stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^n} M^{2s}(\tilde{Y}) \|\psi_Y^w u\|_{L^2(\mathbb{R}^n)}^2 \Delta(Y, \pi(\tilde{Y}))^{-N} d\tilde{Y}. \end{aligned}$$

Integrating in \tilde{y}_1 and changing variable $Z = \pi(\tilde{Y})$, we find that

$$\mathcal{J}_N(Y) \leq \int_{T^*\mathbb{R}^{n-1}} \left(\int_{\tilde{\eta}_1} \frac{1}{M^{2s}(0, z, \tilde{\eta}_1, \zeta)} d\tilde{\eta}_1 \right) \Delta(Y, Z)^{-N} dZ.$$

But by definition of the two weights M and m , we have

$$M(0, z, \tilde{\eta}_1, \zeta)^2 = \tilde{\eta}_1^2 + m^2(z, \zeta).$$

As $s > \frac{1}{2}$, it turns out that

$$\int_{\tilde{\eta}_1} \frac{1}{M^{2s}(0, z, \tilde{\eta}_1, \zeta)} d\tilde{\eta}_1 = \frac{C_s}{m^{2s-1}(Z)} \quad \text{with} \quad C_s = \int_{\mathbb{R}} \frac{dt}{(1+t^2)^s}.$$

So we have

$$\mathcal{J}_N(Y) \leq C_s \int_{T^*\mathbb{R}^{n-1}} \frac{1}{m^{2s-1}(Z)} \Delta(Y, Z)^{-N} dZ.$$

The fact that m is a g -weight and Lemma 2.1 imply that

$$\mathcal{J}_N(Y) \leq \frac{C}{m^{2s-1}(Y)}. \quad (5.5)$$

From this we deduce that

$$\|\gamma(u)\|_{H(m^{s-\frac{1}{2}})}^2 \leq C \int_{T^*\mathbb{R}^{n-1} \times T^*\mathbb{R}^n} \Delta(Y, \pi(\tilde{Y}))^{-N} M^{2s}(\tilde{Y}) \|\psi_{\tilde{Y}}^w u\|_{L^2(\mathbb{R}^n)}^2 dY d\tilde{Y}.$$

By integration on Y , we deduce from Lemma 2.1 that

$$\|\gamma(u)\|_{H(m^{s-\frac{1}{2}})}^2 \leq C \int_{T^*\mathbb{R}^n} M^{2s}(\tilde{Y}) \|\psi_{\tilde{Y}}^w u\|_{L^2(\mathbb{R}^n)}^2 d\tilde{Y}.$$

So the continuity of γ is proved.

5.2. The trace lifting operator. Let us define the trace lifting operator. Let us remember the case of usual Sobolev spaces. If v belongs to $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ with s greater than $1/2$, then we define

$$u = (2\pi)^{-(n-1)} C_s^{-1} \mathcal{F}^{-1} \left(\frac{(1+|\xi'|^2)^{s-\frac{1}{2}}}{(1+|\xi|^2)^s} \hat{v}(\xi') \right).$$

It is obvious that $\|u\|_{H^s} \leq C\|v\|_{H^{s-\frac{1}{2}}}$. Using the fact that the value in 0 is the integral of the Fourier transform of the origin, we have that $\gamma(u) = v$.

Again we have to substitute to $|\xi'|$ and $|\xi|$ the weights m and M . Let χ be a function of $\mathcal{D}(\mathbb{R})$ with value 1 near 0. Let us define the operator R_χ on $\mathcal{S}(\Sigma)$ by the following formula

$$\begin{aligned} (R_\chi v)(x_1, x') &\stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^{n-1}} \mu_Y(x_1) (\phi_Y \#_{T^*\Sigma} \theta_Y)^{w_\Sigma} v(x') dY \quad \text{with} \\ \mu_Y(x_1) &\stackrel{\text{def}}{=} C_s^{-1} m(Y)^{2s-1} \chi(x_1 \langle \eta \rangle^{\frac{1}{2}}) \int_{\mathbb{R}} e^{ix_1 \xi_1} M^{-2s}(0, y, \xi_1, \eta) d\xi_1. \end{aligned} \quad (5.6)$$

As $M^2(0, y, \xi, \eta) = \xi_1^2 + m^2(y, \eta)$, and as $\chi(0) = 1$, we have $\gamma \circ R_\chi = \text{Id}$. In order to prove Theorem 1.2, it is now enough to prove that R_χ can be extended in a continuous operator from $H(m^{s-\frac{1}{2}})$ into $H(M^s)$. The key lemma is the following.

Lemma 5.2. *For any integer N , a constant C exists such that, for any function f in the space $L^2(\mathbb{R}^{n-1})$, we have*

$$\begin{aligned} \|\psi_{\tilde{Y}}^w(\mu_Y(x_1)\phi_Y^{w_\Sigma} f)\|_{L^2(\mathbb{R}^n)} &\leq C\Delta(Y, \pi(\tilde{Y}))^{-N} \frac{m^{2s-1}(Y)}{M^s(\tilde{Y})} \\ &\quad \times \langle \eta \rangle^{-\frac{1}{4}} (1 + \langle \eta \rangle \tilde{y}_1^2)^{-1} \mathcal{J}_Y(\tilde{\eta}_1) \|f\|_{L^2(\mathbb{R}^{n-1})} \end{aligned}$$

where \mathcal{J}_Y is a positive function on \mathbb{R} such that

$$\int_{\mathbb{R}} \mathcal{J}_Y^2(t) dt \leq C \frac{\langle \eta \rangle}{m^{2s-1}(Y)}.$$

The continuity of R_χ follows easily from this lemma. Let us apply it with $f = \theta_Y^{w_\Sigma} v$. It gives us that

$$\begin{aligned} \|\psi_{\tilde{Y}}^w R_\chi v\|_{L^2(\mathbb{R}^n)} &\leq CM^{-s}(\tilde{Y}) \int_{T^*\mathbb{R}^{n-1}} \langle \eta \rangle^{-\frac{1}{4}} (1 + \langle \eta \rangle \tilde{y}_1^2)^{-1} \mathcal{J}_Y(\tilde{\eta}_1) \\ &\quad \times \|\theta_Y^{w_\Sigma} v\|_{L^2(\mathbb{R}^{n-1})} \Delta(Y, \pi(\tilde{Y}))^{-N} m^{2s-1}(Y) dY. \end{aligned}$$

Cauchy-Schwarz inequality for the measure $\Delta(Y, \pi(\tilde{Y}))^{-N} m^{2s-1}(Y) dY$ implies that

$$\begin{aligned} \|\psi_{\tilde{Y}}^w R_\chi v\|_{L^2(\mathbb{R}^n)}^2 &\leq CM^{-2s}(\tilde{Y}) \int_{T^*\mathbb{R}^{n-1}} \|\theta_Y^{w_\Sigma} v\|_{L^2(\mathbb{R}^{n-1})}^2 \Delta(Y, \pi(\tilde{Y}))^{-N} m^{2s-1}(Y) dY \\ &\quad \times \int_{T^*\mathbb{R}^{n-1}} \langle \eta \rangle^{-\frac{1}{2}} (1 + \langle \eta \rangle \tilde{y}_1^2)^{-1} \mathcal{J}_Y^2(\tilde{\eta}_1) \Delta(Y, \pi(\tilde{Y}))^{-N} m^{2s-1}(Y) dY. \end{aligned}$$

By definition of the norm on the space $H(M^s)$, we deduce from this that

$$\begin{aligned} \|R_\chi v\|_{H(M^s)}^2 &\leq C \int_{T^*\mathbb{R}^{n-1}} F_1(Z) F_2(Z) dZ \quad \text{with} \\ F_1(Z) &\stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^{n-1}} \|\theta_Y^{w_\Sigma} v\|_{L^2(\mathbb{R}^{n-1})}^2 \Delta(Y, Z)^{-N} m^{2s-1}(Y) dY \quad \text{and} \\ F_2(Z) &\stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^{n-1}} \langle \eta \rangle^{-\frac{1}{2}} \left(\int_{\mathbb{R}^2} (1 + \langle \eta \rangle t^2)^{-1} \mathcal{J}_Y^2(\tau) \Delta(Y, Z)^{-N} m^{2s-1}(Y) dt d\tau \right) dY. \end{aligned}$$

But we have

$$\int_{\mathbb{R}^2} (1 + \langle \eta \rangle t^2)^{-1} \mathcal{J}_Y^2(\tau) dt d\tau \leq \frac{C \langle \eta \rangle^{\frac{1}{2}}}{m^{2s-1}(Y)}.$$

Lemma 2.1 implies that the function F_2 is bounded on $T^*\mathbb{R}^{n-1}$. Applying again Lemma 2.1, we get

$$\begin{aligned} \|R_\chi v\|_{H(M^s)}^2 &\leq C \int_{T^*\mathbb{R}^{n-1} \times T^*\mathbb{R}^{n-1}} m^{2s-1}(Y) \|\theta_Y^{w_\Sigma} v\|_{L^2(\mathbb{R}^{n-1})}^2 \Delta(Y, Z)^{-N} dY dZ \\ &\leq C \int_{T^*\mathbb{R}^{n-1}} m^{2s-1}(Y) \|\theta_Y^{w_\Sigma} v\|_{L^2(\mathbb{R}^{n-1})}^2 dY \\ &\leq C \|v\|_{H(m^{s-\frac{1}{2}})}^2. \end{aligned}$$

So we have proved that Lemma 5.2 implies that R_χ is continuous which concludes the proof of Theorem 1.2.

Now let us prove Lemma 5.2. Let us state

$$F_{\tilde{Y}, Y} \stackrel{\text{def}}{=} \psi_{\tilde{Y}}^w(\mu_Y(x_1)\phi_Y^{w_\Sigma} f).$$

By Weyl quantization formula and definition of the function μ_Y , we infer that

$$\begin{aligned} F_{\tilde{Y},Y} &= C_s^{-1} m^{2s-1}(Y) \tilde{F}_{\tilde{Y},Y} \quad \text{with} \\ \tilde{F}_{\tilde{Y},Y}(x_1, x') &\stackrel{\text{def}}{=} (2\pi)^{-1} \int_{\mathbb{R}^3} e^{ix_1\tau - it(\tau - \tau')} \frac{\chi(t\langle\eta\rangle^{\frac{1}{2}})}{M^{2s}(0, y, \tau', \eta)} \\ &\quad \times \left(\psi_{\tilde{Y}} \left(\frac{x_1 + t}{2}, \cdot, \tau, \cdot \right) \#_{T^*\Sigma} \phi_Y \right)^{w_\Sigma} f(x') dt d\tau d\tau'. \end{aligned}$$

By integration by part with respect to the vector $\langle\tilde{\eta}'\rangle^{\frac{1}{2}}\partial_\tau$, whose $\tilde{g}_{\tilde{Y}}$ -length is less than 1, it turns out that, for any integer N ,

$$\begin{aligned} \tilde{F}_{\tilde{Y},Y}(x_1, x') &= (2\pi)^{-1} \int_{\mathbb{R}^3} e^{ix_1\tau - it(\tau - \tau')} \frac{\chi(t\langle\eta\rangle^{\frac{1}{2}})}{M^{2s}(0, y, \tau', \eta)} (1 + \langle\tilde{\eta}'\rangle(x_1 - t)^2)^{-N} \\ &\quad \times \left(\psi_{\tilde{Y}}^{(N)} \left(\frac{x_1 + t}{2}, \cdot, \tau, \cdot \right) \#_{T^*\Sigma} \phi_Y \right)^{w_\Sigma} f(x') dt d\tau d\tau'. \end{aligned}$$

where $\psi_{\tilde{Y}}^{(N)} \stackrel{\text{def}}{=} (\text{Id} - \langle\tilde{\eta}'\rangle\partial_\tau^2)^N \psi_{\tilde{Y}}$.

The fact that the metric \tilde{g} is symplectic (i.e. $\tilde{g}^\sigma = \tilde{g}$) allows us to do integration by parts with respect to the vector

$$T_{\tilde{Y},Y} \stackrel{\text{def}}{=} \left(\frac{1}{\langle\tilde{\eta}'\rangle + \langle\eta\rangle} \right)^{\frac{1}{2}} \partial_t$$

whose $\tilde{g}_{\tilde{Y}}$ -length is less than 1. Moreover we have

$$\begin{aligned} |T_{\tilde{Y},Y}^j(\langle\tilde{\eta}'\rangle(x_1 - t)^2)| &\leq (1 + \langle\tilde{\eta}'\rangle(x_1 - t)^2)^{\frac{1}{2}} \quad \text{and obviously} \\ |T_{\tilde{Y},Y}^j(\chi(\langle\eta\rangle^{\frac{1}{2}}t))| &\leq \|\chi^{(j)}\|_{L^\infty}. \end{aligned}$$

From this we deduce that

$$\begin{aligned} \tilde{F}_{\tilde{Y},Y}(x_1, x') &= \int_{\mathbb{R}^3} e^{ix_1\tau - it(\tau - \tau')} \frac{1}{M^{2s}(0, y, \tau', \eta)} (1 + \langle\tilde{\eta}'\rangle(x_1 - t)^2)^{-N} \\ &\quad \times \left(1 + \left(\frac{1}{\langle\tilde{\eta}'\rangle + \langle\eta\rangle} \right) (\tau - \tau')^2 \right)^{-N} (\mathcal{A}_{\tilde{Y},Y}^{(N)} f)(x_1, x') dt d\tau d\tau' \end{aligned}$$

with

$$\begin{aligned} (\mathcal{A}_{\tilde{Y},Y}^{(N)} f)(x_1, x') &= \sum_{j+k \leq 2N} \mathcal{A}_j^{(N)}(t, \tau, \tau', \tilde{Y}, Y) \chi^{(j)}(t\langle\eta\rangle^{\frac{1}{2}}) \\ &\quad \times \left(\psi_{\tilde{Y},Y}^{N,k} \left(\frac{x_1 + t}{2}, \cdot, \tau, \cdot \right) \#_{T^*\Sigma} \phi_Y \right)^{w_\Sigma} f(x') \end{aligned}$$

where the functions $\mathcal{A}_j^{(N)}$ are bounded and positive and where

$$\psi_{\tilde{Y},Y}^{N,k} = T_{\tilde{Y},Y}^k (\text{Id} - \langle\tilde{\eta}'\rangle\partial_\tau^2)^N \psi_{\tilde{Y}}.$$

But we know that M is a \tilde{g} -weight. So a constant C and an integer N_s exist such that

$$M^{-s}(0, y, \tau', \eta) \leq C M^{-s}(\tilde{Y}) \left(1 + \frac{1}{\langle\eta\rangle} |\tau' - \tilde{\eta}_1|^2 \right)^{N_s} (1 + \langle\eta\rangle \tilde{y}_1^2)^{N_s} \Delta(Y, \pi(\tilde{Y}))^{N_s}. \quad (5.7)$$

From estimates (5.4) and (5.7), from the fact that χ is compactly supported, from Lemma 2.2 and from the L^2 estimate, we deduce that, for any couple of integers (N, N') , a constant C exists such that

$$\begin{aligned} \|\tilde{F}_{\tilde{Y}, Y}(x_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})} &\leq C \Delta(Y, \pi(\tilde{Y}))^{-N'} M^{-s}(\tilde{Y}) \|f\|_{L^2(\mathbb{R}^{n-1})} \\ &\quad \times \int_{\mathbb{R}^3} I_{\tilde{Y}, Y}^{1, N}(t) I_{\tilde{Y}, Y}^{2, N}(\tau, \tau') M^{-s}(0, y, \tau', \eta) dt d\tau d\tau' \end{aligned}$$

with

$$\begin{aligned} I_{\tilde{Y}, Y}^{1, N}(t) &\stackrel{\text{def}}{=} (1 + \langle \tilde{\eta}' \rangle (x_1 - t)^2)^{-N} (1 + \langle \eta \rangle t^2)^{-N} (1 + \langle \eta \rangle \tilde{y}_1^2)^{N_s} \\ &\quad \times \left(1 + \langle \tilde{\eta}' \rangle \left(\frac{x_1 + t}{2} - \tilde{y}_1 \right) \right)^{-N} \quad \text{and} \\ I_{\tilde{Y}, Y}^{2, N}(\tau, \tau') &\stackrel{\text{def}}{=} \left(1 + \frac{1}{\langle \tilde{\eta}' \rangle + \langle \eta \rangle} (\tau - \tau')^2 \right)^{-N} \left(1 + \frac{1}{\langle \tilde{\eta}' \rangle} (\tau - \tilde{\eta}_1)^2 \right)^{-N} \\ &\quad \times \left(1 + \frac{1}{\langle \eta \rangle} (\tau' - \tilde{\eta}_1)^2 \right)^{N_s}. \end{aligned}$$

A constant C and an integer N_0 exist such that

$$\left(\frac{\langle \eta \rangle}{\langle \tilde{\eta}' \rangle} \right)^{\pm} \leq C \Delta(\pi(\tilde{Y}), Y)^{N_0}.$$

Thanks to the inequality of the triangle we have for any N ,

$$\begin{aligned} I_{\tilde{Y}, Y}^{1, N}(t) &\leq C^N \Delta(\pi(\tilde{Y}), Y)^{3NN_0} (1 + \langle \eta \rangle x_1^2)^{-N} (1 + \langle \eta \rangle t^2)^{-N} (1 + \langle \eta \rangle \tilde{y}_1^2)^{-N} \quad \text{and} \\ I_{\tilde{Y}, Y}^{2, N}(\tau, \tau') &\leq C^N \Delta(\pi(\tilde{Y}), Y)^{2NN_0} \left(1 + \frac{1}{\langle \eta \rangle} (\tau - \tau')^2 \right)^{N_s - N} \left(1 + \frac{1}{\langle \eta \rangle} (\tau - \tilde{\eta}_1)^2 \right)^{N_s - N}. \end{aligned}$$

Let us apply the above inequalities with $N = N_s + 1$. We deduce from this that for any integer N , a constant C exists such that

$$\begin{aligned} \|\tilde{F}_{\tilde{Y}, Y}(x_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})} &\leq C \Delta(Y, \pi(\tilde{Y}))^{-N} M^{-s}(\tilde{Y}) \|f\|_{L^2(\mathbb{R}^{n-1})} (1 + \langle \eta \rangle x_1^2)^{-1} \\ &\quad \times (1 + \langle \eta \rangle \tilde{y}_1^2)^{-1} \int_{\mathbb{R}^3} I_Y(t, \tau, \tau') M^{-s}(0, y, \tau', \eta) dt d\tau d\tau' \end{aligned}$$

with

$$I_Y(t, \tau, \tau') \stackrel{\text{def}}{=} (1 + \langle \eta \rangle t^2)^{-1} \left(1 + \frac{1}{\langle \eta \rangle} (\tau - \tau')^2 \right)^{-1} \left(1 + \frac{1}{\langle \eta \rangle} (\tau - \tilde{\eta}_1)^2 \right)^{-1}.$$

Cauchy-Schwarz inequality for the measure $I_Y(t, \tau, \tau') dt d\tau d\tau'$ implies that

$$\begin{aligned} \int_{\mathbb{R}^3} I_Y(t, \tau, \tau') M^{-s}(0, y, \tau', \eta) dt d\tau d\tau' &\leq \left(\int_{\mathbb{R}^3} I_Y(t, \tau, \tau') dt d\tau d\tau' \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\mathbb{R}^3} I_Y(t, \tau, \tau') M^{-2s}(0, y, \tau', \eta) dt d\tau d\tau' \right)^{\frac{1}{2}}. \end{aligned}$$

A straightforward computation gives

$$\int_{\mathbb{R}^3} I_Y(t, \tau, \tau') M^{-2s}(0, y, \tau', \eta) dt d\tau d\tau' \leq \mathcal{J}_Y^2(\tilde{\eta}_1) \langle \eta \rangle^{-\frac{1}{2}}$$

with

$$\mathcal{J}_Y(\tilde{\eta}_1) \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^2} \left(1 + \frac{1}{\langle \eta \rangle} (\tau - \tau')^2 \right)^{-1} \left(1 + \frac{1}{\langle \eta \rangle} (\tau - \tilde{\eta}_1)^2 \right)^{-1} M^{-2s}(0, y, \tau', \eta) d\tau d\tau' \right)^{\frac{1}{2}}.$$

It turns out that

$$\begin{aligned} \|\tilde{F}_{\tilde{Y}, Y}(x_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})} &\leq C \Delta(Y, \pi(\tilde{Y}))^{-N'+N_1} M^{-s}(\tilde{Y}) \|f\|_{L^2(\mathbb{R}^{n-1})} (1 + \langle \eta \rangle x_1^2)^{-1} \\ &\quad \times (1 + \langle \eta \rangle \tilde{y}_1^2)^{-1} \mathcal{J}_Y(\tilde{\eta}_1). \end{aligned}$$

Once observed that

$$\int_{\mathbb{R}} \mathcal{J}_Y(t)^2 dt = C m^{-(2s-1)}(Y) \langle \eta \rangle,$$

we get the lemma by integration in x_1 . This completes the proof of Theorem 1.2.

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