REGULARITY OF WEAK SOLUTIONS FOR A CLASS OF INFINTELY DEGENERATE ELLIPTIC SEMILINEAR EQUATIONS

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1. Notations and results

In this work, we study the C^∞ regularity of weak solution of Dirichlet problems for a class of second order semi-linear infinitely degenerate elliptic equation. Consider a system of vector fields $X = (X_1, \cdots, X_m)$ defined on an open domain $\widetilde{\Omega} \subset \mathbb{R}^n$. In the infinite degenerate case, the following is called logarithmic regularity estimate,

$$\|(\log \Lambda)^s u\|_{L^2}^2 \le C\Big\{\sum_{j=1}^m \|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2\Big\}, \qquad \forall u \in C_0^\infty(\widetilde{\Omega}), \tag{1.1}$$

where $\Lambda = (e^2 + |D|^2)^{1/2} =< D >$. If the system X satisfies the finite type of Hörmander's condition then (1.1) holds for any real $s > 0$. On the other hand (1.1) admits the infinite degeneracy of the system X, and the estimate (1.1) with $s > 1$ always implies the interior hypoellipticity of the second order operator $\triangle_X = \sum_{j=1}^m X_j^* X_j$, where X_j^* is the formal adjoint of X_j (see [7]). Some sufficient conditions for this estimate can be seen in the Appendix of [10]. The typical example for (1.1) is the system in $\mathbb{R}^3$ such as $X_1 = \partial_{x_1}, X_2 = \partial_{x_2}$, $X_3 = \exp(-|x_1|^{-1/s})\partial_{x_3}$ with $s > 0$ (see [5, 6, 7]). The operator $\triangle_X$ for this example degenerates infinitely on $\Gamma_0 = \{x_1 = 0\}$.

Associated with the system of vector fields $X = (X_1, \cdots, X_m)$, we define function spaces :

$$H^1_X(\widetilde{\Omega}) = \Big\{u \in L^2(\widetilde{\Omega}); X_j u \in L^2(\widetilde{\Omega}), j = 1, \cdots, m\Big\}.$$

We say that $u \in H^1_{X,\mathrm{loc}}(\widetilde{\Omega})$, if $\alpha u \in H^1_X(\widetilde{\Omega})$ for any $\alpha \in C_0^\infty(\widetilde{\Omega})$. Take $\Omega \subset\subset \widetilde{\Omega}$ and suppose that $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X. Here, for a smooth surface Γ of $\widetilde{\Omega}$, we say that Γ is non characteristic for the system of vector fields X, if for any point $x_0 \in \Gamma$ there exists at least one vector field of $X_1, \cdots, X_m$ which is transversal to Γ at x_0. We define $H^1_{X,0}(\Omega) = \big\{u \in H^1_X(\Omega); u|_{\partial\Omega} = 0\big\}$, as in [10], this is also a Hilbert space, and $C_0^\infty(\Omega)$ is dense in $H^1_{X,0}(\Omega)$.

We consider the following Dirichlet problem;

$$\triangle_X u + X_0 u = F(x, u), \qquad \text{in } \Omega \tag{1.2}$$

$$u|_{\partial\Omega} = g, \tag{1.3}$$

where $F \in C^\infty(\bar{\Omega} \times \mathbb{R})$ and X_0 a vector fields on $\widetilde{\Omega}$. As for the linear hypoellipticity, it is known that the estimate (1.1) with $s = 1$ is not sufficient for hypoellipticity, but the following weak form of estimates is sufficient : For any small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|\log \Lambda\, v\|_{L^2}^2 \le \varepsilon \sum_{j=1}^m \|X_j v\|_{L^2}^2 + C_\varepsilon \|v\|_{L^2}^2, \quad \forall v \in C_0^\infty(\widetilde{\Omega}). \tag{1.4}$$

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The estimate (1.1) with $s > 1$ implies immediately the estimate (1.4) by interpolation. We have a very simple example which satisfies the estimate (1.4), but not (1.1) for any $s > 1$. It is the system in $\mathbb{R}^3$ such as $X_1 = \partial_{x_1}, X_2 = \partial_{x_2},\ X_3 = \exp(-(|x_1||\log|x_1||)^{-1})\partial_{x_3}$, (see [6, 10]).

We have now the following nonlinear hypoelliptic results :

Theorem 1.1. *Suppose that the system of vector fields X satisfy the logarithmic regularity estimate (1.4), and $u \in H^1_{X,\,loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ is a weak solution of equation (1.2). Then $u \in C^\infty(\Omega)$.*

Moreover if $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X, and if $u \in H^1_X(\Omega) \cap L^\infty(\Omega)$ is a weak solution of Dirichlet problem (1.2)–(1.3) with $g \in C^\infty(\partial\Omega)$, then $u \in C^\infty(\bar\Omega)$.

Remark : We get also regularity up to the boundary for linear Dirichlet problem if the function F is linear in (1.2).

We give here an example of equation (1.2) coming from a variational problem. From (1.1), we have the following logarithmic Sobolev inequality(see [10]),

$$\int_\Omega |v|^2 \left|\log\left(e + \frac{|v|^2}{\|v\|^2_{L^2}}\right)\right|^{2s-1} \le C_0\Big\{\sum_{j=1}^m \|X_j v\|^2_{L^2} + \|v\|^2_{L^2}\Big\}, \tag{1.5}$$

for all $v \in H^1_{X,0}(\Omega)$. Suppose that $1 \le k < 2(s-1)$, take $A = (a_1, \cdots, a_k) \in \mathbb{R}^k$, and consider the following variational problems :

$$I_A = \inf_{\|v\|_{L^2}=1,\, v\in H^1_{X,0}(\Omega)} \Big\{\sum_{j=1}^m \|X_j v\|^2_{L^2} - \sum_{j=1}^k a_j \int_\Omega |v|^2\big(\log(e+v^2)\big)^j\Big\}. \tag{1.6}$$

We say that the system of vector fields $X = (X_1, \cdots, X_m)$ satisfies the "**non trapping condition**", if the system of vector fields X satisfies the finite type of Hörmander's condition on $\widetilde\Omega$ except for $\varGamma = \cup_{j\in J}\varGamma_j$, a union of smooth surfaces $\varGamma_j$ in $\widetilde\Omega$, provided that $\varGamma$ is non characteristic for X. Here we say that the union $\varGamma = \cup_{j\in J}\varGamma_j$ of smooth surfaces in $\widetilde\Omega$ is non characteristic for X, if for any point $x_0 \in \varGamma$ there exists at least one vector field of $X_1, \cdots, X_m$ which transverses $\varGamma_j$ at x_0 for all $j \in J_0 = \{k \in J; x_0 \in \varGamma_k\}$. The example with infinite degeneracy on the union $\varGamma = \cup_j \varGamma_j$ is the system in $\mathbb{R}^2$ such as $X_1 = \partial_{x_1}, X_2 = \exp(-(x_1^2\sin^2(\frac{\pi}{x_1}))^{-1/2s})\partial_{x_2}$, and we see that if $\varGamma_j = \{x_1 = \frac{1}{j}\}, j \in \mathbb{Z}\setminus\{0\}, \varGamma_0 = \{x_1 = 0\}$, then X_1 is transversal to all $\varGamma_j, j \in \mathbb{Z}$ and X_2 vanishes infinitely on $\varGamma = \cup_{j\in\mathbb{Z}}\varGamma_j$.

The non trapping condition and Bony's maximal principle implies immediately the following first Poincaré inequality :

$$\|v\|^2_{L^2(\Omega)} \le C_0 \sum_{j=1}^m \|X_j v\|^2_{L^2(\Omega)}, \quad \forall v \in C_0^\infty(\widetilde\Omega). \tag{1.7}$$

We have

Theorem 1.2. *Suppose that $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X. Assume that the system of vector fields X verifies the estimate (1.1) for $s > 3/2$ and satisfies the non trapping condition. Then I_A is an attained minimum in $H^1_{X,0}(\Omega)$, and the minimizer belongs to $C^\infty(\bar\Omega)$.*

In fact, by exactly the same calculus as in [10], the inequality (1.5) and (1.7) give the existence of minimizer $u \in H^1_{X,0}(\Omega)$ for the variational problems (1.6), and the minimizer is a bounded non trivial positive weak solution of the following Euler-Lagrange equation;

$$\triangle_X u = F(u), \qquad u|_{\partial\Omega} = 0, \tag{1.8}$$

with nonlinear term

$$F(t)=\sum_{j=1}^{k}a_j\Big(t\big(\log(e+t^2)\big)^j+\frac{j}{2}\frac{t^3}{e+t^2}\big(\log(e+t^2)\big)^{j-1}\Big)+b_0t\in C^\infty(\mathbb{R}),$$

where b_0 is a constant depending on the minimizer u. See [11] for the detail.

2. Littlewood-Paley theory for logarithmic Sobolev spaces

Let $\ell>0$, and define the following logarithmic Sobolev's space :

$$H^{\log}_\ell(\mathbb{R}^n)=\{u\in L^2(\mathbb{R}^n);\big(\log\langle\xi\rangle\big)^\ell\widehat{u}(\xi)\in L^2(\mathbb{R}^n)\},$$

where $\langle\xi\rangle=(e^2+|\xi|^2)^{1/2}$. We study now the Littlewood-Paley decomposition for this function space as in [1, 13].

Let $\mathcal{C}_0=\{\xi\in\mathbb{R}^n; e<\langle\xi\rangle<e^3\}, \mathcal{C}_k=e^k\mathcal{C}_0,\ k\in\mathbb{N}, \mathcal{C}_{-1}=\{\xi\in\mathbb{R}^n;\langle\xi\rangle<e^2\}$, there exist $\psi\in C^\infty_0(]0,e^2[), \varphi\in C^\infty_0(]e,e^3[)$ such that

$$\psi(\langle\xi\rangle)+\sum_{j=0}^{\infty}\varphi(e^{-j}\langle\xi\rangle)=1,\quad \forall\xi\in\mathbb{R}^n.$$

For $f\in L^2(\mathbb{R}^n)$, we set

$$\Delta_{-1}f=\psi(\Lambda)f,\ \ \Delta_jf=\varphi(e^{-j}\Lambda)f,\ \ j\in\mathbb{N}.$$

Then $f=\sum\Delta_jf$ in $L^2(\mathbb{R}^n)$, and we have the following characterization for function space $H^{\log}_\ell(\mathbb{R}^n)$.

Lemma 2.1. *For $\ell>0$, we have that*

1) if $u\in H^{\log}_\ell(\mathbb{R}^n)$, then

$$\|\Delta_ju\|_{L^2(\mathbb{R}^n)}\le c_jj^{-\ell},\ \ \|\{c_j\}\|_{\ell^2}\le\|u\|_{H^{\log}_\ell(\mathbb{R}^n)}.$$

2) if $u\in L^2(\mathbb{R}^n)$, and

$$\|\Delta_ju\|_{L^2(\mathbb{R}^n)}\le c_jj^{-\ell},\ \ \{c_j\}\in\ell^2,$$

then $u\in H^{\log}_\ell(\mathbb{R}^n)$, and for any $S\ge1$

$$S^{2\ell}\|(\log\Lambda)^\ell u\|^2_{L^2(\mathbb{R}^n)}\le C_1\ell^{2\ell}\|u\|^2_{L^2(\mathbb{R}^n)}+C_2^SS^{2\ell}\|\{c_j\}\|^2_{\ell^2},$$

with C_1,C_2 independent of S,ℓ and u.

Proof : 1) For $u\in H^{\log}_\ell(\mathbb{R}^n)$, we have

$$\|\Delta_ju\|^2_{L^2}=\int\varphi(e^{-j}\langle\xi\rangle)^2|\widehat{u}(\xi)|^2d\xi\le j^{-2\ell}\int_{\mathcal{C}_j}(\log\langle\xi\rangle)^{2\ell}\varphi(e^{-j}\langle\xi\rangle)^2|\widehat{u}(\xi)|^2d\xi.$$

We set

$$c_j^2=\int_{\mathcal{C}_j}(\log\langle\xi\rangle)^{2\ell}\varphi(e^{-j}\langle\xi\rangle)^2|\widehat{u}(\xi)|^2d\xi.$$

Then the fact $\psi^2(\xi)+\sum_{j=0}^\infty\varphi(e^{-j}\langle\xi\rangle)^2\le1$ implies that

$$\sum_{j=-1}^{\infty}c_j^2\le\int_{\mathbb{R}^n}(\log\langle\xi\rangle)^{2\ell}\sum_{j=-1}^{\infty}\varphi(e^{-j}\langle\xi\rangle)^2|\widehat{u}(\xi)|^2d\xi\le\|u\|^2_{H^{\log}_\ell(\mathbb{R}^n)}.$$

2) For $S > 0$, we have

$$
\begin{aligned}
S^{2\ell}\|(\log\Lambda)^\ell u\|_{L^2}^2 &\leq 3\sum_{S(j+3)\leq \ell}(S(j+3))^{2\ell}\int \varphi(e^{-j}\langle\xi\rangle)^2|\widehat{u}(\xi)|^2 d\xi \\
&+ 3\sum_{S(j+3)>\ell}(S(j+3))^{2\ell}\|\Delta_j u\|_{L^2}^2 \\
&\leq 3\ell^{2\ell}\|u\|_{L^2}^2 + 3S^{2\ell}\sum_{S(j+3)>\ell}(j+3)^{2\ell}j^{-2\ell}c_j^2 \\
&\leq 3\ell^{2\ell}\|u\|_{L^2}^2 + 3S^{2\ell}\sum_j (1+3/j)^{2S(j+3)}c_j^2 \\
&\leq 3\ell^{2\ell}\|u\|_{L^2}^2 + 3(e^6 2^6)^S S^{2\ell}\|\{c_j\}\|_{L^2}^2.
\end{aligned}
$$

As in the classical case, for the second part in the preceding lemma, we have more general results

Lemma 2.2. *Suppose that $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of $L^2(\mathbb{R}^n)$, with $\operatorname{Supp}\widehat{u}_k \subset B(0, Ke^k)$ and for $\ell > 1/2$,*

$$\|u_k\|_{L^2(\mathbb{R}^n)} \leq c_k k^{-\ell}, \quad \{c_k\}\in \ell^2.$$

Then $u = \sum_k u_k \in H^{\log}_{\ell-1/2}(\mathbb{R}^n)$ and for any $S \geq 1$,

$$S^{2\ell-1}\|(\log\Lambda)^{\ell-1/2}u\|_{L^2(\mathbb{R}^n)}^2 \leq C_1(\ell-1/2)^{2\ell-1}\|u\|_{L^2(\mathbb{R}^n)}^2 + C_2^S S^{2\ell-1}(2\ell-1)\|\{c_k\}\|_{\ell^2}^2,$$

with C_1, C_2 independent of S, ℓ and u.

Remark : We have a loss of $1/2$ for the index because of the logarithmic sum.

Proof : Since $\ell > 1/2$, we have that $u = \sum_k u_k$ converges in $L^2(\mathbb{R}^n)$, in fact,

$$\|u\|_{L^2} \leq \sum_k \|u_k\|_{L^2} \leq \sum_k c_k k^{-\ell} \leq \|\{c_k\}\|_{\ell^2}\big(\sum_k k^{-2\ell}\big)^{1/2}.$$

We suppose now $S = 1$, since the general case of S is similar as lemma 2.1. We set

$$u = \sum_{j=-1}^{\infty}\Delta_j u = \sum_{j=-1}^{\infty} v_j = \sum_{j=-1}^{\infty}\sum_k \Delta_j u_k.$$

Then

$$
\begin{aligned}
\|u\|_{H^{\log}_{\ell-1/2}(\mathbb{R}^n)}^2 &\leq 2\|\sum_{j+3\leq \ell-1/2}\Delta_j u\|_{H^{\log}_{\ell-1/2}(\mathbb{R}^n)}^2 + 2\|\sum_{j+3>\ell-1/2}\Delta_j u\|_{H^{\log}_{\ell-1/2}(\mathbb{R}^n)}^2 \\
&\leq 2(\ell-1/2)^{2\ell-1}\|u\|_{L^2}^2 + 2\sum_{j+3>\ell-1/2}(j+3)^{2\ell-1}\|\Delta_j u\|_{L^2}^2.
\end{aligned}
$$

On the other hand, there exists $N_1 > 0$ (depending only on K) such that for any $j > k + N_1$, $\mathcal{C}_j \bigcap B(0, Ke^k) = \emptyset$, then $\Delta_j u_k = 0$. We have $v_j = \sum_{k\geq j-N_1}\Delta_j u_k$, and

$$
\begin{aligned}
\|\Delta_j u\|_{L^2}^2 &= \int |\sum_{k\geq j-N_1}\Delta_j u_k|^2 dx \leq \left(\sum_{k\geq j-N_1} k^{-2\ell}\right)\left(\sum_{k\geq j-N_1}\int k^{2\ell}|\Delta_j u_k|^2 dx\right) \\
&\leq (2\ell-1)(j-N_1)^{-2\ell+1}\sum_{k\geq j-N_1} k^{2\ell}\|\Delta_j u_k\|_{L^2}^2.
\end{aligned}
$$

Set now $\widetilde{c}_j^2 = \sum_{k\geq j-N_1} k^{2\ell}\|\Delta_j u_k\|_{L^2}^2$, we have

$$\sum_j \widetilde{c}_j^2 \leq \sum_k k^{2\ell}\|u_k\|_{L^2}^2 \leq \sum_k c_k^2.$$

Finally, for $j+3 > \ell - 1/2$,

$$\left(\frac{j+3}{j-N_1}\right)^{2\ell-1} \leq \left(\frac{j+3}{j-N_1}\right)^{2(j+3)} \leq e^{2(N_1+3)}(N_1+4)^{2(N_1+3)} \leq C_2.$$

We have proved the lemma.

Lemma 2.3. *Suppose that $\{u_k\}$ is a sequence in $C^\infty(\mathbb{R}^n)$ and for $\ell > 1/2$ there exists a function $v \in H_\ell^{\log}(\mathbb{R}^n)$ satisfying the following : For any $\alpha \in \mathbb{N}^n$, there exist $B_{|\alpha|} \geq 0$ such that*

$$\|D^\alpha u_k\|_{L^2(\mathbb{R}^n)} \leq B_{|\alpha|} e^{k|\alpha|}\|\Delta_k v\|_{L^2(\mathbb{R}^n)}.$$

Then $u = \sum_k u_k \in H_{\ell-1/2}^{\log}(\mathbb{R}^n)$ and for any $S \geq 1$,

$$S^{2\ell-1}\|u\|^2_{H^{\log}_{\ell-1/2}(\mathbb{R}^n)} \leq C_S\left((\ell-1/2)^{2\ell-1}\|v\|^2_{L^2(\mathbb{R}^n)} + S^{2\ell-1}(2\ell-1)\|v\|^2_{H^{\log}_\ell(\mathbb{R}^n)}\right),$$

with C_S depending only on $B_0, B_{[S]+2}$ and C_1, C_2 the constants in lemmas 2.1 and 2.2.

Proof : As in the lemma 2.2, we have $u = \sum_k u_k \in L^2$. We decompose,

$$u_k = u_k^1 + u_k^2 = \psi(e^{-k-1}\Lambda)u_k + (1-\psi(e^{-k-1}\Lambda))u_k.$$

Then $u^1 = \sum u_k^1$ satisfies the hypothesis of lemma 2.2, we have for $S \geq 1$,

$$S^{2\ell-1}\|u^1\|^2_{H^{\log}_{\ell-1/2}(\mathbb{R}^n)} \leq C_1(\ell-1/2)^{2\ell-1}B_0^2\|v\|^2_{L^2} + C_2^S B_0^2 S^{2\ell-1}(2\ell-1)\|v\|^2_{H^{\log}_\ell(\mathbb{R}^n)}.$$

We study now $u^2 = \sum u_k^2$, with the conditions

$$\operatorname{Supp} u_k^2 \subset \{\xi \in \mathbb{R}^n; \langle\xi\rangle \geq e^k\},\ \ \|D^\alpha u_k^2\|_{L^2} \leq B_\alpha e^{k|\alpha|}\|\Delta_k v\|_{L^2}.$$

For $k \geq p+3, \mathcal{C}_p \bigcap \{\xi \in \mathbb{R}^n; \langle\xi\rangle \geq e^k\} = \emptyset$, we have $\Delta_p u^2 = \sum_{k\leq p+2}\Delta_p u_k^2$. Then

$$\begin{aligned}\|\Delta_p u^2\|_{L^2}^2 &\leq \left(\sum_{k\leq p+2} e^{2k}\right)\left(\sum_{k\leq p+2} e^{-2k}\|\Delta_p u_k^2\|^2_{L^2}\right)\\ &\leq 2e^{2(p+2)}\sum_{k\leq p+2} e^{-2k}\|\Delta_p u_k^2\|^2_{L^2} \leq 2e^4 p^{-2\ell+1}\sum_{k\leq p+2} e^{-2k}\|\langle D\rangle(\log\Lambda)^{\ell-1/2}\Delta_p u_k^2\|^2_{L^2}.\end{aligned}$$

Set now $\widetilde{c}_p^2 = \sum_{k\leq p+2} e^{-2k}\|\langle D\rangle(\log\Lambda)^{\ell-1/2}\Delta_p u_k^2\|^2_{L^2}$. We have

$$\sum_{p=-1}^{\infty}\widetilde{c}_p^2 \leq \sum_k e^{-2k}\|\langle D\rangle(\log\Lambda)^{\ell-1/2}u_k^2\|^2_{L^2}.$$

By lemma 2.1, we have

$$S^{2\ell-1}\|(\log\Lambda)^{\ell-1/2}(u^2)\|^2_{L^2} \leq C_1(\ell-1/2)^{2\ell-1}\|u\|^2_{L^2} + C_2^S S^{2\ell-1}\|\{\widetilde{c}_p\}\|^2_{\ell^2}.$$

We study now $\|\{\widetilde{c}_p\}\|_{\ell^2}$. For simplicity of the notation, we replace $\ell - 1/2$ by ℓ in what follows,

$$\|\langle D\rangle(\log\Lambda)^\ell u_k^2\|^2_{L^2} = \int \langle\xi\rangle^{-2([S]+1)}\langle\xi\rangle^{2[S]+4}(\log\langle\xi\rangle)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2|\widehat{u}_k(\xi)|^2 d\xi,$$

and if $([S]+1)(k+2)\geq \ell$,

$$\langle\xi\rangle^{-2([S]+1)}(\log\langle\xi\rangle)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2\leq e^{-2([S]+1)(k+2)}(k+2)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2;$$

if $([S]+1)(k+2)<\ell$,

$$\langle\xi\rangle^{-2([S]+1)}(\log\langle\xi\rangle)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2\leq e^{-2([S]+1)(k+2)}\left(\frac{\ell}{[S]+1}\right)^{2\ell}(1-\psi(e^{-k-1}\langle\xi\rangle))^2.$$

Consequently

$$\begin{aligned}\sum_{p=-1}^{\infty}\tilde{c}_p^2 \quad &\leq \quad \sum_{([S]+1)(k+2)<\ell} e^{-2k}e^{-2([S]+1)(k+2)}\left(\frac{\ell}{[S]+1}\right)^{2\ell}\|u_k\|^2_{H^{[S]+2}}\\ &+\quad \sum_{([S]+1)(k+2)\geq\ell} e^{-4([S]+1)}e^{-2k([S]+2)}\left(1+\frac{2}{k}\right)^{2\ell}k^{2\ell}\|u_k\|^2_{H^{[S]+2}},\end{aligned}$$

where $H^{[S]+2}$ is classical Sobolev space on $\mathbb{R}^n$. From the hypothesis of lemma,

$$\|u_k\|_{H^{[S]+2}}\leq B_{[S]+2}e^{k([S]+2)}\|\Delta_k v\|_{L^2},$$

we have

$$\sum_{p=-1}^{\infty}\tilde{c}_p^2\leq B^2_{[S]+2}(S^{-2\ell}\ell^{2\ell}\|v\|^2_{L^2}+\|v\|^2_{H^{\log}_{\ell-1/2}(\mathbb{R}^n)}).$$

We have proved the lemma with the constant C_S depending on $B_0, B_{[S]+2}$ and C_1, C_2.

We study now the non-linear composition for the function of space $H^{\log}_{\ell-1/2}(\mathbb{R}^n)$. We have the following results.

Theorem 2.1. *Suppose that* $F\in C^\infty(\mathbb{R}), F(0)=0$, *and* $u\in H^{\log}_{\ell}(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$ *a real function for* $\ell>1/2$. *Then* $F(u)\in H^{\log}_{\ell-1/2}(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$ *and for any* $S\geq 1$

$$S^{2\ell-1}\|F(u)\|^2_{H^{\log}_{\ell-1/2}(\mathbb{R}^n)}\leq C_S\left(\left(\ell-\frac{1}{2}\right)^{2\ell-1}\|u\|^2_{L^2(\mathbb{R}^n)}+S^{2\ell-1}(2\ell-1)\|u\|^2_{H^{\log}_{\ell}(\mathbb{R}^n)}\right),$$

with C_S *depending only on* $Sup_{|t|\leq\|u\|_{L^\infty}}|F^{(j)}(t)|$ *and* $\|u\|^j_{L^\infty}$ *for* $j=0,1,\cdots,[S]+2$ *.*

Remark : This theorem is still true for the vector value function $u=(u_1,\cdots,u_m)$ and $F(t_1,\cdots,t_m)\in C^\infty(\mathbb{R}^m)$.

Proof : We have firstly

$$\|F(u)\|_{L^2}=\|F(u)-F(0)\|_{L^2}\leq\left(\sup_{|t|\leq\|u\|_{L^\infty}}\right)|F'(t)|\|u\|_{L^2}.$$

We denote, for $k\geq1, S_k u=\sum_{j=-1}^{k-2}\Delta_j u$, then for $u\in H^{\log}_{\ell}(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$, we have $F(u)=\lim_{k\to+\infty}F(S_k u)$ in $L^2(\mathbb{R}^n)$, so that

$$F(u)=F(S_1u)+\sum_{k=2}^{\infty}(F(S_ku)-F(S_{k-1}u))=\sum_{k=1}^{\infty}f_k$$

with $f_1=F(S_1u)$ and for $k>1$

$$f_k=\int_0^1 F'(S_{k-1}u+t\Delta_k u)dt\Delta_k u.$$

Since for any $\alpha\in\mathbb{N}^n$,

$$\|D^\alpha(S_{k-1}u+t\Delta_k u)\|_{L^\infty}\leq C_{|\alpha|}e^{k|\alpha|}\|u\|_{L^\infty},\ \|D^\alpha\Delta_k u\|_{L^2}\leq e^{(k+3)|\alpha|}\|\Delta_k u\|_{L^2},$$

the Faà-di-Bruno formula implies that

$$\|D^\alpha f_k\|_{L^2} \leq B_{|\alpha|} e^{k|\alpha|} \|\Delta_k u\|_{L^2}$$

with $B_{|\alpha|}$ depending only on $\mathrm{Sup}_{|t|\leq \|u\|_{L^\infty}} |F^{(j)}(t)|$ and $\|u\|^j_{L^\infty}$ for $j = 0, 1, \cdots, |\alpha| + 2$.

Then $\sum_k f_k$ satisfies the hypothesis of lemma 2.3, and so we have proved the theorem.

To study the regularity up to the boundary for nonlinear problems, we introduce the following tangential logarithmic Sobolev spaces (see [15]) : For $\ell > 0$, we set

$$H^{\log}_{0,\ell}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n);\ (\log\langle(\xi', 0)\rangle)^\ell \widehat{u}(\xi) \in L^2(\mathbb{R}^n)\},$$

and

$$H^{\log}_{0,\ell}(\mathbb{R}^n_+) = \{u \in L^2(\mathbb{R}^n_+);\ (\log\langle(\xi', 0)\rangle)^\ell \mathcal{F}_{x'} u(\xi', x_n) \in L^2(\mathbb{R}^n_+)\},$$

where $\xi = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, \mathbb{R}^n_+ = \{(x', x_n); x' \in \mathbb{R}^{n-1}, x_n > 0\}$. We have

$$H^{\log}_{0,\ell}(\mathbb{R}^n)|_{\mathbb{R}^n_+} = H^{\log}_{0,\ell}(\mathbb{R}^n_+).$$

We use now the tangential Littlewood-Paley decomposition :

$$\Delta'_{-1} f = \psi(\Lambda') f,\ \ \Delta'_j f = \varphi(e^{-j}\Lambda') f,\ \ j \in \mathbb{N},$$

where $\mathcal{F}(\varphi(\Lambda')f) = \varphi(\langle\langle(\xi', 0)\rangle\rangle)\widehat{f}$, and the function spaces $H^{\log}_{0,\ell}(\mathbb{R}^n_+)$ is characterized by

$$\sum j^{2\ell} \|\Delta'_j u\|^2_{L^2(\mathbb{R}^n_+)} < +\infty.$$

We have the similar results as lemmas 2.1–2.3 and theorem 2.1 for the tangential function spaces.

3. Nonlinear hypoellipticity

Take $\alpha, \beta \in C^\infty_0(\Omega)$ with $\alpha \subset\subset \beta$. By using the theorem 2.1 and its remark, we have the following estimate : Suppose that $\beta u \in H^{\log}_\ell(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for some $\ell > 1/2$, then for any $S \geq 1$, we have

$$\|\big(\log \Lambda^S\big)^{\ell - 1/2} (\alpha F(x, u))\|^2_{L^2} \leq A^2_S \big(\ell^{2\ell-1} \|\beta u\|^2_{L^2} + (2\ell - 1) \|\big(\log \Lambda^S\big)^\ell (\beta u)\|^2_{L^2}\big), \tag{3.1}$$

where A_S depends on S, $\|u\|_{L^\infty}$ and $\|\alpha(x) F(x, t)\|_{C^{[S]+2}(\Omega \times [-\|u\|_{L^\infty}, \|u\|_{L^\infty}])}$, but not on ℓ.

By interpolation, the estimate (1.4) implies that : For any small $\varepsilon > 0$, any $N > 0$, there exists $C_{\varepsilon,N} > 0$ such that

$$\|\log \Lambda\, v\|^2_{L^2} \leq \varepsilon \sum_{j=1}^m \|X_j v\|^2_{L^2} + C_{\varepsilon,N} \|v\|^2_{H^{-N}}, \quad \forall v \in C^\infty_0(\widetilde{\Omega}), \tag{3.2}$$

where H^{-N} is classical Sobolev space. For small $\delta > 0$, we set $\Lambda_\delta = (1 - \delta\triangle)^{-1}$, then this is a uniformly bounded family of operators on $H^m(\mathbb{R}^n)$ for any $m \in \mathbb{R}$, and $\Lambda_\delta(\alpha u) \in H^2(\mathbb{R}^n)$ if $u \in L^2_{\mathrm{loc}}(\Omega)$. We prove now the following proposition.

Proposition 3.1. *Suppose that the system of vector fields X satisfies the logarithmic regularity estimate (1.4), and $u \in H^1_{X,\,loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ is a weak solution of equation (1.2). Then for any $\alpha \in C^\infty_0(\Omega)$ and any $\ell \in \mathbb{N}, S \geq 1$, we have*

$$\|\big(\log \Lambda^S\big)^\ell \Lambda_\delta(\alpha u)\|_{L^2(\mathbb{R}^n)} \leq (M_0 \ell)^\ell \ell^{m_S} R_S, \tag{3.3}$$

where M_0 depends only on Supp α, *m_S depends only on S, R_S depends on A_S of (3.1) and $\|u\|_{L^2(\Omega)}$. Furthermore the constant M_0, m_S and R_S are independent of small $\delta > 0$ and $\ell \in \mathbb{N}$.*

Proof of first part of theorem 1.1 : By using the estimate (3.3) with $S = 4eM_0$, we have

$$\begin{aligned} \|\langle D\rangle^2\Lambda_\delta(\alpha u)\|_{L^2} &\le \sum_{\ell=0}^{\infty}\|\big(\log\Lambda^2\big)^\ell\Lambda_\delta(\alpha u)\|_{L^2}(\ell!)^{-1} \\ &\le \sum_{\ell=0}^{\infty}\|\big(\log\Lambda^S\big)^\ell\Lambda_\delta(\alpha u)\|_{L^2}\left(\frac{2}{S}\right)^\ell(\ell!)^{-1} \\ &\le R_S\sum_{\ell=1}^{\infty}\left(\frac{1}{2}\right)^\ell\ell^{m_S}+\|\alpha u\|_{L^2}<+\infty, \end{aligned}$$

where we have used the estimate $\ell^\ell \le e^\ell \ell!$. Since R_S, m_S independent of δ, we have proved $\alpha u \in H^2(\mathbb{R}^n)$. Now $\Lambda_\delta(\alpha u)\in H^4$, the similar calculus as above give that $\alpha u \in H^4(\mathbb{R}^n)$ if we take $S = 2\times 4eM_0$ in (3.3). By recurrence we get that $\alpha u \in H^m(\mathbb{R}^n)$ for any $m\in\mathbb{N}$. It follows from the Sobolev embedding theorem that $\alpha u\in C^\infty(\mathbb{R}^n)$. Since $\alpha\in C_0^\infty(\Omega)$ is arbitrary, we have proved $u\in C^\infty(\Omega)$.

Proof of proposition 3.1 For $\ell \ge 1$ fixed, we choose the functions of $C_0^\infty(\Omega)$ as in [6, 7],

$$\alpha=\alpha_\ell\subset\subset\alpha_{\ell-1}\subset\subset\cdots\subset\subset\alpha_1\subset\subset\alpha_0=\beta,$$

such that

$$\|D^\lambda\alpha_j\|_{L^\infty}\le C_\lambda\ell^{|\lambda|},\quad\forall\lambda\in\mathbb{N}^n. \tag{3.4}$$

For the proof of proposition 3.1, we prove the following estimate : for any $1\le j\le\ell$, and any $j\le k\le\ell$, we have

$$\|\big(\log\Lambda^S\big)^j\Lambda_\delta(\alpha_k u)\|_{L^2}\le(M_0\ell)^j\ell^{m_S}R_S \tag{3.5}$$

with the constant as in proposition 3.1.

We need also the following two classicals results about pseudo-differential calculus (see [4]).

First result is about the pseudo-differential operators as a regularlizer.

Proposition 3.2. *For any $m, m'\in\mathbb{N}$, we have*

$$\|(\alpha_k-1)\big(\log\Lambda^S\big)^j\Lambda_\delta(\alpha_{k+1}u)\|^2_{H^m}\le C_{S,m,m'}(j!\ell^{3m+2m'+2S+3n+4})^2\|\beta u\|^2_{H^{-m'}},$$

with $C_{S,m,m'}$ independent of ℓ, j and δ, and

$$\|\alpha_k(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|^2_{H^{-S}}\le C_S(j!\ell^{2S+3n+7})^2\|\beta u\|^2_{L^2}.$$

with C_S independent of ℓ, j and δ.

For the commutators, we have

Proposition 3.3. *Let X be vector fields, $1\le j\le\ell$, $j\le k\le\ell$, we have*

$$\|[X,\alpha_k\big(\log\Lambda^S\big)^j\Lambda_\delta\alpha_{k+1}]\alpha_k u\|^2_{L^2}\le C_S\big(\ell^2|||u|||^2_{j,k,S}+(j!)^2\ell^{10(S+n+2)}\|\beta u\|^2_{L^2}\big),$$

and

$$\|[X,[X,\alpha_k\big(\log\Lambda^S\big)^j\Lambda_\delta\alpha_{k+1}]]\alpha_k u\|^2_{L^2}\le C_S\big(\ell^4|||u|||^2_{j,k,S}+(j!)^2\ell^{10(S+n+2)}\|\beta u\|^2_{L^2}\big),$$

with C_S independent of j, k, ℓ and δ, where

$$|||u|||^2_{j,k,S}=\sum_{0\le j'\le\min\{j,S+2\}}\left(\frac{j!}{(j-j')!}\right)^2\|\big(\log\Lambda^S\big)^{j-j'}\Lambda_\delta\alpha_{k-j'}u\|^2_{L^2}.$$

We prove now (3.5) by induction on j.

1) For $j=1,\ 1\le k\le \ell-1$, take $\alpha_{k+1}\Lambda_\delta\alpha_k^2\Lambda_\delta(\alpha_{k+1}u)\in H_0^1(\Omega)$ as test function in (1.2),

$$\sum_{p=1}^m\int_\Omega(X_pu)X_p(\alpha_{k+1}\Lambda_\delta\alpha_k^2\Lambda_\delta(\alpha_{k+1}u))dx=\int_\Omega\alpha_{k+1}\big(F(x,u)-X_0u\big)(\Lambda_\delta\alpha_k^2\Lambda_\delta(\alpha_{k+1}u))dx.$$

Then it follows from Cauchy-Schwarz inequality and (3.4) that

$$\sum_{p=1}^m\|X_p\alpha_k\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2\le C_1\|\alpha_{k+1}u\|_{L^2}^2+C_2\ell^2\|\alpha_ku\|_{L^2}^2,$$

where C_1 and C_2 are the constants in (3.1) and (3.4). On the other hand, (1.4) gives that

$$\|\log\Lambda(\alpha_k\Lambda_\delta(\alpha_{k+1}u))\|_{L^2}^2\le\varepsilon\|X(\alpha_k\Lambda_\delta(\alpha_{k+1}u))\|_{L^2}^2+C_\varepsilon\|\alpha_{k+1}u\|_{L^2}^2.$$

We have for any $S\ge1$,

$$\begin{aligned}\|\log(\Lambda^S)\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2&\le S^2\varepsilon\|X(\alpha_k\Lambda_\delta(\alpha_{k+1}u))\|_{L^2}^2+S^2C_\varepsilon\|\alpha_{k+1}u\|_{L^2}^2\\&+\|\log(\Lambda^S)(\alpha_k-1)\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2\\&\le S^2\varepsilon\big(C_1\|\alpha_{k+1}u\|_{L^2}^2+C_2\ell^2\|\alpha_ku\|_{L^2}^2\big)\\&+S^2C_\varepsilon\|\alpha_{k+1}u\|_{L^2}^2+\|\log(\Lambda^S)(\alpha_k-1)\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2.\end{aligned}$$

For the last term of right hand side, the proposition 3.2 gives

$$\|\log(\Lambda^S)(\alpha_k-1)\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2\le C_S\ell^{4S+6n+8}\|\beta u\|_{L^2}^2.$$

We have proved (3.5) for $j=1$ if we choose $\varepsilon>0$ small such that $\varepsilon S^2\le1$ and

$$M_0^2\ge C_1+C_2+C_3+1,\ R_S^2\ge(SC_{1/S}+C_S)\|\beta u\|_{L^2}^2,\ 2m_S\ge10(S+n+2).$$

2) Suppose now that there exists a $j\le\ell-1$ such that (3.5) is true for any $p\le j$. We shall prove (3.5) for $j+1$. Firstly, take $\delta\to0$, we have for any $p\le j$ and $p\le k\le\ell$

$$\|\big(\log\Lambda^S\big)^p(\alpha_ku)\|_{L^2}\le(M_0\ell)^p\ell^{m_S}R_S,\tag{3.6}$$

and

$$\sum_{0\le j'\le\min\{j,S+2\}}\left(\frac{j!}{(j-j')!}\right)^2\|\big(\log\Lambda^S\big)^{j-j'}\Lambda_\delta\alpha_{k-j'}u\|_{L^2}^2\le C_S(M_0\ell)^{2j}\ell^{2m_S}R_S^2.\tag{3.7}$$

For $j\le k\le\ell-1$, set

$$v=\alpha_{k+1}\Lambda_\delta(\log\Lambda^S)^j\alpha_k^2(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u),$$

then $v\in H_0^1(\Omega)$, using v as test function in (1.2),

$$\int_\Omega\big(\sum_{p=1}^mX_p^*X_pu\big)vdx=\int_\Omega\big(F(x,u)-X_0u\big)vdx.$$

By using integration by part, Cauchy-Schwarz inequality, induction hypothesis, Proposition 3.2 and Proposition 3.3, we have that, if $m_S\ge5(S+n+2)$,

$$\begin{aligned}\sum_{p=1}^m\|X_p\alpha_k(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2&\le\frac12\|\log(\Lambda^S)\alpha_k(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|_{L^2}^2\\&+\widetilde C_S(M_0\ell)^{2j+2}\ell^{2m_S}R_S^2.\end{aligned}$$

Using (3.2) with $N = S$ and the proposition 3.2, we get

$$\|\log\Lambda\,\alpha_k(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|^2_{L^2}$$
$$\le \varepsilon\sum_{p=1}^m \|X_p\alpha_k(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|^2_{L^2} + C_{\varepsilon,S}\|\alpha_k(\log\Lambda^S)^j\Lambda_\delta(\alpha_{k+1}u)\|^2_{H^{-S}}.$$

Choose $\varepsilon > 0$ small enough such that $\varepsilon S^2 \le 1,\ \varepsilon S^2 C_S \le 1/4$, we get (3.5) if we take

$$R_S^2 \ge 2(2S^2C_{\varepsilon,S+1} + 3C_S)\|\beta u\|^2_{L^2}.$$

Regularity up to the boundary

As in the classical case, we transform firstly the non homogeneous Dirichlet problem (1.2)–(1.3) into a homogeneous Dirichlet problem, i. e. we suppose that $g = 0$ in (1.3). Suppose now that $u \in L^\infty(\Omega)\cap H^1_{X,0}(\Omega)$ is a weak solution of Dirichlet problem (1.2))–(1.3), we have already the interior regularity $u \in C^\infty(\Omega)$, and so we want to prove here that $u \in C^\infty(\bar\Omega)$, that is C^∞ regularity up to the boundary.

Since $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X, near a point of $\partial\Omega$, we use the standard process of localization and a C^∞ change of variable to flatten out the boundary (we keep the same notation for the solution u), then the weak solution u satisfies the following equation (see [2, 3, 15]) :

$$\text{(3.8)}\qquad \begin{cases} \partial^2_{x_n}(\alpha u) - \sum_{j=1}^{m-1} Y_j^* Y_j(\alpha u) = \partial_{x_n}(a_0\beta u) + Y_0(\beta u) + \widetilde{F}(x,\beta u), \text{ in } \mathbb{R}^n_+ \\ \beta u(x',0) = 0, \qquad \text{for } x' \in \mathbb{R}^{n-1} \end{cases}$$

where $\alpha,\beta,a_0 \in C_0^\infty(\overline{\mathbb{R}^n_+})$, $\alpha \subset\subset \beta$ with $\operatorname{Supp}\beta$ a neighborhood of 0 in $\mathbb{R}^n$, and $Y_j = \sum_{k=1}^{n-1} a_{jk}(x',x_n)\partial_{x_k}, j = 0,1,\cdots,m-1$ are the tangential vector fields. We have that the system of vector fields $Y = (\partial_{x_n}, Y_1,\cdots,Y_{m-1})$ satisfies still the logarithmic regularity estimates (1.1) or (1.4) on a neighborhood $\mathcal{O} \subset \mathbb{R}^n$ of 0. Remark that we have $\beta u \in L^\infty(\mathbb{R}^n_+)\cap H^1_{Y,0}(\mathbb{R}^n_+)$.

Let $\Lambda' = (e + |D'|^2)^{1/2}$ with $D' = (D_{x_1},\cdots,D_{x_{n-1}})$. On account of (1.4), for any small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\text{(3.9)}\qquad \|(\log\Lambda')v\|^2_{L^2(\mathbb{R}^n)} \le \varepsilon\Big(\sum_{j=1}^{m-1}\|Y_jv\|^2_{L^2(\mathbb{R}^n)} + \|\partial_{x_n}v\|^2_{L^2(\mathbb{R}^n)}\Big) + C_\varepsilon\|v\|^2_{L^2(\mathbb{R}^n)},$$

for all $v \in C_0^\infty(\mathcal{O}\cap\mathbb{R}^n_+)$. By density, this is true for $v \in H^1_{Y,0}(\mathcal{O}\cap\mathbb{R}^n_+)$.

Firstly for the nonlinear term we have the similar result as in (3.1) : Suppose that $\beta u \in H^{\log}_{0,\ell}(\mathbb{R}^n_+)\cap L^\infty(\mathbb{R}^n_+)$ for some $\ell > 1/2$, then for any $S \ge 1$, we have

$$\text{(3.10)}\qquad \begin{aligned} \|\big(\log{\Lambda'}^S\big)^{\ell-1/2}(\alpha\widetilde{F}(x,\beta u))\|^2_{L^2(\mathbb{R}^n_+)} \ &\le\ A_S^2\big(\ell^{2\ell-1}\|\beta u\|^2_{L^2(\mathbb{R}^n_+)} \\ &+\ (2\ell-1)\|\big(\log{\Lambda'}^S\big)^\ell(\beta u)\|^2_{L^2(\mathbb{R}^n_+)}\big), \end{aligned}$$

where A_S depends on $S, \|u\|_{L^\infty}$ and $\|\alpha(x)\widetilde{F}(x,t)\|_{C^{[S]+2}((\overline{\mathbb{R}^n_+}\times[-\|u\|_{L^\infty},\|u\|_{L^\infty}])}$, but not on ℓ.

If the equation (1.2) is linear, we use the following estimate : If $f \in C^\infty(\overline{\mathbb{R}^n_+})$, then for any $\ell \in \mathbb{N}$, any $S \ge 1$, and $\alpha \in C_0^\infty(\overline{\mathbb{R}^n_+})$,

$$\|(\log{\Lambda'}^S)^\ell(\alpha f)\|_{L^2(\mathbb{R}^n_+)} \le \ell!\|{\Lambda'}^S(\alpha f)\|_{L^2(\mathbb{R}^n_+)}.$$

For small $\delta > 0$, we set $\Lambda'_\delta = (1 - \delta\triangle_{x'})^{-1}$, with $\triangle_{x'} = \sum_{j=1}^{n-1}\partial^2_{x_j}$, this is a tangential regularization operators. As for the proposition 3.1, we have that for any $\ell \in \mathbb{N}$, and any

$S \geq 1$,

$$\| \big(\log {\Lambda'}^S \big)^\ell \Lambda'_\delta(\alpha u)\|_{L^2(\mathbb{R}^n_+)} \leq (M_0 \ell)^\ell \ell^{m_S} R_S, \tag{3.11}$$

with the same constants as in (3.3). By using the estimates (3.9) and (3.10), the proof of this estimations is exactly as that of proposition 3.1, for example in the step 2 of proof for the proposition 3.1, we take here

$$v = \alpha_{k+1}\Lambda'_\delta(\log {\Lambda'}^S)^j \alpha_k^2 (\log {\Lambda'}^S)^j \Lambda'_\delta(\alpha_{k+1}u),$$

as test function in (3.8). In fact, we have $v, \partial_{x_n} v, \Lambda' v \in L^2(\mathbb{R}^n_+)$ and $v(x',0) = 0$, then $v \in H^1_0(\mathbb{R}^n_+)$. Moreover, $\alpha_{k+1}\Lambda'_\delta(\log {\Lambda'}^S)^j \alpha_k^2 (\log {\Lambda'}^S)^j \Lambda'_\delta \alpha_{k+1}$ is a tangential pseudo-differential operators, thus all pseudo-differential calculus in the proof is tangential, and the integration by part for the variable x_n take only once.

Now the estimate (3.11) implies that ${\Lambda'}^m(\alpha u) \in L^2(\mathbb{R}^n_+)$ for any $m \in \mathbb{N}$ and any $\alpha \in C_0^\infty(\mathcal{O} \cap \overline{\mathbb{R}^n_+})$, and we have already $\partial_{x_n}(\alpha u) \in L^2(\mathbb{R}^n_+)$, so that we have $\alpha u \in H^1(\mathbb{R}^n_+)$. For $m \geq 2$, we have, by using the equation (3.8),

$$\partial^2_{x_n}(\alpha u) = \sum_{j=1}^{m-1} Y_j^* Y_j(\alpha u) + \partial_{x_n}(a_0 \beta u) + Y_0(\beta u) + \widetilde{F}(x, \beta u) \in L^2(\mathbb{R}^n_+),$$

then, we have $\alpha u \in H^2(\mathbb{R}^n_+)$. By induction we prove that $\alpha u \in H^m(\mathbb{R}^n_+)$ for any $m \in \mathbb{N}$. We have proved finally $\alpha u \in C^\infty(\overline{\mathbb{R}^n_+})$ by Sobolev embedding theorem. Take $\alpha = 1$ near $0 \in \mathbb{R}^n$, we have proved $u \in C^\infty(\widetilde{\mathcal{O}} \cap \overline{\mathbb{R}^n_+})$ for $\widetilde{\mathcal{O}}$ a neighborhood of 0 in $\mathbb{R}^n$. So that we get the C^∞ regularity of solution up to the boundary.

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