Cauchy problem for viscous shallow water equations

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Abstract : *In this paper, we study the Cauchy problems for viscous shallow water equations. We work in the Sobolev spaces of index $s > 2$, we obtain the local solutions for any initial data, and global solution for small initial data.*

Key words Shallow water equation, Littlewood-Paley decomposition, global solution.

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1 Introduction

We consider in this work the Cauchy problems for viscous shallow water equations as follows:

$$h(u_t + (u \cdot \nabla)u) - \nu\nabla \cdot (h\nabla u) + h\nabla h = 0, \tag{1.1}$$

$$h_t + \text{div}(hu) = 0, \tag{1.2}$$

$$u|_{t=0} = u_0, \quad h|_{t=0} = h_0; \tag{1.3}$$

where $h(x,t)$ is the height of fluid surface, $u(x,t) = (u^1(x,t), u^2(x,t))^t$ is the horizontal velocity field, $x = (x_1, x_2) \in \mathbf{R}^2$, and $0 < \nu < 1$ is the viscous coe□cients.

The equations form a quasi-linear hyperbolic-parabolic system. For the initial data $h_0(x)$, we suppose that it is a small perturbation of some positive constant $\bar{h}_0$. We study the Cauchy problem (1.1)-(1.3) in the Sobolev function space.

The main theorems of this paper is :

Theorem 1.1 *Let $s > 0, u_0, h_0 - \bar{h}_0 \in H^{2+s}(\mathbf{R}^2), \|h_0 - \bar{h}_0\|_{H^{2+s}} << \bar{h}_0$. Then there exists a positive time T, an unique solution (u, h) of Cauchy problem (1.1)-(1.3) such that*

$$u, h - \bar{h}_0 \in L^\infty([0,T]; H^{2+s}), \quad \nabla u \in L^2([0,T]; H^{2+s}).$$

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Furthermore, there exists a constant c such that if $\|h_0 - \bar{h}_0\|_{H^{2+s}} + \|u_0\|_{H^{2+s}} \leq c$, then we can choose $T = +\infty$.

The local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for the shallow water equations using Lagrangian coordinates and Hölder space estimates with initial data in $C^{2+\alpha}$ was studied in [2]. Kloeden [5] and Sundbye [10] proved global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem using Sobolev space estimates by following the energy method of Matsumura and Nisshida [7, 8, 9]. Sundbye [11] proved also the existence and uniqueness of classical solutions to the Cauchy problem using method of [7, 8, 9]. But all of those results consider only the problems for small initial date. In general, the problems of existence of solutions for large initial date is difficult, since there are stronger non-linearization than small initial date. We use the Littlewood-Paley decomposition theory (see [1, 3]) for Sobolev space to obtain the losing energy estimates in H^{s+2} for any $s > 0$, and we get the local existence of solution for all size initial data. Moreover, we also improve the global existence of solution and regularity for small initial date. Basing the result of global existence, we can give some decay estimate as in [6, 12] by the method of Green function. For brevity, we left it to the future.

The structure of the paper is the following:

1. In the second section we recall Littlewood-Paley theory for Sobolev space.
2. In section 3, we prove the first part of main theorem : local existence of solution for all size initial date.
3. In section 4, we prove the global existence of solution for small initial data.
4. Finally in section 5, we prove the losing energy estimates for nonolinear terms.

2 Littlewood-Paley theory

Let us recall the Sobolev space defined by Littlewood-Paley theory (see J.-M. Bony [1] and J.-Y. Chemin [3]). There exist the functions φ and ψ in $C_0^\infty(\mathbf{R}^d)$ with respectively in a fixed ring of $\mathbf{R}^d$ far from the origin $\mathcal{C} = \{\xi; 5/6 \leq |\xi| \leq 12/5\}$, and in a fixed ball containing the origin $B = \{|\xi| \leq 2\}$, such that

$$\forall \xi \in \mathbf{R}^d \setminus \{0\},\ \sum_{j\in\mathbf{Z}} \varphi(2^{-j}\xi) = 1 \ \text{ and } \ \forall \xi \in \mathbf{R}^d,\ \psi(\xi) + \sum_{j\in\mathbf{N}} \varphi(2^{-j}\xi) = 1.$$

Let us note that if $|j - j'| \geq 2$, then $\text{supp}\varphi(2^{-j}\cdot) \cap \text{supp}\varphi(2^{-j'}\cdot) = \emptyset$. We define the following operators of localization in Fourier space

$$\dot{\Delta}_j u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\hat{u}(\cdot)) = 2^{jd} \int_{\mathbf{R}^d} f(2^j y) u(x-y) dy, \text{ for } j \in \mathbf{Z},$$

and

$$\Delta_{-1} u = \mathcal{F}^{-1}(\psi(\cdot)\hat{u}(\cdot)), \ \ \Delta_j = \dot{\Delta}_j, \text{ for } j \in \mathbf{N},$$

where $\hat{u}$ denote the Fourier transformation of u, and $f = \mathcal{F}^{-1}(\varphi)$. So that for $u \in \mathcal{S}'$, we have that $\Delta_j u, \Delta_{-1} u \in C^\infty \cap L^2$. Then the Sobolev space con be defined as following, for $s \in \mathbf{R}$,

$$H^s(\mathbf{R}^2) = \{u \in \mathcal{S}'(\mathbf{R}^2); \|u\|_{H^s}^2 = \sum_{j=-1}^{\infty} 2^{2js} \|\Delta_j u\|_{L^2}^2 < +\infty\}.$$

In the low vertical frequencies estimates, we have to use the homogeneous Sobolev spaces,

$$\dot{H}^s(\mathbf{R}^2) = \{u \in \mathcal{S}'(\mathbf{R}^2);\, \|u\|_{\dot{H}^s}^2 = \sum_{j\in\mathbf{Z}} 2^{2js}\|\dot{\Delta}_j u\|_{L^2}^2 < +\infty\}.$$

For $d = 2$, we have that $H^{2+s}(\mathbf{R}^2) \subset L^\infty(\mathbf{R}^2)$ for any $s > -1$, and

$$\|f\|_{L^\infty(\mathbf{R}^2)} \leq C_s\|f\|_{H^{2+s}(\mathbf{R}^2)},$$

C_s is the so-called Sobolev constant in $\mathbf{R}^2$. We have also that, for any $q \geq 0$,

$$\|\Delta_q f\|_{L^\infty} \leq C\|\nabla(\Delta_q f)\|_{L^2},$$

and

$$\|\Delta_q f\|_{L^2} \leq C2^{-q}\|\nabla(\Delta_q f)\|_{L^2}.$$

We set $S_q(u) = \sum_{-1\leq p\leq q-2} \Delta_p u$, then $S_q : H^s \to H^{+\infty}$ and

$$\Delta_p(S_q(u)\Delta_q u) = 0,\ \text{if } |p-q| \geq 4;\ \text{and}\ \|S_q(\nabla u)\|_{L^\infty} \leq 2^q\|S_q u\|_{L^\infty}$$

For the product of two functions, we have the decomposition :

$$uv = \sum_q S_{q-1}(u)\Delta_q v + \sum_q S_{q-1}(v)\Delta_q u + \sum_{|p-q|<2} \Delta_p u\Delta_q v = T_u v + T_v u + R(u,v).$$

Here T_u is a linear operator, and we have that :
if $u \in L^\infty$, then for all $s \in \mathbf{R}$,

$$\|T_u\|_{\mathcal{L}(H^s,H^s)} \leq C\|u\|_{L^\infty};$$

if $u \in H^\tau, \tau < d/2$, then for all $s \in \mathbf{R}$,

$$\|T_u\|_{\mathcal{L}(H^s,H^{s+\tau-d/2})} \leq C\|u\|_{H^\tau};$$

if $u \in H^{s_1}, v \in H^{s_2}, s_1 + s_2 - d/2 > 0$, then

$$\|R(u,v)\|_{H^{s_1+s_2-d/2}} \leq C\|u\|_{H^{s_1}}\|v\|_{H^{s_2}}.$$

For the nonlinear composition, we have that, for $F \in C^\infty(I)$ such that $F(0) = 0$, $u \in H^\tau(\mathbf{R}^2), \tau > 1$ with $u(x) \in I$ for all $x \in \mathbf{R}^2$. Then there exists a function of one variable B_0 depending only on τ, F, I such that

$$\|F(u)\|_{H^\tau} \leq B_0(\|u\|_{L^\infty})\|u\|_{H^\tau}. \tag{2.1}$$

In our equation, we have the products of 3 functions, so that we need the following precise estimates :

$$\begin{aligned} |(ab,c)_{L^2}| &\leq C\|a\|_{L^\infty}\|b\|_{L^2}\|c\|_{L^2}, \\ |(ab,c)_{L^2}| &\leq C\|a\|_{\dot{H}^{1/2}}\|b\|_{L^2}\|c\|_{\dot{H}^{1/2}}, \\ \|a\|_{\dot{H}^{1/2}}^2 &\leq \|a\|_{L^2}\|\nabla a\|_{L^2}. \end{aligned} \tag{2.2}$$

For the detail of those results, we send to the reference [3].

In the prove of main theorem, we need to estimate the nonlinear term in the equations, this is so-called “Losing energy estimates”

Lemma 2.1 *Let $\tau > 1$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $v, \nabla v, g, \nabla g \in H^\tau$, we have*

$$|\int_{\mathbf{R}^2} \Delta_k((v\cdot\nabla)g)\Delta_k g dx| \le C_0 d_k^2 2^{-2k\tau}\|v\|_{H^{\tau+1}}\|g\|_{H^\tau}^2,$$

with $\{d_k\} \in \ell^2$ and $\|\{d_k\}\|_{\ell^2} \le 1$.

Lemma 2.2 *(a) Let $\tau > 2$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $f, v, g, u, \nabla u \in H^\tau$, with $\|g\|_{L^\infty} \le 1/4$, we have*

$$|\int_{\mathbf{R}^2} \Delta_k(\frac{\nabla f}{1+g}\nabla v)\Delta_k u dx| \le C_0 d_k^2 2^{-2k\tau}\|f\|_{H^\tau}\|v\|_{H^\tau}(1+\|g\|_{H^\tau})\|u\|_{H^{\tau+1}},$$

where $\|\{d_k\}\|_{\ell^2} \le 1$.

(b) Let $1 < \tau < 2$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $f, g, u, \nabla u, v, \nabla v \in H^\tau$, with $\|g\|_{L^\infty} \le 1/4$, we have

$$|\int_{\mathbf{R}^2} \Delta_k(\frac{\nabla f}{1+g}\nabla v)\Delta_k u dx| \le C_0 d_k^2 2^{-2k\tau}\|f\|_{H^\tau}(1+\|g\|_{H^\tau})U_\tau(u,v),$$

with $\|\{d_k\}\|_{\ell^2} \le 1$, and

$$U_\tau(u,v) =: \|\nabla v\|_{L^\infty}\|u\|_{H^{\tau+1}} + \|\nabla v\|_{H^\tau}(\|\nabla u\|_{H^1} + \|u\|_{H^\tau}).$$

Lemma 2.3 *(a) Let $\tau > 2$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $f, v, u, \nabla u, g_1, g_2 \in H^\tau$, with $\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty} \le 1/4$, we have*

$$\begin{aligned} &|\textstyle\int_{\mathbf{R}^2} \Delta_k(\frac{(g_1-g_2)}{(1+g_1)(1+g_2)}\nabla f\nabla v)\Delta_k u dx| \\ \le\ & C_0 d_k^2 2^{-2k\tau}\|f\|_{H^\tau}\|v\|_{H^\tau}\|g_1-g_2\|_{H^\tau}\|u\|_{H^{\tau+1}}, \end{aligned}$$

with $\|\{d_k\}\|_{\ell^2} \le 1$.

(b) Let $1 < \tau < 2$ and $-1 \le k < +\infty$, then there exists $C_0 > 0$ such that for all $f, v, g_1, g_2, u, \nabla u, v, \nabla v \in H^\tau$, with $\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty} \le 1/4$, we have

$$\begin{aligned} &|\textstyle\int_{\mathbf{R}^2} \Delta_k(\frac{(g_1-g_2)}{(1+g_1)(1+g_2)}\nabla f\nabla v)\Delta_k u dx| \\ \le\ & C_0 d_k^2 2^{-2k\tau}\|f\|_{H^\tau}\|g_1-g_2\|_{H^\tau}U_\tau(u,v), \end{aligned}$$

with $\|\{d_k\}\|_{\ell^2} \le 1$, and $U_\tau(u,v)$ is as in Lemma 2.2, (b).

In the proofs of existence of global solutions, we need the following high vertical frequencies estimates.

Lemma 2.4 *Let $\tau > 0$, then there exists $M > 0, C_0 > 0$ such that for all $h, u, v, \nabla h, \nabla u \in H^\tau, M \le k < +\infty$, with $\|h\|_{L^\infty} \le 1/4$, we have*

$$\begin{aligned} &|\textstyle\int_{\mathbf{R}^2} \Delta_k(\frac{1}{1+h}\nabla h\nabla u)\Delta_k v dx| \\ \le\ & C_0 d_k^2 2^{-2k\tau}(1+\|h\|_{H^{\tau+1}})\|\triangle u\|_{H^\tau}\|\nabla h\|_{H^\tau}\|v\|_{H^\tau}, \end{aligned}$$

with $\|\{d_k\}\|_{\ell^2} \le 1$.

Lemma 2.5 *Let $\tau > 0$, then there exists $M > 0, C_0 > 0$ such that for all $h \in H^{\tau+1}$ with $\|h\|_{L^\infty} \le 1/4$, and $u \in H^{\tau+2}, M \le k < +\infty$, we have*

$$\begin{aligned} &|\textstyle\int_{\mathbf{R}^2} \Delta_k(\mathrm{div}(hu))\Delta_k(\Delta h) dx| \\ \le\ & C_0 d_k^2 2^{-2k\tau}\|\nabla h\|_{H^\tau}(\|\nabla h\|_{H^\tau}^2 + \|\nabla u\|_{H^{\tau+1}}^2), \end{aligned}$$

with $\|\{d_k\}\|_{\ell^2} \le 1$.

We will prove this five lemmas in the last section.

3 The local existence of solution

In order to study the local existence of solution, we define the function set, $(f,g) \in \mathcal{X}([t_1,t_2],\sigma,E_1,E_2)$ if $(f,g) \in L^\infty([t_1,t_2],H^\sigma(\mathbf{R}^2)), \nabla f \in L^2([t_1,t_2],H^\sigma(\mathbf{R}^2))$ and

$$\|f\|^2_{L^\infty([t_1,t_2],H^\sigma(\mathbf{R}^2))} + \nu\|\nabla f\|^2_{L^2([t_1,t_2],H^\sigma(\mathbf{R}^2))} \leq E_1^2$$
$$\|g\|_{L^\infty([t_1,t_2],H^\sigma(\mathbf{R}^2))} \leq E_2.$$

The main result of this section is the following local existence theorem for any initial data :

Theorem 3.1 *Let $s > 0, (u_0, h_0 - \overline{h}_0) \in H^{s+2}(\mathbf{R}^2)$ with $\|h_0 - \overline{h}_0\|_{H^{2+s}} \leq \frac{\bar{h}_0}{4C_s}$, then there exist a positive time T and a solution*

$$(u, h - \overline{h}_0) \in \mathcal{X}([0,T], s+2, E_1, E_2)$$

for the Cauchy problem (1.1)-(1.3). Here $E_1 = 2\|u_0\|_{H^{s+2}}, E_2 = 2\|h_0 - \overline{h}_0\|_{H^{s+2}}$, and C_s is the Sobolev constant.

For convenience sake, we take $\overline{h}_0 = 1$. Substitute h by $1 + h$ in (1.1)-(1.3), we have

$$\begin{aligned} u_t + (u\cdot\nabla)u - \nu\tfrac{\nabla\cdot((1+h)\nabla u)}{1+h} + \nabla h &= 0, \\ h_t + \mathrm{div} u + \mathrm{div}(hu) &= 0, \\ u(x,0) = u_0(x), h(x,0) &= h_0(x). \end{aligned}$$

We suppose now $h_0 \in H^{s+2}(\mathbf{R}^2), \|h_0\|_{H^{2+s}} \leq \frac{1}{4C_s}$, and $E_1 = 2\|u_0\|_{H^{s+2}}, E_2 = 2\|h_0\|_{H^{s+2}}$.

The proof of Theorem 3.1 involves the method of successive approximations. We define the sequence $\{u_n, h_n\}$ by following linear systems:

$$(u_1, h_1) = S_2(u_0, h_0),$$
$$\partial_t u_{n+1} - \nu\triangle u_{n+1} = G_1(u_n, h_n), \tag{3.1}$$
$$\partial_t h_{n+1} + u_n\nabla h_{n+1} = G_2(u_n, h_n), \tag{3.2}$$
$$(u_{n+1}, h_{n+1})|_{t=0} = S_{n+2}(u_0, h_0), \tag{3.3}$$

where

$$\begin{aligned} G_1(u_n, h_n) &= \tfrac{\nu}{1+h_n}\nabla h_n \nabla u_n - u_n\nabla u_n + \nabla h_n \\ G_2(u_n, h_n) &= -(1 + h_n)\mathrm{div} u_n. \end{aligned}$$

Since S_q are smooth operators, the initial data $S_{n+2}(u_0, h_0)$ are smooth functions. If $(u_n, h_n) \in \mathcal{X}([0,T], s+2, E_1, E_2)$ and smooth, we have

$$\|h_n\|_{L^\infty} \leq C_s\|h_n\|_{H^{2+s}} \leq C_s E_2 = 2C_s\|h_0\|_{H^{2+s}} \leq \frac{2C_s}{4C_s} \leq \frac{1}{2},$$

then $G_1(u_n, h_n)$ and $G_2(u_n, h_n)$ are also smooths functions. Note that (3.1) is heat equations for u_{n+1}, (3.2) is transport equation for h_{n+1}, then the existence of the smooth solutions for the Cauchy problems (3.1)-(3.3) is evident. We denote by P_n the application from (u_n, h_n) to (u_{n+1}, h_{n+1}) the solution of problem (3.1)-(3.3).

Now the proof of theorem 3.1 is in two steps : “Estimation for big norms” and “convergence for small norms”.

Estimation for big norms

Proposition 3.1 *Suppose that* $(u_0, h_0) \in H^{s+2}(\mathbf{R}^2)$ *for* $s > 0$ *and* $\|h_0\|_{H^{s+2}} \leq \frac{1}{4C_s}$*, then there exists a positive time* T_1 *such that for any* $n \in \mathbf{N}$*,* P_n *is an application from* $\mathcal{X}([0, T_1], s+2, E_1, E_2)$ *to* $\mathcal{X}([0, T_1], s+2, E_1, E_2)$ *for* $E_1 = 2\|u_0\|_{H^{s+2}}, E_2 = 2\|h_0\|_{H^{s+2}}$*.*

Proof : For convenience sake, we suppose that $1 \leq E_1$ (the proof for $E_1 < 1$ is easy), and remark that $0 < E_2 < 1, 0 < \nu < 1$. We take now

$$T_1 = \min\{(\frac{12}{5}K)^{-2}, \frac{\nu E_2^2}{16C_0^2 E_1^4}\},$$

where $K = \|\mathcal{F}^{-1}(\varphi)\|_{L^1}$. We prove the proposition by induction. Firstly, $(u_1, h_1) = S_2(u_0, h_0)$, then

$$\begin{aligned}
&\|u_1\|_{H^{s+2}} \leq \|u_0\|_{H^{s+2}}, \quad \|h_1\|_{H^{s+2}} \leq \|h_0\|_{H^{s+2}},\\
&\nu \int_0^{T_1} \|\nabla u_1\|_{H^{s+2}}^2 d\tau \leq \nu T_1 (\tfrac{12}{5}K)^2 \|u_0\|_{H^{s+2}}^2 \leq \|u_0\|_{H^{s+2}}^2.
\end{aligned}$$

Thus $(u_1, h_1) \in \mathcal{X}([0, T_1], s+2, E_1, E_2)$.

Now, we assume that $(u_n, h_n) \in \mathcal{X}([0, T_1], s+2, E_1, E_2)$ is valid, and prove that $P_n(u_n, h_n) = (u_{n+1}, h_{n+1}) \in \mathcal{X}([0, T_1], s+2, E_1, E_2)$ is also valid.

Applying the operator Δ_k to the equations (3.1), (3.2), multiplying the first by $\Delta_k u_{n+1}$, and the second by $\Delta_k h_{n+1}$, integration over $\mathbf{R}^2$ yields

$$\begin{aligned}
\partial_t \|\Delta_k u_{n+1}\|_{L^2}^2 + 2\nu \|\nabla \Delta_k u_{n+1}\|_{L^2}^2 &= 2\int_{\mathbf{R}^2} \Delta_k G_1(u_n, h_n) \Delta_k u_{n+1} dx,\\
\partial_t \|\Delta_k h_{n+1}\|_{L^2}^2 - 2\int_{\mathbf{R}^2} \Delta_k(u_n \nabla h_{n+1}) \Delta_k h_{n+1} dx &= 2\int_{\mathbf{R}^2} \Delta_k G_2(u_n, h_n) \Delta_k h_{n+1} dx.
\end{aligned}$$

By using Lemma 2.1, Lemma 2.2 (a) and hypotheses on (u_n, h_n), we obtain

$$\begin{aligned}
&\partial_t \|\Delta_k u_{n+1}\|_{L^2}^2 + 2\nu \|\Delta_k(\nabla u_{n+1})\|_{L^2}^2 \leq C_0 d_k^2 2^{-2k(s+2)}\\
&\times(\|h_n\|_{H^{s+2}} \|\nabla u_{n+1}\|_{H^{s+2}} + V_1(t)(\|u_{n+1}\|_{H^{s+2}} + \|\nabla u_{n+1}\|_{H^{s+2}})), \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
&\partial_t \|\Delta_k h_{n+1}\|_{L^2}^2 \leq C_0 d_k^2 2^{-2k(s+2)}\\
&\times(\|u_n\|_{H^{s+2}} \|h_{n+1}\|_{H^{s+2}}^2 + (1 + \|h_n\|_{H^{s+2}}) \|\nabla u_n\|_{H^{s+2}} \|h_{n+1}\|_{H^{s+2}}), \qquad (3.5)
\end{aligned}$$

where

$$V_1(t) = \|h_n(t)\|_{H^{s+2}} \|u_n(t)\|_{H^{s+2}} (1 + \|h_n(t)\|_{H^{s+2}}) + \|u_n(t)\|_{H^{s+2}}^2 \leq \frac{3}{4} E_1^2.$$

Multiplying both sides of (3.4) and (3.5) by $2^{2k(s+2)}$, and taking the sum over k gives respectively

$$\begin{aligned}
&\partial_t \|u_{n+1}\|_{H^{s+2}}^2 + \nu \|\nabla u_{n+1}\|_{H^{s+2}}^2 \leq \|u_{n+1}\|_{H^{s+2}}^2 + 2C_0^2 E_1^4 \nu^{-1},\\
&\partial_t \|h_{n+1}\|_{H^{s+2}}^2 \leq \tfrac{\nu E_2^2}{4E_1^2} \|\nabla u_n\|_{H^{s+2}}^2 + \tfrac{5C_0^2 E_1^2}{\nu E_2^2} \|h_{n+1}\|_{H^{s+2}}^2.
\end{aligned}$$

Taking integration from 0 to t yields

$$\begin{aligned}
&\|u_{n+1}(t)\|_{H^{s+2}}^2 + \nu \int_0^t e^{t-\tau} \|\nabla u_{n+1}(\tau)\|_{H^{s+2}}^2 d\tau \leq \|u_{n+1}(0)\|_{H^{s+2}}^2 e^t + te^t 2C_0^2 E_1^4 \nu^{-1},\\
&\|h_{n+1}(t)\|_{H^{s+2}}^2 \leq e^{t5C_0^2 E_1^2 \nu^{-1} E_2^{-2}} (\|h_{n+1}(0)\|_{H^{s+2}}^2 + \tfrac{\nu E_2^2}{4E_1^2} \int_0^t \|\nabla u_n(t')\|_{H^{s+2}}^2 dt').
\end{aligned}$$

By the definition of $(u_{n+1},h_{n+1})|_{t=0}$ we know that

$$\|u_{n+1}(0)\|_{H^{s+2}} \le \|u_0\|_{H^{s+2}}, \|h_{n+1}(0)\|_{H}$$ [redacted]

Thus, the choose of T_1 give that

$$\|u_{n+1}(t)\|^2_{L^\infty([0,T_1],H^{s+2})} + \nu\|\nabla u_{n+1}(\tau)\|^2_{L}$$ [redacted]

$$\|h_{n+1}(t)\|^2_{L^\infty([0,T_1],H^{s+2})} \le$$ [redacted]

We have proved the proposition 3.1.

Convergence for small norm

Proposition 3.2 *Let* $(u_0(x),h_0(x)) \in H^{s+2}(\mathbf{R}^2)$ *for* $s >$ [redacted] *exists a positive time* $T_2(\le T_1)$ *which independent of* n, [redacted] *s a Cauchy sequence in* $\mathcal{X}([0,T_2],s+1,E_1,E_2)$ *if* $s \neq 1$, *and* [redacted] *all* $1 > \varepsilon > 0$ *if* $s = 1$.

Proof : From the equations(3.1) and (3.2), we have

$$\partial_t(u_{n+1}-u_n) - \nu\triangle(u_{n+1}-u_n) =$$ [redacted] (3.6)

$$\partial_t(h_{n+1}-h_n) + u_n\nabla($$ [redacted] (3.7)

where

$\sum_{j=1}^{6} F_j = \frac{1}{1+h_n}\nabla h_n\nabla(u_n-u_{n-1})$

$+\frac{1}{1+h_n}\nabla(h_n-h_{n-1})\nabla u_{n-1} + (\frac{1}{1+h_n}$ [redacted]

$-u_n\nabla(u_n-u_{n-1}) - (u_n-u_{n-1})\nabla u_n$ [redacted]

$\sum_{j=1}^{3} J_j = (u_n-u_{n-1})\nabla h_n + (1+h_n$ [redacted]

As in the proofs of proposition 3.1, applying the operato [redacted] multiplying the first by $\Delta_k(u_{n+1}-u_n)$, and the second [redacted] over $\mathbf{R}^2$, we obtain

$$\partial_t\|\Delta_k(u_{n+1}-u_n)\|^2_{L^2} + 2\nu\|\nabla\Delta_k(u_{n+1}-u_n)\|^2_{L^2} =$$ [redacted]

$$\partial_t\|\Delta_k(h_{n+1}-h_n)\|^2_{L^2} = \sum_{j=1}^{3}\int_{\mathbf{R}^2}\Delta_k J_j\Delta_k(h_{n+1}$$ [redacted]

Below we only consider the case of $0 < s < 1$. B [redacted] ma 2.1, Lemma 2.2, Lemma 2.3 and the fact of $\|u_n(t)\|_{H^{s+1}} \le E_1$ and $\|h_n(t)\|$ [redacted] en $t \le T$

$$\partial_t \|h_{n+1} - h_n\|^2_{H^{s+1}} \le \widetilde{A}_0^2 \|h_{n+1} - h_n\|^2_{H^{s+1}}$$
$$+ \frac{E_2^2}{4E_1^2}\|u_n - u_{n-1}\|^2_{H^{s+1}} + \frac{\nu E_2^2}{4E_1^2}\|\nabla(u_n - u_{n-1})\|^2_{H^{s+1}} + \|h_n - h_{n-1}\|^2_{H^{s+1}}, \tag{3.9}$$

where $\widetilde{A}_0^2 = 4A_0^2(1 + \frac{E_1^2}{\nu E_2^2})$.

We prove now that there exist a positive time $T_2(\le T_1)$, such that for any n

$$\|u_n - u_{n-1}\|_{L^\infty([0,T_2],H^{s+1})} + \nu\|\nabla(u_n - u_{n-1})\|_{L^2([0,T_2],H^{s+1})} \le E_1 2^{-n},$$
$$\|h_n - h_{n-1}\|_{L^\infty([0,T_2],H^{s+1})} \le E_2 2^{-n}. \tag{C_n}$$

We will prove (C_n) by induction on n. In fact, it is easy to see that (C_1) is valid if $T_2 \le T_1$.

We suppose now that (C_n) is holds and prove that (C_{n+1}) is valid by using the estimations (3.8) and (3.9). Taking integration from 0 to t on both side of (3.8), we deduce

$$\|(u_{n+1} - u_n)(t)\|^2_{H^{s+1}} + \nu \int_0^t e^{t-t'}\|\nabla(u_{n+1} - u_n)(t')\|^2_{H^{s+1}}dt'$$
$$\le e^t\|(u_{n+1} - u_n)(0)\|^2_{H^{s+1}} + te^t\frac{A_0^2}{\nu}(E_1^2 + E_2^2)2^{-2n}.$$

If $T_2 = \min\{T_1, \nu(6A_0^2)^{-1}\}$ and $t \le T_2$, we have $e^t \le 3/2, te^t 2\frac{A_0^2}{\nu} \le 3/2$, we have also

$$\|(u_{n+1} - u_n)(0)\|_{H^{s+1}} \le 2^{-(n+1)}\|\Delta_{n+1}u_0\|_{H^{s+2}} \le \frac{1}{2}E_1 2^{-(n+1)}.$$

Using (C_n), we obtain

$$\|(u_{n+1} - u_n)\|^2_{L^\infty([0,T_2],H^{s+1})} + \nu\|\nabla(u_{n+1} - u_n)\|^2_{L^2([0,t_2],H^{s+1})} \le E_1^2 2^{-2(n+1)}. \tag{3.10}$$

Taking integration from 0 to t on both side of (3.9), and using $((C_n)$, we deduce

$$\begin{aligned} &\|(h_{n+1} - h_n)(t)\|^2_{H^{s+1}} \\ \le\ & e^{\widetilde{A}_0^2 t}\|(h_{n+1} - h_n)(0)\|^2_{H^{s+1}} + 2te^{\widetilde{A}_0^2 t}E_2^2 2^{-2n}. \end{aligned}$$

Finally if $T_2 = \min\{T_1, \nu(6A_0^2)^{-1}, \widetilde{A}_0^{-2}\}$ and $t \le T_2$, we obtain

$$\|(h_{n+1} - h_n)\|^2_{L^\infty([0,T_2],H^{s+1})} \le E_2^2 2^{-2(n+1)}. \tag{3.11}$$

Proposition 3.2 is proved now with $T_2 = O(E_1^{-10}\nu^2 E_2^6)$.

Regularity and uniqueness of solutions

From the Proposition 3.2, the approximative sequence (u_n, h_n) of problems (3.1)-(3.3) is a Cauchy sequence in $\mathcal{X}([0,T_2], s+1, E_1, E_2)$ with $s > 0$. So that the limiter (u,h) is a solution of Cauchy problem (1.1)-(1.3). From the Proposition 3.1, this sequence is bounded in $\mathcal{X}([0,T_1], s+2, E_1, E_2)$, so that it is also the Cauchy sequence in $\mathcal{X}([0,T_2], s'+2, E_1, E_2)$ for all $s' < s$ by interpolation, and the limiter is in $\mathcal{X}([0,T_2], s+2, E_1, E_2)$. So we have proved the existence of solution for Theorem 3.1.

The proofs of uniqueness of solution is similar with the proofs for the convergence of approximative sequence. In fact, we consider

$$\partial_t(u - v) - \nu\triangle(u - v) = G_1(u,h) - G_1(v,g), \tag{3.12}$$
$$\partial_t(h - g) - u\nabla(h - g) = (u - v)\nabla g + G_2(u,h) - G_2(v,g), \tag{3.13}$$

with initial data $u(x,0)=v(x,0)=u_0(x)\in H^{s+2}$ and $h(x,0)=g(x,0)=h_0(x)\in H^{s+2}$.

As in the proofs of Proposition 3.2, we obtain that

$$\begin{aligned} & \|(u-v)\|^2_{L^\infty([0,T_2],H^{s+1})}+\nu\|\nabla(u-v)\|^2_{L^2([0,T_2],H^{s+1})} \\ \leq\ & 2\|(u-v)(0)\|^2_{H^{s+1}}+\frac{1}{16}(\|u-v\|^2_{L^\infty([0,T_2],H^{s+1})} \\ & +\|h-g\|^2_{L^\infty([0,T_2],H^{s+1})}+\nu\|u-v\|^2_{L^2([0,T_2],H^{s+1})}), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} & \|(g-h)(t)\|^2_{H^{s+1}}\leq 2\|(h-g)(0)\|^2_{H^{s+1}} \\ & +\frac{1}{16}(\|u-v\|^2_{L^\infty([0,T_2],H^{s+1})}+\|h-g\|^2_{L^\infty([0,T_2],H^{s+1})}+\nu\|u-v\|^2_{L^2([0,T_2],H^{s+1})}). \end{aligned} \tag{3.15}$$

Which give the uniqueness of solution.

4 The global existence for small initial data

We prove firstly *a priori estimates* for local solutions.

Theorem 4.1 *(a priori estimate) Suppose that the problem (1.1)-(1.3) has a solution* $(u,h)\in L^\infty([0,T],H^{s+1}),\nabla u\in L^2([0,T],H^{s+1}(\mathbf{R}^2))$ *for some* $T>0$ *with initial data* $u_0,h_0\in H^{s+1}(\mathbf{R}^2), s>0$*, and*

$$N(T)=(\|u\|^2_{L^\infty([0,T];H^{s+1})}+\|h\|^2_{L^\infty([0,T];H^{s+1})}+\nu\|\nabla u\|^2_{L^2([0,T];H^{s+1})})^{1/2}\leq E_0.$$

Then there exist positive constants ε *and* C_1 *with* $\varepsilon C_1\leq E_0$*, which are independent of* T *such that, if* $N(T)\leq\varepsilon$*, then*

$$N(T)\leq C_1N(0). \tag{4.1}$$

A combination of local existence theorem 3.1 and above *a priori* estimate give the following theorem.

Theorem 4.2 *Suppose that* $u_0,h_0\in H^{s+2}(\mathbf{R}^2), s>0$*. Then there exists* $\varepsilon>0$ *such that if*

$$\|u_0\|_{H^{s+2}}+\|h_0\|_{H^{s+2}}\leq\varepsilon,$$

then the Cauchy problems (3.1)-(3.1) has a unique global solution $(u,h)\in L^\infty([0,+\infty[,H^{s+2}(\mathbf{R}^2)),\nabla u\in L^2([0,\infty[,H^{s+2}(\mathbf{R}^2))$*.*

For the proof of this theorem see for example Sundbye [11].

Remark: We get the global solution with index $s+2$, since we have only the local solution with index $s+2$ in theorem 3.1. But we have proved the *a priori* estimate for small index $s+1$, so if we can get the local solution for $s+1$, we get also the global solution for small index $s+1$.

We prove now the theorem 4.1. We linearize the equations (1.1) and (1.2) on $(h,u)=(1,0)$ as following

$$\begin{cases} u_t-\nu\triangle u+\nabla h=H_1, \\ h_t+\mathrm{div}u=H_2, \end{cases} \tag{4.2}$$

where

$$\begin{cases} H_1 = & \frac{1}{1+h}\nabla h\nabla u-(u\cdot\nabla)u, \\ H_2 = & -\mathrm{div}(hu), \end{cases}$$

In the following, we will estimate (u,h) under the *a priori* assumption

$$\widetilde{N}(T)=\|h\|^2_{L^\infty([0,T],H^{s+1})}+\|u\|^2_{L^\infty([0,T],H^{s+1})}\le\delta_0 \tag{4.3}$$

where $s>0$ and $0<\delta_0<<1$.

Applying the operator Δ_k on (4.2), and multiplying first equation of (4.2) by $\Delta_k(u-\triangle u+\lambda\nabla h)$, second equation by $\Delta_k(h-\triangle h)$, then summing them and integrating over $\mathbf{R}^2$ yields

$$\begin{aligned}&\tfrac{1}{2}\partial_t(\|u_k\|^2_{H^1}+\|h_k\|^2_{H^1})+\nu\|\nabla u_k\|^2_{L^2}+\nu\|\triangle u_k\|^2_{L^2}+\lambda\|\nabla h_k\|^2_{L^2}\\ &=\int_{\mathbf{R}^2}(\Delta_kF_1\Delta_k(u-\triangle u+\lambda\nabla h)+\Delta_kF_2\Delta_k(h-\triangle h))dx\\ &-\lambda\int_{\mathbf{R}^2}(\partial_tu_k\nabla h_k-\nu\triangle u_k\nabla h_k)dx,\end{aligned} \tag{4.4}$$

where $0<\lambda<<1, u_k=\triangle_ku, h_k=\triangle_kh$.

High vertical frequencies estimates

Now we will give some estimates to the right hand of (4.4) for the case of high vertical frequencies. That means for certain M big enough, we study (4.4) for $k>M$. By lemma 2.4 we have

$$\begin{aligned}&|\int_{\mathbf{R}^2}\Delta_k(\tfrac{1}{1+h}\nabla h\nabla u)u_kdx|\\ \le\ & C_0d_k^22^{-2ks}\|u\|_{H^s}(1+\|h\|_{H^{s+1}})(\|\triangle u\|^2_{H^s}+\|\nabla h\|^2_{H^s}).\end{aligned} \tag{4.5}$$

Similar we have

$$\begin{aligned}&|\int_{\mathbf{R}^2}\Delta_k(\tfrac{1}{1+h}\nabla h\nabla u)\triangle u_kdx|\\ \le\ & C_0d_k^22^{-2ks}\|\nabla h\|_{H^s}(1+\|h\|_{H^{s+1}})\|\triangle u\|^2_{H^s},\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}&\lambda|\int_{\mathbf{R}^2}\Delta_k(\tfrac{1}{1+h}\nabla h\nabla u)\nabla h_kdx|\\ \le\ & C_0\lambda d_k^22^{-2ks}\|\nabla h\|_{H^s}(1+\|h\|_{H^{s+1}})(\|\triangle u\|^2_{H^s}+\|\nabla h\|^2_{H^s}).\end{aligned} \tag{4.7}$$

We have also

$$\begin{aligned}&|\int_{\mathbf{R}^2}\Delta_k((u\cdot\nabla)u)\triangle u_kdx|,\\ =\ &(|\int_{\mathbf{R}^2}\Delta_k(T_u\nabla u+T_{\nabla u}u)\triangle u_k+R(u,\nabla u)\triangle u_kdx|\\ \le\ &\textstyle\sum_{|q-k|\le N_1}(|\int_{\mathbf{R}^2}\Delta_k(S_qu\nabla u_q)\triangle u_kdx|+|\int_{\mathbf{R}^2}\Delta_k(S_q(\nabla u)u_q)\triangle u_kdx|\\ &+\textstyle\sum_{q\ge k-N_2,j\in\{-1,0,1\}}|\int_{\mathbf{R}^2}\Delta_k(u_q\nabla u_{q-j})\triangle u_kdx|\\ \le\ &C_0d_k^22^{-2ks}(\|u\|_{L^2}\|\triangle u\|^2_{H^s}+\|\nabla u\|_{L^2}\|\nabla u\|_{H^s}\|\triangle u\|_{H^s})\\ \le\ &C_0d_k^22^{-2ks}\|u\|_{H^{s+1}}\|\nabla u\|^2_{H^{s+1}}.\end{aligned}$$

Similarly

$$\begin{aligned}&\lambda|\int_{\mathbf{R}^2}\Delta_k((u\cdot\nabla)u)\nabla h_kdx|\\ \le\ &C_0\lambda d_k^22^{-2ks}\|u\|_{H^{s+1}}(\|\nabla u\|^2_{H^{s+1}}+\|\nabla h\|^2_{H^s}).\end{aligned}$$

Since $\|f_q\|_{H^s}\le\|\nabla f_q\|_{H^s}$ for $q\ge0$, we can obtain that

$$|\int_{\mathbf{R}^2}\Delta_k((u\cdot\nabla)u)u_kdx|\le Cd_k^22^{-2ks}\|u\|_{H^s}\|\nabla u\|^2_{H^s},$$

and

$$\begin{array}{rl} & |\int_{\mathbf{R}^2} \mathrm{div}\Delta_k(uh)h_k dx| = |\int_{\mathbf{R}^2} \Delta_k(uh)\nabla h_k dx| \\ \leq & Cd_k^2 2^{-2ks}(\|u\|_{L^2} + \|h\|_{L^2})(\|\nabla u\|_{H^s}^2 + \|\nabla h\|_{H^s}^2). \end{array}$$

By lemma 2.5, we have

$$|\int_{\mathbf{R}^2} \Delta_k(\mathrm{div}(hu)\triangle h_k dx| \leq Cd_k^2 2^{-2ks}\|\nabla h\|_{H^s}(\|\nabla h\|_{H^s}^2 + \|\nabla u\|_{H^{s+1}}^2).$$

It is easy to see

$$\lambda\nu|\int_{\mathbf{R}^2} \triangle u_k \nabla h_k dx| \leq C\lambda\nu d_k^2 2^{-2ks}(\varepsilon^{-1}\|\triangle u\|_{H^s}^2 + \varepsilon\|\nabla h\|_{H^s}^2).$$

Noting

$$\int_{\mathbf{R}^2}(\partial_t u_k)(\nabla h_k)dx = \partial_t \int_{\mathbf{R}^2} u_k \nabla h_k dx - \int_{\mathbf{R}^2} u_k \partial_t(\nabla h_k)dx,$$

we have

$$\begin{array}{rl} & \lambda|\int_{\mathbf{R}^2} u_k \partial_t(\nabla h_k)dx| \\ \leq & C\lambda d_k^2 2^{-2ks}(\|\nabla u\|_{H^s}^2 + (\|h\|_{H^{s+1}} + \|u\|_{H^{s+1}})(\|\nabla u\|_{H^s}^2 + \|\nabla h\|_{H^s}^2)). \end{array}$$

and

$$\begin{array}{rl} & \lambda|\int_0^t \partial_\tau(\int_{\mathbf{R}^2} u_k \nabla h_k dx)d\tau| \\ \leq & C\lambda d_k^2 2^{-2ks}(\|u(t)\|_{H^s}\|\nabla h(t)\|_{H^s} + \|u(0)\|_{H^s}\|\nabla h(0)\|_{H^s}). \end{array}$$

Multiplying inequality (4.4) by 2^{2ks} and integrating over $(0,t)$, we obtain

$$\begin{array}{l} \|u_k(t)\|_{H^{s+1}}^2 + \|h_k(t)\|_{H^{s+1}}^2 + \int_0^t(\nu\|\nabla u_k(\tau)\|_{H^{s+1}}^2 + \lambda\|\nabla h_k(\tau)\|_{H^s}^2)d\tau \\ \leq Cd_k^2(\|u(0)\|_{H^{s+1}}^2 + \|h(0)\|_{H^{s+1}}^2) + C\lambda\nu d_k^2 \int(\varepsilon^{-1}\|\nabla u\|_{H^{s+1}}^2 + \varepsilon\|\nabla h\|_{H^s})d\tau \\ +Cd_k^2(\|h\|_{L^\infty([0,T],H^{s+1})} + \|h\|_{L^\infty([0,T],H^{s+1})}^2 + \|u\|_{L^\infty([0,T],H^{s+1})} + \|u\|_{L^\infty([0,T],H^{s+1})}^2) \\ \times \int_0^t(\|\nabla u\|_{H^{s+1}}^2 + \|\nabla h\|_{H^s}^2)d\tau + C\lambda d_k^2(\|u(t)\|_{H^s}^2 + \|h(t)\|_{H^{s+1}}^2). \end{array} \tag{4.8}$$

Low vertical frequencies estimates

Now we will consider the low vertical frequencies. Denoting $S_M = \sum_{k<M}\Delta_k$, and applying the operator S_M on (4.2), and multiplying first equation of (4.2) by $S_k(u + \lambda\nabla h)$, second equation by $S_k h$, then summing them and integrating over $\mathbf{R}^2$ yields

$$\begin{array}{rl} & \frac{1}{2}\partial_t(\|S_M u\|_{L^2}^2 + \|S_M h\|_{L^2}^2) + \nu\|\nabla S_M u\|_{L^2}^2 + \lambda\|\nabla S_M h\|_{L^2}^2 \\ = & \int_{\mathbf{R}^2}(S_M(F_1)S_M(u + \lambda\nabla h) + S_M(F_2)S_M h)dx \\ & -\lambda\int_{\mathbf{R}^2}(\partial_t S_M u \nabla S_M h - \nu\triangle S_M u \nabla S_M h)dx, \end{array} \tag{4.9}$$

where $0 < \lambda << 1$. As in the proofs of (4.8), we will give some estimates to the right hand of (4.9). It is easy to see that

$$|\int_{\mathbf{R}^2} S_M(\frac{\nabla h}{1+h}\nabla u)S_M u dx| \leq C\|\frac{1}{1+h}\|_{L^\infty}\|\nabla h\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^2},$$

and

$$\begin{array}{rl} & \lambda|\int_{\mathbf{R}^2} S_M(\frac{\nabla h}{1+h}\nabla u - (u\cdot\nabla)u)S_M(\nabla h)dx| \\ \leq & C\lambda(\|\frac{1}{1+h}\|_{L^\infty}\|\nabla h\|_{L^2}^2\|\nabla u\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla h\|_{L^2}). \end{array}$$

Using the estimates (2.2), we have

$$|\int_{\mathbf{R}^2} S_M((u\cdot\nabla)u)S_M u dx| \le C\|u\|^2_{\dot H^{1/2}}\|\nabla u\|_{L^2} \le C\|u\|_{L^2}\|\nabla u\|^2_{L^2},$$

and

$$\begin{aligned} |\textstyle\int_{\mathbf{R}^2} S_M(\mathrm{div}(uh))S_M h dx| \le &\; C\|u\|_{\dot H^{1/2}}\|h\|_{\dot H^{1/2}}\|\nabla h\|_{L^2} \\ \le &\; C(\|u\|_{L^2}+\|h\|_{L^2})(\|\nabla u\|^2_{L^2}+\|\nabla h\|^2_{L^2}). \end{aligned}$$

Note that $\|S_M\triangle u\|_{L^2} \le 2^M\|S_M\nabla u\|_{L^2}$, we have

$$|\lambda\nu\int_{\mathbf{R}^2} S_M(\triangle u)S_M(\nabla h)dx| \le C\lambda\nu(\varepsilon^{-1}\|\nabla u\|^2_{L^2}+\varepsilon\|\nabla h\|^2_{L^2}).$$

As in the above proofs, we have

$$\begin{aligned} \lambda|\textstyle\int_{\mathbf{R}^2} S_M u S_M(\partial_t(\nabla h))dx| \le &\; C\lambda(\|h\|_{L^2}+\|u\|_{L^2})(\|\nabla u\|^2_{L^2}+\|\nabla h\|^2_{L^2}) \\ &+C\lambda(\varepsilon^{-1}\|\nabla u\|^2_{L^2}+\varepsilon\|\nabla h\|^2_{L^2}), \end{aligned}$$

and

$$\lambda|\int_0^t\partial_t\int_{\mathbf{R}^2} S_M u S_M(\nabla h)dxd\tau| \le C\lambda(\|u\|_{L^2}\|\nabla h\|_{L^2}+\|u(0)\|_{L^2}\|\nabla h(0)\|_{L^2}).$$

Integrating both side of (4.9) over $(0,t)$, and summing above estimates to the right hand of (4.9), we have

$$\begin{aligned} &(\|S_M u\|^2_{L^2}+\|S_M h\|^2_{L^2}) + \textstyle\int_0^t(\nu\|\nabla S_M u\|^2_{L^2}+\lambda\|\nabla S_M h\|^2_{L^2})d\tau \\ \le\; & C(\|u\|_{L^\infty([0,T],H^{s+1})}+\|h\|_{L^\infty([0,T],H^{s+1})})\textstyle\int_0^t(\|\nabla u\|^2_{H^s}+\|\nabla h\|^2_{H^s})d\tau \\ &+C\lambda(\|u(t)\|^2_{L^2}+\|\nabla h(t)\|^2_{L^2})+C(\|u(0)\|^2_{L^2}+\|\nabla h(0)\|^2_{L^2}) \\ &+C\lambda\nu\textstyle\int_0^t(\varepsilon^{-1}\|\nabla u\|^2_{L^2}+\varepsilon\|\nabla h\|^2_{L^2})d\tau. \end{aligned}$$

Since $\|S_M f\|_{H^s} \le 2^{Ms}\|S_M f\|_{L^2}$, we can write

$$\begin{aligned} &(\|S_M u\|^2_{H^{s+1}}+\|S_M h\|^2_{H^{s+1}}) + \textstyle\int_0^t(\|\nabla S_M u\|^2_{H^{s+1}}+\lambda\|\nabla S_M h\|^2_{H^s})d\tau \\ \le\; & C(\|u\|_{L^\infty([0,T],H^{s+1})}+\|h\|_{L^\infty([0,T],H^{s+1})})\textstyle\int_0^t(\|\nabla u\|^2_{H^{s+1}}+\|\nabla h\|^2_{H^s})d\tau \\ &+C\lambda(\|u(t)\|^2_{H^s}+\|\nabla h(t)\|^2_{H^s})+C(\|u(0)\|^2_{H^s}+\|\nabla h(0)\|^2_{H^s}) \\ &+\lambda\textstyle\int_0^t(\varepsilon^{-1}\|\nabla u\|^2_{H^s}+\varepsilon\|\nabla h\|^2_{H^s})d\tau. \end{aligned}$$

Thus we obtain from (4.8) and above inequality

$$\begin{aligned} &\|u(t)\|^2_{H^{s+1}}+\|h(t)\|^2_{H^{s+1}}+\textstyle\int_0^t(\|\nabla u(\tau)\|^2_{H^{s+1}}+\lambda\|\nabla h(\tau)\|^2_{H^s})d\tau \\ \le\; & C(\|h\|_{L^\infty([0,T],H^{s+1})}+\|u\|_{L^\infty([0,T],H^{s+1})}+\|h\|^2_{L^\infty([0,T],H^{s+1})}+\|u\|^2_{L^\infty([0,T],H^{s+1})}) \\ &\times\textstyle\int_0^t(\|\nabla u\|^2_{H^{s+1}}+\|\nabla h\|^2_{H^s})d\tau+C(\|u(0)\|^2_{H^{s+1}}+\|h(0)\|^2_{H^{s+1}}) \\ &+C\lambda(\|u(t)\|^2_{H^{s+1}}+\|h(t)\|^2_{H^{s+1}})+C\lambda\nu\textstyle\int_0^t(\varepsilon^{-1}\|\nabla u\|^2_{H^{s+1}}+\varepsilon\|\nabla h\|^2_{H^s})d\tau. \end{aligned}$$

Taking ε and λ small enough, such that $C\varepsilon=\frac{1}{2}$ and $C\lambda\varepsilon^{-1}=\frac{1}{4}$, we get

$$\begin{aligned} &\|u(t)\|^2_{H^{s+1}}+\|h(t)\|^2_{H^{s+1}}+\textstyle\int_0^t(\nu\|\nabla u(\tau)\|^2_{H^{s+1}}+\lambda\|\nabla h(\tau)\|^2_{H^s})d\tau \\ \le\; & C(\|h\|_{L^\infty([0,T],H^{s+1})}+\|h\|^2_{L^\infty([0,T],H^{s+1})}+\|u\|_{L^\infty([0,T],H^{s+1})}+\|u\|^2_{L^\infty([0,T],H^{s+1})}) \\ &\times\textstyle\int_0^t(\nu\|\nabla u\|^2_{H^{s+1}}+\lambda\|\nabla h\|^2_{H^s})d\tau+C(\|u(0)\|^2_{H^{s+1}}+\|h(0)\|^2_{H^{s+1}}). \end{aligned}$$

For fixed λ, taking δ_0 small enough, such that $C\delta_0<\frac{1}{5}\min\{1,\lambda\}$, we obtain

$$\begin{aligned} &\|u(t)\|^2_{H^{s+1}}+\|h(t)\|^2_{H^{s+1}}+\textstyle\int_0^t(\nu\|\nabla u(\tau)\|^2_{H^{s+1}}+\lambda\|\nabla h(\tau)\|^2_{H^s})d\tau \\ \le\; & C(\|u(0)\|^2_{H^{s+1}}+\|h(0)\|^2_{H^{s+1}}). \end{aligned}$$

Which prove the theorem 4.1

5 Losing energy estimates

We prove now the losing energy estimates of the section 2, the proofs in this section are technique.

Lemma 5.1 *Let $\tau > 1$ and $-1 \leq k < +\infty$, then there exists $C > 0$ such that for all $v, \nabla v, g, \nabla g \in H^\tau$, we have*

$$|\int_{\mathbf{R}^2} \Delta_k(v\nabla g)\Delta_k g dx| \leq C d_k^2 2^{-2k\tau}\|v\|_{H^{\tau+1}}\|g\|_{H^\tau}^2,$$

with $\{d_k\} \in \ell^2$ and $\|\{d_k\}\|_{\ell^2} \leq 1$.

Proof. By the paraproduct calculation, we have

$$\begin{array}{l}\int_{\mathbf{R}^2} \Delta_k(v\nabla g)\Delta_k g dx = \int_{\mathbf{R}^2} \Delta_k(T_{\nabla g}v)\Delta_k g dx + \int_{\mathbf{R}^2} \Delta_k(T_v\nabla g)\Delta_k g dx \\ + \int_{\mathbf{R}^2} \Delta_k R(v, \nabla g)\Delta_k g dx = I_1 + I_2 + I_3.\end{array}$$

Then there exists $N_1 > 0$ such that for any fixed $M > N_1$ and $k > M$,

$$\begin{array}{rl}|I_1| \leq & \sum_{|q-k|\leq N_1} \|S_q(\nabla g)\|_{L^\infty}\|\Delta_q v\|_{L^2}\|\Delta_k g\|_{L^2} \\ \leq & \sum_{|q-k|\leq N_1} \|S_q g\|_{L^\infty}\|\Delta_q(\nabla v)\|_{L^2}\|\Delta_k g\|_{L^2} \leq C d_k^2 2^{-2k\tau}\|g\|_{H^\tau}^2\|v\|_{H^{\tau+1}}.\end{array}$$

Here we have used Sobolev inequality for $\|g\|_{L^\infty}$ since $\tau > 1$. For $k \leq M$, by using $\sum_{|q-k|\leq N_1} \|S_q(\nabla g)\|_{L^\infty} \leq C 2^M \|g\|_{L^\infty}$, we can get the same results.

For the term I_2, since we want pass the operator ∇ from g to v, we rewrite

$$\begin{array}{rl}I_2 & = \sum_{|q-k|\leq N_1} \int_{\mathbf{R}^2} \Delta_k(S_q v \Delta_q(\nabla g))\Delta_k g dx \\ & = \sum_{|q-k|\leq N_1} (\int_{\mathbf{R}^2} [\Delta_k, S_q v]\Delta_q(\nabla g)\Delta_k g dx \\ & + \int_{\mathbf{R}^2} (S_q - S_k)v\Delta_k\Delta_q(\nabla g)\Delta_k g dx) + \int_{\mathbf{R}^2} S_k v\Delta_k(\nabla g)\Delta_k g dx).\end{array}$$

Note that the operator Δ_k are convolution operators in $\mathbf{R}^2$, so

$$[\Delta_k, S_q v]\Delta_q(\nabla g) = 2^{2k}\int_{\mathbf{R}^2} (S_q v(x) - S_q v(y)) f(2^k(x-y))\Delta_q(\nabla g)(y) dy,$$

where $f(x) = (\mathcal{F}^{-1}\varphi)(x)$. Using the fact $|q-k| \leq N_1$ and Hausdorff-Yang inequality, we have

$$\begin{array}{rl} & \sum_{|q-k|\leq N_1} \|[\Delta_k, S_q v]\Delta_q(\nabla g)\|_{L^2} \\ \leq & C\sum_{|q-k|\leq N_1} 2^{2k}\|\nabla(S_q v)\|_{L^\infty} 2^{-k}\|(2^k\cdot)f(2^k\cdot)\|_{L^1}\|\Delta_q(\nabla g)\|_{L^2} \\ \leq & C d_k 2^{-k\tau}\|\nabla v\|_{L^\infty}\|g\|_{H^\tau},\end{array}$$

Similar calculus for the other terms, we get

$$|I_2| \leq C d_k^2 2^{-2k\tau}\|\nabla v\|_{H^\tau}\|g\|_{H^\tau}^2.$$

Finally, for I_3, there exists $N_1 > 0$ such that

$$\begin{array}{rl}|I_3| \leq & \sum_{q\geq k-N_2, j\in\{-1,0,1\}} |\int_{\mathbf{R}^2} \Delta_k(\Delta_q v\Delta_{q-j}(\nabla g))\Delta_k g dx| \\ \leq & C\sum_{q\geq k-N_2} \|\Delta_q v\|_{L^2}\|\Delta_{q-j}(\nabla g)\|_{L^\infty}\|\Delta_k g\|_{L^2} \\ \leq & C d_k 2^{-2k\tau}(\sum_{q\geq k-N_2} d_q 2^{-(q-k)\tau})\|v\|_{H^{\tau+1}}\|g\|_{H^\tau}^2.\end{array}$$

Denote $d_k' = (\sum_{q\geq k-N_2} d_q 2^{-(q-k)\tau})$, then $\{d_k'\} \in l^2$ since $q > k$ and $\tau > 1$. For convenience sake, we also denote d_k' by d_k below. Thus

$$|I_3| \leq C d_k^2 2^{-2k\tau}\|v\|_{H^{\tau+1}}\|g\|_{H^\tau}^2.$$

The lemma 5.1 is proved.

$u\{d$

Lemma 5.2 *(a) Let $\tau > 2$ and $-1 \le k <$ + [redacted] for all $f, v, g, u, \nabla u \in H^\tau$, we have*

$$|\int_{\mathbf{R}^2} \Delta_k(\frac{\nabla f}{1+g}\nabla v)\Delta_k \ldots Cd_k^2 2^{-2k\tau} \ldots$$

with $\{d_k\} \in \ell^2$, where

$$H_0(g) \ldots + g) \ldots$$

and B_0 is the function give in (2. … et 1 … $C > 0$ such that for all $f, g, u, \nabla u, \ldots H^\tau$, …

$$|\int_{\mathbf{R}^2} \Delta_k \ldots \Delta_k \ldots$$
$$\ldots d_k^2 2^{-2k} \ldots f\|_{H} \ldots$$

with $\{d_k\} \in \ell^2$, where

$$U_\tau(u, v) =: \|\nabla v\|_{\ldots} + \ldots$$

Proof. (a) As the proof …

$$\int_{\mathbf{R}^2} \Delta_k(\frac{\nabla f}{1+g}\nabla v)\Delta_k u dx = \ldots T_{\frac{\nabla f}{1+g}}\nabla v + T_{\nabla v}\frac{\nabla f}{1+g} + R(\frac{\nabla f}{1+ \ldots} \ldots u dx.$$

For $k > M$, we have

$$
\begin{aligned}
|\int_{\mathbf{R}^2} \Delta_k(T_{\frac{\nabla f}{1+g}}\nabla v)\Delta_k \ldots & \ldots \sum_{|q-k|\le N_1} \|\frac{\nabla f}{1+g}\|_{L^\infty}\|\Delta_q(\nabla v)\| \ldots L^2 \\
& \ldots \sum_{|q-k|\le N_1} \|\frac{\nabla f}{1+g}\|_{L^\infty} 2^q \|\Delta_q v\| \ldots \\
& \le Cd_k^2 2^{-2k\tau}\|\frac{1}{1+g}\|_{L^\infty}\|f\|_{H^\tau}\|v\| \ldots \\
& \ldots \|_{kH'},
\end{aligned}
$$

f

(b) Firstly, we write

$$\begin{aligned} & |\int_{\mathbf{R}^2} \Delta_k(\tfrac{\nabla f}{1+g}\nabla v)\Delta_k u dx| \leq |\int_{\mathbf{R}^2} \Delta_k(T_{\frac{\nabla v}{1+g}}\nabla f)\Delta_k u dx| \\ + & |\int_{\mathbf{R}^2} \Delta_k(T_{\nabla f}\tfrac{\nabla v}{1+g})\Delta_k u dx| + |\int_{\mathbf{R}^2} \Delta_k R(\tfrac{\nabla v}{1+g}, \nabla f))\Delta_k u dx|, \end{aligned}$$

The estimation for the first term and the third term is easy, so we discuss only the second term, we consider also two cas, if $k > M$, we have

$$|\int_{\mathbf{R}^2} \Delta_k(T_{\nabla f}\frac{\nabla v}{1+g})\Delta_k u dx| \leq \sum_{|q-k|\leq N_1} \|S_q(\nabla f)\|_{L^\infty}\|\Delta_q(\frac{\nabla v}{1+g})\|_{L^2}\|\Delta_k u\|_{L^2}.$$

Since $1 < \tau < 2$, we have

$$\|S_q(\nabla f)\|_{L^\infty} \leq \sum_{p\leq q+2} 2^{2p}\|\Delta_p f\|_{L^2} \leq C2^{-q(\tau-2)}\|f\|_{H^\tau},$$

and

$$\|\frac{\nabla v}{1+g}\|_{H^\tau} \leq \|\nabla v\|_{H^\tau}(1 + \|\frac{g}{1+g}\|_{H^\tau}) \leq \|\nabla v\|_{H^\tau}(1 + \|(\frac{1}{1+g})^2\|_{L^\infty}\|g\|_{H^\tau}),$$

one has

$$|\int_{\mathbf{R}^2} \Delta_k(T_{\nabla f}\frac{\nabla v}{1+g})\Delta_k u dx| \leq Cd_k^2 2^{-2k\tau} H_0(g)(1 + \|g\|_{H^\tau})\|f\|_{H^\tau}\|\nabla v\|_{H^\tau}\|\nabla u\|_{H^1}.$$

If $k \leq M$, it is easy to see that

$$|\int_{\mathbf{R}^2} \Delta_k(T_{\nabla f}\frac{\nabla v}{1+g})\Delta_k u dx| \leq Cd_k^2 2^{-2k\tau} H_0(g)(1 + \|g\|_{H^\tau})\|f\|_{H^\tau}\|\nabla v\|_{H^\tau}\|u\|_{H^\tau}.$$

The lemma is proved.

Lemma 5.3 *(a) Let $\tau > 2$ and $-1 \leq k < +\infty$, then there exists $C > 0$ such that for all $f, v, u, \nabla u, g_1, g_2 \in H^\tau$, we have*

$$\begin{aligned} & |\int_{\mathbf{R}^2} \Delta_k(\tfrac{(g_1-g_2)}{(1+g_1)(1+g_2)}\nabla f \nabla v)\Delta_k u dx| \\ \leq & \; Cd_k^2 2^{-2k\tau} H_1(g_1, g_2)\|f\|_{H^\tau}\|v\|_{H^\tau}\|g_1 - g_2\|_{H^\tau}(\|\nabla u\|_{H^\tau} + \|u\|_{H^\tau}), \end{aligned}$$

with $\{d_k\} \in \ell^2$, and

$$\begin{aligned} H_1(g_1, g_2) = & \; (1 + \|(1 + g_1)^{-1}\|^2_{L^\infty}\|g_1\|_{H^\tau})(1 + \|(1 + g_2)^{-1}\|^2_{L^\infty}\|g_2\|_{H^\tau}) \\ & + \|(1 + g_1)^{-1}\|^2_{L^\infty}\|(1 + g_2)^{-1}\|^2_{L^\infty}. \end{aligned}$$

(b) Let $1 < \tau < 2$ and $-1 \leq k < +\infty$, then there exists $C > 0$ such that for all $f, v, g_1, g_2, u, \nabla u, v, \nabla v \in H^\tau$, we have

$$\begin{aligned} & |\int_{\mathbf{R}^2} \Delta_k(\tfrac{(g_1-g_2)}{(1+g_1)(1+g_2)}\nabla f \nabla v)\Delta_k u dx| \\ \leq & \; Cd_k^2 2^{-2k\tau} H_1(g_1, g_2)\|f\|_{H^\tau}\|g_1 - g_2\|_{H^\tau} U_\tau(u, v), \end{aligned}$$

with $\{d_k\} \in \ell^2$, and $U_\tau(u, v)$ as in Lemma 5.2, (b).

The proof of this lemma is similar to Lemma 5.2, just remark that if $F_j = \frac{1}{1+g_j}, \bar{F_j} = \frac{g_j}{1+g_j} (j = 1, 2)$, we have

$$F = \frac{g_1 - g_2}{(1 + g_1)(1 + g_2)} = (g_1 - g_2)F_1F_2 = (g_1 - g_2)(1 - \bar{F_1} - \bar{F_2} + \bar{F_1}\bar{F_2}).$$

And we have the following estimates

$$\begin{aligned} \|F\|_{L^\infty} &\leq C\|g_1 - g_2\|_{L^\infty}\|F_1\|_{L^\infty}\|F_2\|_{L^\infty}, \\ \|\Delta_q F\|_{L^2} &\leq Cd_q^2 2^{-2q\tau}\|g_1 - g_2\|_{H^\tau}(1 + \|F_1\|^2_{L^\infty}\|g_1\|_{H^\tau})(1 + \|F_2\|^2_{L^\infty}\|g_2\|_{H^\tau}). \end{aligned}$$

Below we will consider the losing energy estimate for the case of high vertical frequencies, i. e., $k > M$. Here, we assume that $M > N_1 + N_2$.

Lemma 5.4 *Let $\tau > 0$ and $M \leq k < \infty$, then there exists $C > 0$ such that for all $g, u, v, \nabla g, \nabla u \in H^\tau$, we have*

$$\begin{aligned} & |\textstyle\int_{\mathbf{R}^2} \Delta_k(\frac{1}{1+h}\nabla h\nabla u)\Delta_k v dx| \\ \leq\ & Cd_k^2 2^{-2k\tau}H_0(h)(1 + \|h\|_{H^{\tau+1}})\|\triangle u\|_{H^\tau}\|\nabla h\|_{H^\tau}\|v\|_{H^\tau}, \end{aligned}$$

with $\{d_k\} \in l^2$.

The proof of this lemma is similar with lemma 5.2 and the following lemma.

Lemma 5.5 *Let $\tau > 0$ and $M \leq k < \infty$, then there exists $C > 0$ such that for all $g \in H^{\tau+1}$ and $u \in H^{\tau+2}$, we have*

$$\begin{aligned} & |\textstyle\int_{\mathbf{R}^2} \Delta_k(\mathrm{div}(hu))\Delta(\Delta_k h)dx| \\ \leq\ & Cd_k^2 2^{-2k\tau}\|\nabla h\|_{H^\tau}(\|\nabla h\|^2_{H^\tau} + \|\nabla u\|^2_{H^{\tau+1}}), \end{aligned}$$

with $\{d_k\} \in l^2$.

Proof. Firstly we write

$$\begin{aligned} & |\textstyle\int_{\mathbf{R}^2} \Delta_k(\mathrm{div}(hu))\Delta(\Delta_k h)dx| \\ \leq\ & |\textstyle\int_{\mathbf{R}^2} \Delta_k(\nabla h\mathrm{div}u)\Delta_k(\nabla h)dx| + |\textstyle\int_{\mathbf{R}^2} \Delta_k(h\nabla(\mathrm{div}u)\Delta_k(\nabla h)dx| \\ & + |\textstyle\int_{\mathbf{R}^2} \Delta_k(u\nabla h)\Delta_k(\triangle h)dx|. \end{aligned}$$

It is easy to estimate the first and the second terms by

$$Cd_k^2 2^{-2k\tau}(\|\nabla u\|_{L^\infty}\|\nabla h\|^2_{H^\tau} + \|\nabla h\|_{L^2}\|\triangle u\|_{H^\tau}\|\nabla h\|_{H^\tau}),$$

for the third term, since we can't control $\triangle h$, we first need to write

$$\begin{aligned} & \textstyle\int_{\mathbf{R}^2} \Delta_k(T_u\nabla h)\Delta_k(\triangle h)dx = \sum_{|q-k|\leq N_1}\int_{\mathbf{R}^2} \Delta_k(S_q u\nabla h_q)\Delta_k(\triangle h)dx \\ =\ & \textstyle\sum_{|q-k|\leq N_1}\int_{\mathbf{R}^2}((S_q u\Delta_k\Delta_q(\nabla h)) + [\Delta_k, S_q u]\Delta_q(\nabla h))\Delta_k(\triangle h)dx \\ =\ & \textstyle\sum_{|q-k|\leq N_1}\int_{\mathbf{R}^2}((S_q - S_k)u\Delta_k\Delta_q(\nabla h) + [\Delta_k, S_q u]\Delta_q(\nabla h))\Delta_k(\triangle h)dx \\ & + \textstyle\int_{\mathbf{R}^2} S_k u\nabla(\Delta_k h)\triangle(\Delta_k h)dx = K_1 + K_2 + K_3. \end{aligned}$$

Since $(S_q - S_k)u = -\sum_{q\leq p\leq k-1}\Delta_p u$,

$$\begin{aligned} K_1 &\leq C\textstyle\sum_p\|\Delta_p u\|_{L^\infty}\|\nabla\Delta_k h\|_{L^2}\|\triangle\Delta_k h\|_{L^2} \\ &\leq Cd_k^2 2^{-2k\tau}\|\triangle u\|_{H_\tau}\|\nabla h\|_{L^2}\|\nabla h\|_{H^\tau}. \end{aligned}$$

As the proof of lemma 5.1, we have

$$K_2 \leq C d_k^2 2^{-2k\tau} \|\nabla u\|_{L^\infty} \|\nabla h\|_{H^\tau}^2.$$

Using the following calculus

$$\int_{\mathbf{R}^2} (S_k u \nabla(\Delta_k h)) \triangle (\Delta_k h) dx = \int_{\mathbf{R}^2} (\frac{1}{2} \mathrm{div}(S_k u) |\nabla(\Delta_k h)|^2 - \sum_{i,j} \partial_j (S_k u^i) \partial_i (\Delta_k h) \partial_j (\Delta_k h)) dx,$$

we get immediately

$$K_3 \leq C d_k^2 2^{-2k\tau} \|\nabla u\|_{L^\infty} \|\nabla h\|_{H^\tau}^2.$$

The Lemma has been proved.

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