Two-scale contact process

L. Belhadji and N. Lanchier

Abstract Most particle systems are described by local interactions between neighboring sites of the lattice $\mathbb{Z}^d$, the neighborhood of each site being deduced from the neighborhood of 0 by translation. In this article, we turn $\mathbb{Z}^d$ into a "chessboard" by superposing a mesoscopic lattice on the usual microscopic lattice, and investigate an extension of the contact process in which interactions occur at both site (microscopic) level and square (mesoscopic) level. The superposition of two interaction levels induces two birth rates, called within and outside birth rates, respectively. We study the effect of patchiness, defined as the ratio of the mesoscopic scale over the microscopic scale, on the survival probability of the particle system. We find that particles are more likely to spread out as patchiness increases, even if the outside birth rate decreases significantly with the square size.

1. Introduction

The contact process is a stochastic model including a spatial structure in the form of local interactions (Harris, 1974). Each site of the d-dimensional integer lattice is either empty or occupied by a particle. Empty sites become occupied at a rate proportional to the number of particles present in some interaction neighborhood. The proportionality constant is usually called the birth rate of the particles. Occupied sites become empty at rate 1, regardless of the state of their neighbors. In any dimension, it is known that the system exhibits a phase transition, that is, starting from the "all occupied" configuration, each site of the lattice gets occupied infinitely many often if and only if the birth rate exceeds some threshold (Liggett, 1999). The smaller the dimension, the greater the threshold for the particles to spread.

In this article, we investigate an extension of the contact process with interactions at two scale levels. More precisely, we think of the integer lattice as an infinite "chessboard", each square of the chessboard containing the same number of sites, and assume that interactions occur at both site level (or microscopic level) and square level (or mesoscopic level). The superposition of microscopic and mesoscopic lattices induces two parameters, namely the within birth rate (the rate at which a particle gives birth onto adjacent sites), and the outside birth rate (the rate at which a particle gives birth onto adjacent squares). Our first results show that, like the basic contact process, our "two-scale" contact process exhibits phase transitions in the sense that, for each fixed within birth rate (respectively, outside birth rate), the particles can spread if and only if the outside birth rate (respectively, the within birth rate) exceeds some critical value. But our main objective is to determine the effect of patchiness, defined as the ratio of the mesoscopic scale over the microscopic scale, on the interacting particle system. We find that the particles are more likely to spread out as patchiness increases.

To define the mesoscopic structure, we let $N \geq 1$ be an integer referred to as the space scale, and let $L_N = (2N-1)\,\mathbb{Z}^d$ denote the mesoscopic lattice. Then, we tile $\mathbb{Z}^d$ with squares of length side $2N-1$ by setting

$$\mathcal{H} \;=\; (-N,N)^d \qquad \text{and} \qquad \mathcal{H}_z \;=\; (2N-1)\,z \;+\; \mathcal{H} \quad \text{for any } z \in \mathbb{Z}^d.$$

We shall use the 2-dimensional terminology like "chessboard" or "square" even if our results hold in any dimension. The previous partition allows us to introduce the equivalence relation

$$x \asymp y \quad \text{if and only if there exists } z \in \mathbb{Z}^d \text{ such that } x,\, y \in \mathcal{H}_z.$$

That is, the class of site $x \in \mathbb{Z}^d$, denoted by $\dot{x}$ later, is the square containing site x. To describe the contacts between adjacent sites, we also introduce an interaction neighborhood: $x \sim y$ indicates

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that site y is one of the $2d$ nearest neighbors of site x. To take into account the interactions between adjacent squares, we extend the binary relation $\sim$ to $\mathbb{Z}^d / \asymp$ by

$$\dot x \sim \dot y \quad \text{if and only if there exist } w, z \in \mathbb{Z}^d \text{ with } w \sim z \text{ such that } \dot x = \mathcal{H}_w \text{ and } \dot y = \mathcal{H}_z.$$

The two-scale contact process is a continuous-time Markov process in which the state at time t is a function $\eta_t : \mathbb{Z}^d \longrightarrow \{0, 1\}$. A site $x \in \mathbb{Z}^d$ is said to be empty if $\eta_t(x) = 0$, and occupied by a particle otherwise. The state of site x flips according to the following transition rates:

$$\begin{array}{lll} 0 \ \to \ 1 & \text{at rate} & \displaystyle \frac{\alpha}{Z_N} \sum_{\dot y \sim \dot x} \sum_{z \in \dot y} \eta(z) \ \mathbb{1}_{\{x \in L_N\}} \ + \ \beta \sum_{z \sim x} \eta(z) \ \mathbb{1}_{\{z \in \dot x\}} \\ 1 \ \to \ 0 & \text{at rate} & 1 \end{array}$$

where Z_N is a positive polynomial in N. In other words, each particle tries to give birth at rate α / Z_N (respectively, at rate β) to a particle which is then sent to the center of a randomly chosen adjacent square (respectively, to a randomly chosen adjacent site). In both cases, the birth actually occurs if and only if the offspring is sent to an empty site. Each particle dies at rate 1. The parameters α and β are the (intrinsic) outside birth rate, and the within birth rate, respectively. The presence of the function Z_N means that large squares make it harder for particles to invade adjacent squares. We are interested in the special cases $Z_N \equiv 1$ (no dependency on the square size), $Z_N = 2N - 1$ (the rate decreases with the distance between centers), and $Z_N = (2N - 1)^d$ (the rate decreases with the volume of the squares).

We now discuss the existence of a nontrivial stationary distribution (where nontrivial means different from the "all 0" configuration). Monotonicity is our key tool. If we think of the process as being generated by a collection of independent Poisson processes (Harris, 1972), a standard coupling argument allows us to prove that the process η_t is attractive regadless of the value of Z_N and that

Lemma 1.1 *$P_0^{\eta}(|\eta_t| \geq 1$ for all $t \geq 0)$ is nondecreasing with respect to α and β,*

where P_0^{η} denotes the law of the process starting with a single particle at site 0.

We observe that, when $N = 1$, the value of β is irrelevant and the process reduces to a contact process with parameter α / Z_1. In this case, there exists a critical value $\lambda_c \in (0, \infty)$ such that the following holds: If $\alpha / Z_1 \leq \lambda_c$ the process converges in distribution to the "all 0" configuration, while if $\alpha / Z_1 > \lambda_c$ there is a nontrivial stationary distribution. See Liggett (1999), Theorem 2.25, or Bezuidenhout and Grimmett (1990). In the same way, when $\beta = 0$ and $N \geq 2$, the particles concentrate on the mesoscopic lattice L_N and the process $\hat\eta_t(x) = \eta_t((2N-1)\, x)$ becomes a contact process with parameter α / Z_N, which implies that there exists a nontrivial stationary distribution if and only if $\alpha / Z_N > \lambda_c$.

We now deal with the case $N \geq 2$ and $\beta > 0$. For each fixed birth rate β (respectively, α), we find that survival is possible if and only if α (respectively, β) exceeds some critical value. The result is rigorously stated in the following two theorems.

Theorem 1 *For all $N \geq 2$ and $\beta > 0$, there is $0 < \alpha_c \leq Z_N \, \lambda_c$ such that if $\alpha < \alpha_c$ the process converges to the "all 0" configuration, while if $\alpha > \alpha_c$ there is a nontrivial stationary distribution.*

Theorem 2 *For all $N \geq 2$ and $0 < \alpha < Z_N \, \lambda_c$, there is $\beta_c > 0$ such that if $\beta < \beta_c$ the process converges to the "all 0" configuration, while if $\beta > \beta_c$ there is a nontrivial stationary distribution.*

The key of our proof is to consider the two-scale contact process as an extension of the individual recovery process (Belhadji and Lanchier, 2006) in which clusters are "spatialized". The basic idea is to construct the individual recovery process from the two-scale contact process through a "destruction" of the spatial structure of each square by turning our interacting finite state contact processes into interacting random walks (see Section 3). This trick allows us to deduce quite easily

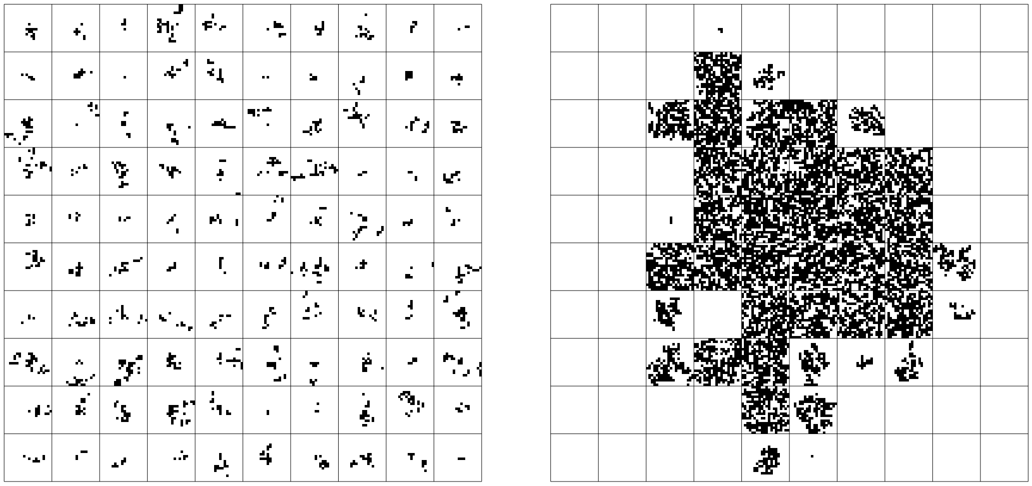

Figure 1. *Two-scale contact process on the* 200×200 *square with periodic boundary conditions. Picture on the left:* $Z_N \equiv 1$, $\alpha = 3$ *and* $\beta = 1.5$ ($\alpha > \lambda_c > \beta$). *Picture on the right:* $Z_N \equiv 1$, $\alpha = 0.002$ *and* $\beta = 3$ ($\alpha < \lambda_c < \beta$).

our first two theorems from their analogues in Belhadji and Lanchier (2006). Theorems 1 and 2 are illustrated by the numerical simulations given in Figure 1.

The most interesting feature of the two-scale contact process is the connection between the patchiness of the environment, modeled by N, and the survival probability of the particle system. When $Z_N \equiv 1$, it turns out that the particles are more likely to spread out as N increases.

Lemma 1.2 *Let* $Z_N \equiv 1$. *Then* $P_0^{\eta}\,(|\eta_t| \geq 1$ *for all* $t \geq 0)$ *is nondecreasing with respect to* N.

When Z_N is rapidly increasing, typically $Z_N = (2N-1)^d$, the contacts between adjacent squares are weaker and weaker as N increases, so we expect that increasing patchiness drives the process to extinction. Theorem 3, however, reveals that

Theorem 3 *For all* $\alpha > 0$ *and* $\beta > \lambda_c$ *there exists a nontrivial stationary distribution provided the space scale* N *is sufficiently large. In particular, when* $Z_N \equiv 1$ *and* $\alpha < \lambda_c$, *there exists* $N_c \geq 2$ *such that if* $N < N_c$ *the process converges to the "all 0" configuration, while if* $N > N_c$ *there is a nontrivial stationary distribution.*

When Z_N is a general polynomial in N, the result is not really inconsistent with the fact that invasion of the neighboring squares is harder when N is large since it does not tell us that survival is possible *only* if N is sufficiently large. But the combination of Theorems 1 and 3 implies that

Theorem 4 *For all* $N_1 \geq 1$ *and* $\beta > \lambda_c$ *there exist* $\alpha_0 > 0$ *and* $N_2 \geq N_1$ *such that the process with parameters* (α_0, β, N) *converges to the "all 0" configuration if* $N < N_1$, *while there is a nontrivial stationary distribution if* $N > N_2$.

The proof is quite easy. If $N \leq N_1$ then Theorem 1 implies the existence of a certain $\alpha_0(N) > 0$ such that the process with parameters $(\alpha_0(N), \beta, N)$ converges in distribution to the "all 0" configuration. Theorem 4 then follows by setting $\alpha_0 = \min\{\alpha_0(N),\ N \leq N_1\}$ and applying Theorem 3 with $\alpha = \alpha_0$ to get the existence of N_2.

In conclusion, for fixed birth and death rates, the patchiness of the environment tends to make the particle system more likely to survive even when the mesoscopic interactions may be significantly smaller. This somewhat interesting feature has already been observed for another spatially explicit model in which particles evolve on a chessboard as well (see Theorem 3 in Lanchier and Neuhauser, 2005). Even if technicalities are quite different due to the evolution mechanism of both processes, the intuitive background is similar. The extinction time of the process restricted to a single square of length side $2N - 1$ turns out to be exponential in N with some positive probability which is basically independent of the patchiness. When restoring the interactions between adjacent squares by setting $\alpha > 0$, the time required to invade an empty square with a particle whose offspring will live an exponentially long time is typically equal to a constant multiple of Z_N. That is, the invasion time can be made much shorter than the extinction time by increasing the patchiness. In particular, by dividing the time of the process into slices of height T_N and pretending that a cube in the resulting space-time partition is occupied if it contains at least one particle (see Figure 4), we find that, for a suitable T_N and all N sufficiently large, the set of occupied cubes percolate with positive probability.

The rest of the article is devoted to proofs. In Section 2, we rely on coupling arguments to investigate the monotonicity of the process. In Section 3, we introduce a mesoscopic or "semi-spatialized" version of our process to deduce Theorems 1 and 2 from their analogues for the individual recovery process. Finally, in Section 4, we construct the percolation structure described above, and prove Theorem 3.

2. Construction of the process. Proof of Lemmas 1.1 and 1.2

This section is devoted to the proof of Lemmas 1.1 and 1.2 whose first step is to construct the process from collections of independent Poisson processes, which is referred to as Harris' graphical representation (Harris, 1972). For each x, $z \in \mathbb{Z}^d$ with $x \sim z$ and $\dot{x} = \dot{z}$, we let $\{T_n(x,z) : n \geq 1\}$ denote the arrival times of independent Poisson processes with rate β, and draw an arrow from site x to site z at time $T_n(x,z)$ to indicate that a within birth may occur. To take into account the interactions between neighboring squares, we introduce, for any $\dot{x}$, $\dot{z} \in \mathbb{Z}^d / \asymp$ with $\dot{x} \sim \dot{z}$, and for any $i = 1, 2, \ldots, \kappa_N$, with $\kappa_N = (2N-1)^d$, a further collection of independent Poisson processes, denoted by $\{U_n(\dot{x}, \dot{z}, i) : n \geq 1\}$, each of them having rate α / Z_N. We put a i-arrow from square $\dot{x}$ to square $\dot{z}$ at time $U_n(\dot{x}, \dot{z}, i)$ to indicate that a birth from the outside may occur. Finally, for each $x \in \mathbb{Z}^d$, we let $\{V_n(x) : n \geq 1\}$ be the arrival times of independent rate 1 Poisson processes, and put a × at site x at time $V_n(x)$ to indicate that a death may occur.

Given an initial configuration η_0, and the graphical representation introduced above, the process can be constructed as follows. If there is a particle at site x at time $T_n(x,z)$ then site z becomes occupied if it is not already, that is the state of z flips from 0 to 1. In the same way, if there are at least i particles in $\dot{x}$ at time $U_n(\dot{x}, \dot{z}, i)$ then the center of square $\dot{z}$ becomes occupied if it is not already. Finally, a particle at site x at time $V_n(x)$, if it exists, is killed, that is the state of x flips from 1 to 0, regardless of the state of its neighbors. A nice feature of the graphical representation is that it allows us to couple processes with different parameters by using the same collections of Poisson processes. See Durrett (1995), page 119, and Liggett (1999), page 32.

With the graphical representation in hands, we can now prove the lemmas. Let $\eta_t^{^1}$ and $\eta_t^{^2}$ denote the two-scale contact processes with parameters (α_1, β_1, N_1) and (α_2, β_2, N_2), respectively, and assume that $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$ and $N_1 \leq N_2$. Since when $N_1 \neq N_2$ the processes $\eta_t^{^1}$ and $\eta_t^{^2}$ are defined on somewhat different spatial structures, the monotonicity with respect to N is not straightforward due to overlapping of squares. To fix the problem, the basic idea is to inject the small squares (those of length $2N_1 - 1$) into the large ones (those of length $2N_2 - 1$). Let $\dot{x}$ (respectively, $\ddot{x}$) be the class of x when the space scale is equal to N_1 (respectively, N_2), and consider the process $\eta_t^{^2}$ constructed from the collections of Poisson processes $T_n(x,z)$, $U_n(\ddot{x}, \ddot{z}, i)$ and $V_n(x)$ introduced above for $\alpha / Z_N = \alpha_2 / Z_{N_2}$ and $\beta = \beta_2$. To prove that

$$P_0^{^1}(|\eta_t^{^1}| \geq 1 \text{ for all } t \geq 0) \ \leq \ P_0^{^2}(|\eta_t^{^2}| \geq 1 \text{ for all } t \geq 0)$$

where $P_0^{^1}$ and $P_0^{^2}$ denote the laws of the processes $\eta_t^{^1}$ and $\eta_t^{^2}$ starting with a single particle at site 0, the next step is to construct the process $\eta_t^{^1}$ from the previous graphical representation in such a way that if $|\eta_t^{^1}| \geq 1$ then $|\eta_t^{^2}| \geq 1$ with probability 1.

For any $x \in \mathbb{Z}^d$, there exist a unique $q(x) \in \mathbb{Z}^d$ and a unique $r(x) \in (-N_1, N_1)^d$ such that

$$x \ = \ (2N_1 - 1)\, q(x) + r(x).$$

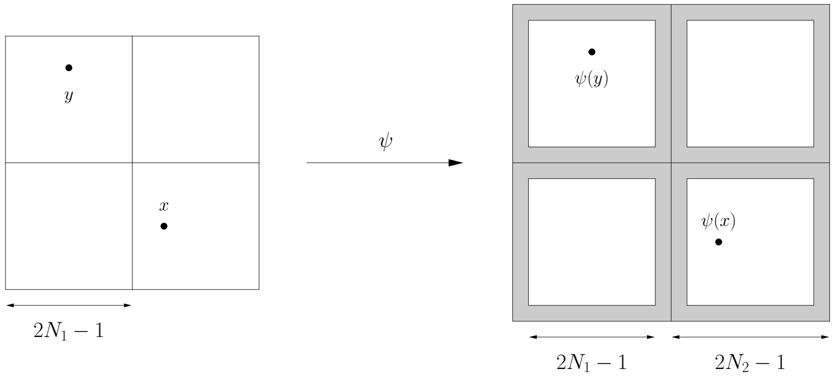

Figure 2. *Picture of the coupling argument. The process* η_t^1 *(on the left) is constructed from the Poisson processes attached to the white squares of the right picture. The Poisson processes attached to the grey part of the right picture have no effect on the process* η_t^1*.*

We then define a function $\psi : \mathbb{Z}^d \longrightarrow \mathbb{Z}^d$ by setting

$$\psi\,(x) \;=\; \psi\,((2N_1 - 1)\,q(x) + r(x)) \;=\; (2N_2 - 1)\,q(x) + r(x).$$

See Figure 2 for a picture. If we assume that $N_1 = N_2$ ($\psi = \mathrm{id}$) or $Z_N \equiv 1$ so that $Z_{N_1} = Z_{N_2}$, the two-scale contact process η_t^1 can be constructed from the graphical representation of η_t^2 as follows. If there is a particle at site x at time $T_n(\psi(x), \psi(z))$, we toss a coin with success probability β_1/β_2. If there is a success then site z becomes occupied if it is not already. In the same way, if there are at least i particles in $\dot{x}$ at time $U_n(\psi(\ddot{x}), \psi(\ddot{z}), i)$, we toss a coin with success probability α_1/α_2. If there is a success then the center of square $\dot{z}$ becomes occupied if it is not already. In particular, i-arrows with $i > (2N_1 - 1)^d$ have no effect on the process η_t^1 when $N_1 < N_2$. Finally, a particle at site x at time $V_n(\psi(x))$, if it exists, is killed regardless of the state of its neighbors.

When $N_1 = N_2$ (in particular $\psi = \mathrm{id}$), one can check from the previous coupling that for any site $x \in \mathbb{Z}^d$, $\eta_t^1(x) \leq \eta_t^2(x)$ with probability 1 at any time $t > 0$, provided the inequalities hold at time 0. This proves the monotonicity with respect to α and β.

When $Z_N \equiv 1$ and $N_1 \neq N_2$, the properties

$$\psi(\dot{x}) \;\subset\; \psi(\ddot{x}) \quad \text{for any } x \in \mathbb{Z}^d \qquad \text{and} \qquad \psi(L_{N_1}) \;=\; L_{N_2}$$

imply that, if $\eta_0^1(x) \leq \eta_0^2(\psi(x))$ for any $x \in \mathbb{Z}^d$ at time 0, then $\eta_t^1(x) \leq \eta_t^2(\psi(x))$ for any $x \in \mathbb{Z}^d$ with probability 1 at any later time $t > 0$. This completes the proof of Lemma 1.2.

From the graphical representation introduced above, we have the attractivity of the process η_t that is if $\eta_0^1(x) \leq \eta_0^2(x)$ for any $x \in \mathbb{Z}^d$ at time 0 then η_t^1 and η_t^2 can be constructed in the same probability space in such a way that

$$P^{1,2}\,(\forall\; x \in \mathbb{Z}^d,\; \eta_t^1(x) \leq \eta_t^2(x)) \;=\; 1 \qquad \text{for any} \quad t \geq 0,$$

with $P^{1,2}$ denoting the law of the coupled process starting from (η_0^1, η_0^2).

3. Proof of Theorems 1 and 2

The proof essentially relies on the analogous results for the individual recovery process introduced in Belhadji and Lanchier (2006), namely the continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \longrightarrow \{0, 1, \ldots, \kappa\}$ with $\xi_t(x)$ denoting the number of particles at x at time $t \geq 0$, and whose evolution at site x is given by the transitions

$$\begin{array}{llll} 0 \ \to \ 1 & \text{at rate} & \lambda \displaystyle\sum_{x \sim z} \xi(z) & \\ i \ \to \ i+1 & \text{at rate} & i\,\phi & i = 1, 2, \ldots, \kappa - 1 \\ i \ \to \ i-1 & \text{at rate} & i & i = 1, 2, \ldots, \kappa. \end{array}$$

The first step of the proof is to exhibit the connections between the individual recovery process and the two-scale contact process. The basic idea is that we can turn the first one into the second one thanks to a "spatialization" of the cluster of particles at each site, and the second one into the first one thanks to a "destruction" of the spatial structure of each square, which suggests a correspondence between interacting random walks and interacting finite state space contact processes. To figure out the evolution of the two-scale contact process at a mesoscopic scale, we introduce the process $\bar\eta_t : \mathbb{Z}^d \longrightarrow \{0, 1, \ldots, \kappa_N\}$ with $\kappa_N = (2N-1)^d$ defined by

$$\bar\eta_t(x) \ = \ \sum_{z \in \mathcal{H}_x} \eta_t(z).$$

In other words, $\bar\eta_t$ counts the number of particles in each square. Unfortunately, in view of the loss of information due to the destruction of the spatial structure, $\bar\eta_t$ is not a Markov process. In particular, the rate of the transition $i \to i+1$ at site $x \in \mathbb{Z}^d$ cannot be computed since it depends on the spatial configuration of the particles in square $\mathcal{H}_x$. If square $\mathcal{H}_x$ is empty, however, it may get invaded only from the outside which gives rise to the transition

$$0 \ \to \ 1 \qquad \text{at rate} \qquad \frac{\alpha}{Z_N} \sum_{x \sim z} \bar\eta(z)$$

at site $x \in \mathbb{Z}^d$. Moreover, since the superposition of i Poisson processes with rate 1 is a Poisson process with rate i, the first death in a square with i particles occurs at rate i so that

$$i \ \to \ i-1 \qquad \text{at rate} \qquad i \qquad i = 1, 2, \ldots, \kappa_N.$$

In particular, when $\lambda = \alpha / Z_N$ and $\kappa = \kappa_N$, the transitions $0 \to 1$ and $i \to i-1$ at site $x \in \mathbb{Z}^d$ occur at the same rate in the individual recovery process and the two-scale contact process.

To prove the convergence to the "all 0" configuration in Theorem 1, we first observe that, when square $\mathcal{H}_x$ has i particles, the number of empty sites with an occupied neighbor can be bounded by $2di$ so that, at site $x \in \mathbb{Z}^d$,

$$i \ \to \ i+1 \qquad \text{at rate at most} \qquad 2di\,\beta \ + \ \frac{\alpha}{Z_N} \sum_{x \sim z} \bar\eta(z) \qquad i = 1, 2, \ldots, \kappa_N - 1.$$

In particular, when $\lambda = \alpha = 0$ and $\phi = 2d\beta$, the processes ξ_t and $\bar\eta_t$ can be defined on the same space in such a way that, for any $x \in \mathbb{Z}^d$, $\xi_t(x) \geq \bar\eta_t(x)$ with probability 1 at any time, provided the inequalities hold at time 0. Theorem 1 in Belhadji and Lanchier (2006) then implies that, for any $x \in \mathbb{Z}^d$ and any $z \in \mathcal{H}_x$

$$\liminf_{t \to \infty} \ P^{\eta}(\eta_t(z) = 0) \ \geq \ \liminf_{t \to \infty} \ P^{\bar\eta}(\bar\eta_t(x) = 0) \ \geq \ \liminf_{t \to \infty} \ P^{\xi}(\xi_t(x) = 0) \ = \ 1$$

when $\alpha = 0$. The proof relies on a comparison of the particle system viewed on suitable length and time scale with a 1-dependent oriented percolation process (Bramson and Durrett, 1988), which provides a good enough rate of convergence of $P^{\eta}(\eta_t(z) = 0)$ so that a perturbation argument can be applied. In particular, since the transition rates of η_t are continuous with respect to α, we can conclude that there is a small enough $\alpha_1 > 0$ depending on β and N so that the two-scale process converges to the "all 0" configuration when $\alpha < \alpha_1$.

To prove the convergence to the "all 0" configuration in Theorem 2, we follow the same strategy as above. First of all, when $\alpha \in (0, Z_N \, \lambda_c)$ and $\beta = 0$, the process $\bar{\eta}_t$ reduces to a subcritical contact process so that, for any $x \in \mathbb{Z}^d$ and any $z \in \mathcal{H}_x$

$$\liminf_{t \to \infty} \ P^{\,\eta} \, (\eta_t(z) = 0) \ \geq \ \liminf_{t \to \infty} \ P^{\,\bar{\eta}} \, (\bar{\eta}_t(x) = 0) \ = \ 1$$

with a good enough rate of convergence (thanks to a comparison with an oriented percolation process) so that a perturbation argument can be applied (see, e.g., Schinazi, 2002). As previously, the existence of a small $\beta_1 > 0$ so that η_t converges in distribution to the "all 0" configuration when $\beta < \beta_1$ follows from the continuity of the transition rates with respect to β.

The existence of a nontrivial stationary distribution in Theorems 1 and 2 relies on a coupling argument. Since a site in $\mathcal{H}_x$ has at least d neighbors belonging to $\mathcal{H}_x$ the transition $i \to i+1$ at site $x \in \mathbb{Z}^d$ occurs at rate at least $d\beta$ for the process $\bar{\eta}_t$ provided $i \geq 1$. In particular,

$$i \ \to \ i+1 \qquad \text{at rate at least} \qquad d i \, \beta / \kappa_N \qquad i = 1, 2, \ldots, \kappa_N - 1.$$

It follows that, when $\lambda = \alpha / Z_N$, $\phi = d \, \beta / \kappa_N$, and $\kappa = \kappa_N$,

$$\liminf_{t \to \infty} \ P_1^{\,\bar{\eta}} \, (\bar{\eta}_t(x) = 1) \ \geq \ \liminf_{t \to \infty} \ P_1^{\,\xi} \, (\xi_t(x) = 1) \qquad \text{for any } x \in \mathbb{Z}^d$$

where $P_1^{\,\bar{\eta}}$ and $P_1^{\,\xi}$ denote the laws of the processes starting from the "all 1" configuration. This, together with Theorem 1 (respectively, 3) in Belhadji and Lanchier (2006), implies the existence of a $\alpha_2 < Z_N \, \lambda_c$ depending on β and N (respectively, a $\beta_2 < \infty$ depending on α and N) such that, when $\alpha > \alpha_2$ (respectively, $\beta > \beta_2$), the process has a nontrivial stationary distribution.

Finally, we apply Lemma 1.1 to get the existence of $\alpha_c \in [\alpha_1, \alpha_2]$ and $\beta_c \in [\beta_1, \beta_2]$ satisfying the statements of Theorems 1 and 2.

4. Proof of Theorem 3

First of all, note that we do the proof for any value of Z_N. To prove Theorem 3, the strategy is to compare the particle system viewed on suitable length and time scales with a 1-dependent oriented percolation process on

$$\mathcal{L} \ = \ \{(z, n) \in \mathbb{Z}^2 : z + n \text{ is even and } n \geq 0\}.$$

To compare both processes, we let T_N be a large integer depending on N to be fixed later and say that a site $(z, n) \in \mathcal{L}$ is occupied if at any time $t \in [n \, T_N, (n+1) \, T_N]$ there is at least one particle in square $\mathcal{H}_{ze_1}$, where e_1 denotes the first unit vector of the d-dimensional integer lattice. Theorem 3 then follows from

Lemma 4.1 *Let $\alpha > 0$ and $\beta > \lambda_c$. Then, for any $\varepsilon > 0$, there exists a large enough N such that the set of occupied sites dominates the set of wet sites in a 1-dependent oriented percolation process with parameter $1 - \varepsilon$.*

The proof of Lemma 4.1 essentially relies on estimates on the extinction time of the process restricted to a single square (when the outside birth rate α is equal to 0).

The contact process on a finite set

To describe the behavior of the process in an isolated square, we consider the basic contact process ζ_t^N evolving on the finite set $B_N = (-N, N)^d$ and let

$$\tau_N \ = \ \inf \{t \geq 0 : \zeta_t^N \equiv 0\}$$

denote the extinction time of the process. Since ζ_t^N is a finite state Markov process that is irreducible, it converges in distribution to its single absorbing state, the "all 0" configuration, so that we have $\tau_N < \infty$ with probability 1. The aim is to prove that, when N is large, the process starting with a single particle at site 0 dies out exponentially fast or lives an exponentially long time. To avoid cumbersome notations, we prove the result in two dimensions only but the proof is easy to extend to any dimension $d \geq 1$. First of all, we tile B_N with $a \times a$ squares by setting

$$\pi(w) \ = \ (aw_1, aw_2), \quad D \ = \ (-a/2, a/2\,]^2 \quad \text{and} \quad D(w) \ = \ \pi(w) \ + \ D$$

for $w \in \mathbb{Z}^2$, and assume, for more convenience, that $q_N = (2N-1)/a$ is an integer. We say that a site w with $D(w) \subset B_N$ is good if the square $D(w)$ contains at least $\sqrt{a}$ particles, and that a configuration $\zeta^N : B_N \to \{0,1\}$ is good if, in the configuration ζ^N, there are at least $\sqrt{N}$ good sites w. Let $\mathcal{G}_N$ denote the set of good configurations, and Ω_N the event

$$\Omega_N \ = \ \{\zeta_t^N \in \mathcal{G}_N \text{ for some } t \geq 0\}.$$

All along this section, P_ζ^N will denote the law of the contact process ζ_t^N starting from the configuration $\zeta_0^N = \zeta$ with the notation P_0^N for the process starting with a single particle at 0. The first step is to prove that the process starting from a good configuration lives an exponentially long time. More precisely, we have the following

Lemma 4.2 *Let $\beta > \lambda_c$ and $\kappa_N = (2N-1)^2$. Then there exist $\delta > 0$ and $a > 0$ such that*

$$\sup_{\zeta \in \mathcal{G}_N} \ P_\zeta^N\,(\tau_N < 2\,e^{\delta \kappa_N}) \ \leq \ C_1 \exp(-\gamma_1\, q_N^2)$$

for suitable constants $C_1 < \infty$ and $\gamma_1 > 0$.

Proof. The first step is to set the parameter a sufficiently large so that the process viewed on the macroscopic $q_N \times q_N$ lattice dominates a (supercritical) oriented percolation process. Let $\varepsilon > 0$ small. Since $\beta > \lambda_c$ there exist a $\Gamma > 0$ and a large $a > 0$, fixed from now on, such that the following holds: If $w \sim w'$ with $D(w)$, $D(w') \subset B_N$, and w is good at time 0, then

$$P\,(w' \text{ is good at time } \Gamma a) \ \geq \ 1 - \varepsilon. \tag{1}$$

See Bezuidenhout and Grimmett (1990). To complete the proof, we use an idea introduced in Liggett (1999), p 76, to compare the contact process on B_N with an oriented percolation process with parameter $1-\varepsilon$ evolving in a linear tube of length q_N^2 embedded in B_N. The basic construction is shown in Figure 3. The comparison is obtained by assuming that births through the continuous thick lines of the picture are not allowed. That is, if x, $z \in B_N$ with $x \sim z$ belong to adjacent $a \times a$ squares whose joint side is drawn in thick line, then we suppress the interactions between sites x and z. For any $n \geq 0$, let W_n denote the set of wet sites in the percolation process with parameter $1-\varepsilon$ at level n, and set

$$X_n \ = \ \{w \in \mathbb{Z}^2 : D(w) \subset B_N \text{ and } w \text{ is good at time } n\,\Gamma a\}.$$

Then, the lower bound in (1) implies that the set of open sites in the percolation process can be chosen in such a way that $W_n \subset X_n$ at any level $n \geq 1$ provided $W_0 \subset X_0$ (see Bramson and Durrett, 1988). In particular, the stopping time τ_N is bounded from below by the extinction time of the percolation process. Since the set of open sites at time 0 can be made arbitrarily large by choosing N large enough (recall that $\zeta_0 \in \mathcal{G}_N$ implies that $|W_0| \geq \sqrt{N}$), the lemma follows from the analogous result for the percolation process evolving on the segment $\{0, 1, \ldots, q_N^2 - 1\}$. See Durrett and Schonmann (1988) for a proof of this last result. $\square$

Lemma 4.3 *Let $\beta > \lambda_c$. Then there exist $C_2 < \infty$ and $\gamma_2 > 0$ such that*

$$P_0^N\,(\tau_N > t \ ; \ \Omega_N^c) \ \leq \ C_2 \exp(-\gamma_2\, N) \ + \ C_2 \exp(-\gamma_2\, t).$$

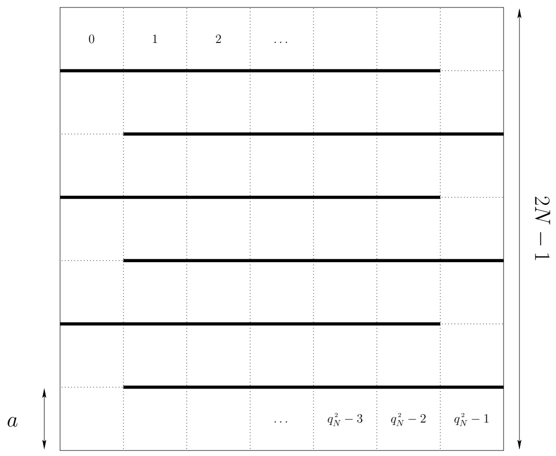

Figure 3.

Proof. The basic idea is to decompose the event to be estimated according to whether the stopping time π_N is finite or infinite, where

$$\pi_N \ = \ \inf\,\{t \geq 0 : \zeta_t^N\,(x) = 1 \text{ for some } x \in B_N \setminus B_{N-1}\}$$

denotes the first time the process reaches the boundary of B_N. To begin with, the Shape Theorem in Liggett (1999), p 128, implies that, on the event $\{\pi_N < \infty\}$, there exists a constant $c > 0$ such that, with probability at least $1 - C_3\,\exp\,(-\gamma_3\,N)$, the process ζ_t^N is distributed according to the upper invariant measure $\bar\mu$ of the basic contact process in $(-c\,N, c\,N)^2$ at time π_N. Since the measure $\bar\mu$ is ergodic (see Proposition 2.16, p 143, in Liggett, 1985), there are at least $\sqrt{N}$ good sites $w \in \mathbb{Z}^2$ with $D(w) \subset B_N$ at time π_N for all N sufficiently large. It follows that

$$P_0^N\,(\Omega_N \mid \pi_N < \infty) \ \geq \ 1 - C_3\,\exp\,(-\gamma_3\,N).$$

To complete the proof, we let ζ_t denote the contact process with parameter β and τ the extinction time of ζ_t. Bezuidenhout and Grimmett (1990) have shown that, provided $\beta > \lambda_c$, the contact process viewed on suitable length and time scales dominates oriented percolation. Combining this with the analogous result for oriented percolation in Durrett (1984), Section 12, proves that there are constants $C_4 < \infty$ and $\gamma_4 > 0$ such that

$$P_0\,(\tau > t \mid \tau < \infty) \ \leq \ C_4\,\exp\,(-\gamma_4\,t)$$

where P_0 denotes the law of the process ζ_t starting with a single particle at site 0. See for instance Theorem 2.30 in Liggett (1999). In other respects, both processes ζ_t and ζ_t^N can be constructed on the same probability space in such a way that, provided $\pi_N = \infty$, $\zeta_t = \zeta_t^N$ a.s. at any time $t \geq 0$ so that $\{\pi_N = \infty\} \subset \{\tau < \infty\}$. This, together with the previous estimate, implies that

$$P_0^N\,(\tau_N > t \mid \pi_N = \infty) \ \leq \ C_4\,\exp\,(-\gamma_4\,t).$$

In conclusion, we get

$$\begin{aligned} P_0^N\,(\tau_N > t\ ;\ \Omega_N^c) \ &\leq \ P_0^N\,(\Omega_N^c\ ;\ \pi_N < \infty) \ + \ P_0^N\,(\tau_N > t\ ;\ \pi_N = \infty) \\ &\leq \ C_3\,\exp\,(-\gamma_3\,N) \ + \ C_4\,\exp\,(-\gamma_4\,t). \end{aligned}$$

This completes the proof. $\square$

Lemma 4.4 *Let* $\beta > \lambda_c$ *and* $\kappa_N = (2N-1)^2$. *Then there exist* $C_5 < \infty$ *and* $\gamma_5 > 0$ *such that*

$$P_0^N\,(t < \tau_N < 2\,e^{\delta\kappa_N}) \ \leq\ C_5\,\exp\,(-\gamma_5\,N) \ +\ C_5\,\exp\,(-\gamma_5\,t)$$

where δ *is the constant introduced in Lemma 4.2.*

Proof. To begin with, we observe that, on the event Ω_N, the first time the process ζ_t^N hits the "all 0" configuration when starting from a good configuration is bounded in distribution by the first time it hits the "all 0" configuration when starting with a single particle at site 0 so that

$$P_0^N\,(\tau_N < 2\,e^{\delta\kappa_N}\ ;\ \Omega_N) \ \leq\ \sup_{\zeta\in\mathcal{G}_N}\ P_\zeta^N\,(\tau_N < 2\,e^{\delta\kappa_N}).$$

In particular, by decomposing the event to be estimated according to whether Ω_N occurs or not, it follows from Lemma 4.2 and Lemma 4.3 that

$$\begin{array}{rcl} P_0^N\,(t < \tau_N < 2\,e^{\delta\kappa_N}) & \leq & P_0^N\,(\tau_N < 2\,e^{\delta\kappa_N}\ ;\ \Omega_N) \ +\ P_0^N\,(\tau_N > t\ ;\ \Omega_N^c) \\ & \leq & C_1\,\exp\,(-\gamma_1\,q_N^2) \ +\ C_2\,\exp\,(-\gamma_2\,N) \ +\ C_2\,\exp\,(-\gamma_2\,t) \\ & \leq & C_5\,\exp\,(-\gamma_5\,N) \ +\ C_5\,\exp\,(-\gamma_5\,t) \end{array}$$

for suitable $C_5 < \infty$ and $\gamma_5 > 0$. This completes the proof of the lemma. □

Percolation structure. Proof of Theorem 3

With Lemma 4.4 in hands, we are now ready to prove Lemma 4.1 and Theorem 3. The proof is similar to the proof of Theorem 5 in Belhadji and Lanchier (2006). First of all, since the evolution rules of η_t are invariant by translation of vector a constant multiple of $(2N-1)\,e_1$, it suffices to prove the following statement: If there is at least one particle in square $\mathcal{H}_0$ from time 0 to time T_N then, with probability at least $1-\varepsilon$ when N is sufficiently large, there is at least one particle in square $\mathcal{H}_{e_1}$ from time T_N to time $2\,T_N$. See Figure 4 for a picture. Moreover, to facilitate the application of the restart argument described in Lemma 4.5 below, we prove the result for the process η_t modified so that births from the outside are not allowed in nonempty squares, that is squares in which there is at least one particle. Each time a particle in $\mathcal{H}_0$ gives birth to a particle which is sent to the (empty) square $\mathcal{H}_{e_1}$, we call this particle a good particle if its offspring lives at least $2\,T_N$ units of time, and let

$$\sigma_{e_1} \ =\ \inf\,\{t \geq 0 : \bar\eta_s(e_1) \neq 0 \text{ for any } t \leq s \leq t + 2\,T_N\}$$

denote the first time a good particle is sent to square $\mathcal{H}_{e_1}$. Then,

Lemma 4.5 *Let* $\alpha > 0$, $\beta > \lambda_c$ *and* $T_N = e^{\delta\kappa_N}$. *If site* $(0,0)$ *is occupied, then*

$$P^{\eta}\,(\sigma_{e_1} > T_N) \ \leq\ \varepsilon \quad \textit{for all } N \textit{ sufficiently large.}$$

Proof. For any $i \geq 1$, we introduce the stopping times

$$\rho_i \ =\ \inf\,\{t \geq \bar\rho_{i-1} : \bar\eta_t(e_1) = 1\} \qquad \text{and} \qquad \bar\rho_i \ =\ \inf\,\{t \geq \rho_i : \bar\eta_t(e_1) = 0\}$$

with the convention $\bar\rho_0 = 0$. That is, ρ_i is the i-th time a particle born in $\mathcal{H}_0$ is sent to the empty square $\mathcal{H}_{e_1}$ and $\bar\rho_i$ the i-th time square $\mathcal{H}_{e_1}$ becomes empty. Let

$$K \ =\ \inf\,\{i \geq 1 : \bar\rho_i - \rho_i > 2\,T_N\}$$

so that $\rho_K = \sigma_{e_1}$. By decomposing according to the value of K, we get

$$\begin{aligned} P^{\eta}(\sigma_{e_1} > T_N) &= \sum_{n=1}^{\infty} P^{\eta}(\rho_n > T_N \mid K = n)\, P^{\eta}(K = n) \\ &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} P^{\eta}(\rho_{i+1} - \rho_i > T_N/n \text{ and } \bar{\rho}_i - \rho_i \leq 2\,T_N)\, P^{\eta}(K = n) \\ &\leq \sum_{n=1}^{\infty} n\, P^{\eta}(\rho_2 - \rho_1 > T_N/n \text{ and } \bar{\rho}_1 - \rho_1 \leq 2\,T_N)\, P^{\eta}(K = n). \end{aligned}$$

First of all, Lemma 4.2 implies that, for N sufficiently large,

$$P^{\eta}(K = 1) \;\geq\; \inf_{\zeta \in \mathcal{G}_N} P_{\zeta}^{N}(\tau_N > 2\,T_N)\, P_0^{N}(\Omega_N) \;\geq\; p_{\beta}$$

where $p_{\beta} > 0$ provided $\beta > \lambda_c$. In particular, for N large, there is a large enough n_{ε} such that

$$\sum_{n=n_{\varepsilon}}^{\infty} n\, P^{\eta}(K = n) \;=\; \sum_{n=n_{\varepsilon}}^{\infty} n\, p_{\beta}\,(1 - p_{\beta})^{n-1} \;\leq\; \varepsilon/2.$$

To show that the first n_{ε} terms can be made smaller then $\varepsilon/2$, we first observe that

$$\begin{aligned} P^{\eta}(\rho_2 - \rho_1 > T_N/n \text{ and } \bar{\rho}_1 - \rho_1 \leq 2\,T_N) \;&\leq\; P^{\eta}(\rho_2 - \bar{\rho}_1 > T_N/2n) \\ &\qquad + P^{\eta}(T_N/2n < \bar{\rho}_1 - \rho_1 \leq 2\,T_N). \end{aligned}$$

Since Z_N is polynomial in N and that there is at least one particle in $\mathcal{H}_0$, we get

$$P^{\eta}(\rho_2 - \bar{\rho}_1 > T_N/2n) \;\leq\; \exp(-\alpha\, e^{\delta \kappa_N}/2 Z_N n_{\varepsilon}) \;\leq\; \varepsilon/4n_{\varepsilon}$$

for $n \leq n_{\varepsilon}$ and N sufficiently large. Finally, Lemma 4.4 implies that

$$\begin{aligned} P^{\eta}(T_N/2n < \bar{\rho}_1 - \rho_1 \leq 2\,T_N) \;&\leq\; P_0^{N}(T_N/2n_{\varepsilon} < \tau_N < 2\,T_N) \\ &\leq\; 2\,C_5\, \exp(-\gamma_5\, N) \;\leq\; \varepsilon/4n_{\varepsilon} \end{aligned}$$

for all $n \leq n_{\varepsilon}$ and all N sufficiently large. This completes the proof of the lemma. $\square$

Lemma 4.1 is a straightforward consequence of Lemma 4.5. To construct our nontrivial stationary distribution, we start η_t from the "all 1" configuration. Theorem 3.9 in Liggett (1999) implies that there are infinitely many $z \in 2\mathbb{Z}$ such that $(z, 0)$ is occupied. By using the attractivity of the process we obtain that η_t converges in distribution to the upper stationary measure μ. See Theorem 2.7 in Durrett (1995). Finally, since percolation occurs with positive probability when $\varepsilon > 0$ is small, it follows from Lemma 4.1 that

$$\liminf_{n \to \infty} \;\mu((z, n) \text{ is occupied}) \;\geq\; \liminf_{n \to \infty} \;P_{\varepsilon}((z, n) \text{ is wet}) \;>\; 0 \qquad \text{if } N \geq N_{\varepsilon}$$

where P_{ε} denotes the law of the percolation process with parameter $1 - \varepsilon$. This implies that μ concentrates on configurations with infinitely many 1's. When $Z_N \equiv 1$, due to lemme 1.2 the existence of N_c follows and completes the proof of Theorem 3.

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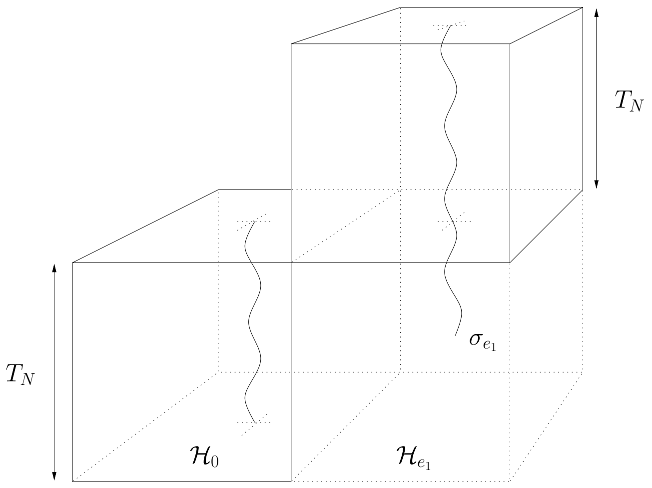

Figure 4.

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Laboratoire de Mathématiques Raphaël Salem,
UMR 6085, CNRS - Université de Rouen,
Avenue de l'Université, BP. 12,
76801 Saint Etienne du Rouvray, France.